P1)

Auxiliar 13

(a)

- Ecuaciones de movimiento:

$$m\ddot{y}_1 = -k(y_1 - l_0) + k\left((y_2 - y_1) - l_0\right) + mg \qquad (1)$$

$$m\ddot{y}_2 = -k\left((y_2 - y_1) - l_0\right) + mg \qquad (2)$$

- Equilibrio ssi $\ddot{y}_1 = \ddot{y}_2 = 0$

$$(2) \Rightarrow \quad y_2^* - y_1^* = l_0 + \frac{mg}{k} \Rightarrow \boxed{y_1^* = l_0 + \frac{2mg}{k}}$$

$$\Rightarrow \boxed{y_2^* = 2l_0 + \frac{3mg}{k}}$$

(b)

- Usando $\delta_1 = y_1 - y_1^*$; $\delta_2 = y_2 - y_2^*$:

$$\Rightarrow m\ddot{\delta}_1 = -k\left(\delta_1 + \cancel{\frac{2mg}{k}}\right) + k\left(\delta_2 - \delta_1 + \cancel{\frac{mg}{k}}\right) + \cancel{mg}$$

$$m\ddot{\delta}_2 = -k\left(\delta_2 - \delta_1 + \cancel{\frac{mg}{k}}\right) + \cancel{mg}$$

- Definiendo $\omega_0^2 = \frac{k}{m}$:

$$\Rightarrow \boxed{\begin{pmatrix} \ddot{\delta}_1 \\ \delta_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_2 \\ \delta_1 \end{pmatrix}}$$

Scanned by CamScanner

(c)

- Como $\delta_i = A_i \sin(\omega t) + B_i \cos(\omega t) \Rightarrow \ddot{\delta}_i = -\omega^2 \delta_i$

$$\Rightarrow -\omega^2 \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = 0$$

- Polinomio característico:

$$(2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = 0$$

$$\Rightarrow \omega^4 - 3\omega_0^2\omega^2 + 2\omega_0^4 - \omega_0^4 = 0$$

$$\Rightarrow (\omega^2)^2 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

$$\Rightarrow \omega_\pm^2 = \frac{3 \pm \sqrt{9-4}}{2}\omega_0^2 \Rightarrow \boxed{\omega_\pm^2 = \frac{\sqrt{9} \pm \sqrt{5}}{2}\omega_0^2}$$

(d)

- La relación entre las amplitudes corresponde a los vectores propios:

$$(2\omega_0^2 - \omega_\pm^2)\delta_1 - \omega_0^2\delta_2 = 0$$

$$\Rightarrow \delta_2 = \frac{2\omega_0^2 - 3\omega_0^2 \mp \sqrt{5}\,\omega_0^2}{\omega_0^2}\delta_1 \Rightarrow \delta_2 = (-1 \mp \sqrt{5})\delta_1$$

$$\Rightarrow \omega_+^2 = \frac{3+\sqrt{5}}{2}\omega_0^2 \text{ tiene asociado } \begin{pmatrix} 1 \\ -1-\sqrt{5} \end{pmatrix}$$

$$\omega_-^2 = \frac{3-\sqrt{5}}{2}\omega_0^2 \text{ tiene asociado } \begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix}$$

P1

Auxiliar 13

(a)

$$\boxed{U(x,y) = \frac{1}{2}k(x-L)^2 + \frac{1}{2}k\left(\sqrt{x^2+y^2} - \sqrt{2}L\right)^2}$$

(b)

- Los equilibrios del sistema se encuentran en $\nabla U = 0$

$$\nabla U = \left[k(x-L) + k\left(\sqrt{x^2+y^2} - \sqrt{2}L\right)\frac{x}{\sqrt{x^2+y^2}}\right]\hat{x}$$
$$+ \left[k\left(\sqrt{x^2+y^2} - \sqrt{2}L\right)\frac{y}{\sqrt{x^2+y^2}}\right]\hat{y}$$

- $\nabla U = 0$ si $y = 0$, o bien $\sqrt{x^2+y^2} = \sqrt{2}L$

1. $y = 0$:

$$\Rightarrow k(x-L) + k(x - \sqrt{2}L) = 0 \Rightarrow x = \frac{1+\sqrt{2}}{2}L$$

$$\Rightarrow \boxed{(x_1^*, y_1^*) = \left(\frac{1+\sqrt{2}}{2}L, 0\right)}$$

2. $\sqrt{x^2+y^2} = \sqrt{2}L$:

$$\Rightarrow x = L \Rightarrow y = L \Rightarrow \boxed{(x_2^*, y_2^*) = (L, L)}$$

- Para analizar la estabilidad, hay que ver si el Hessiano es definido positivo (estable) o definido negativo (inestable)
  - ↳ una matriz es definida positiva si todos sus valores propios son positivos y definida negativa si son negativos.

$$H = \begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x \partial y} \\ \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial y^2} \end{bmatrix} \longrightarrow \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \quad pq \quad U \in C^\infty$$

$$\frac{\partial^2 U}{\partial x^2} = k + \frac{k x^2}{x^2+y^2} + k\left(\sqrt{x^2+y^2} - \sqrt{2}L\right) \cdot \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{k y^2}{x^2+y^2} + k\left(\sqrt{x^2+y^2} - \sqrt{2}L\right) \cdot \frac{\sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{k x y}{x^2+y^2} - k\left(\sqrt{x^2+y^2} - \sqrt{2}L\right) \cdot \frac{x y}{(x^2+y^2)^{3/2}}$$

- Reemplazando las derivadas en (L,L) se tiene que:

$$H(L,L) = \frac{k}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

- Resolviendo el problema de valores propios:

$$(\lambda - 3)(\lambda - 1) - 1 = 0 \Rightarrow \lambda^2 - 4\lambda + 2 = 0$$

$$\Rightarrow \lambda_\pm = \frac{4 \pm \sqrt{16-8}}{2}$$

$$\Rightarrow \lambda_\pm = \frac{\sqrt{16} \pm \sqrt{8}}{2} \Rightarrow \lambda_\pm > 0 \Rightarrow H(L,L) \text{ es definida positiva}$$

$\Rightarrow$ $\boxed{(L,L) \text{ es estable}}$

- Como $U(x,y)$ es continua y $U \in C^\infty$, no pueden haber dos mínimos seguidos $\Rightarrow$ $\boxed{\left(\frac{1+\sqrt{2}}{2}L, 0\right) \text{ es inestable.}}$

Scanned by CamScanner

(c)

$$x = L + \delta_x \quad , \quad y = L + \delta_y$$

$$\Rightarrow U(x,y) = \frac{k}{2}\delta_x^2 + \frac{k}{2}\left[\sqrt{2L^2 + 2L(\delta_x+\delta_y) + \delta_x^2 + \delta_y^2} - \sqrt{2}L\right]^2$$

· Se sabe que cuando $\xi \ll 1$, se tiene que:

$$\sqrt{A+\xi} = \sqrt{A} + \frac{\xi}{2\sqrt{A}} - \frac{\xi^2}{8\sqrt{A}^3} + \cdots$$

$$\Rightarrow U(x,y) \approx \frac{k}{2}\delta_x^2 + \frac{k}{2}\left[\cancel{\sqrt{2}L} + \frac{2L(\delta_x+\delta_y) + \delta_x^2 + \delta_y^2}{2\sqrt{2}L} - \frac{(2L)^2(\delta_x+\delta_y)^2}{8\sqrt{2}^3L^3} - \cancel{\sqrt{2}L}\right]^2$$

$$\Rightarrow U(x,y) \approx \frac{k}{2}\delta_x^2 + \frac{k}{2}\cdot\frac{\cancel{4}\cancel{L^2}(\delta_x+\delta_y)^2}{\underset{2}{\cancel{8}}\cancel{L^2}}$$

$$\Rightarrow \boxed{U(x,y) \approx \frac{k}{2}\delta_x^2 + \frac{k}{4}(\delta_x+\delta_y)^2}$$

(d)

$$\nabla U = \left[k\delta_x + \frac{k}{2}(\delta_x+\delta_y)\right]\hat{x} + \left[\frac{k}{2}(\delta_x+\delta_y)\right]\hat{y}$$

$$\Rightarrow \begin{aligned} m\ddot{\delta}_x &= -k\delta_x - \frac{k}{2}\delta_x - \frac{k}{2}\delta_y \\ m\ddot{\delta}_y &= -\frac{k}{2}\delta_x - \frac{k}{2}\delta_y \end{aligned}$$

· Definiendo $\omega_0^2 = \frac{k}{m}$:

$$\Rightarrow \boxed{\begin{pmatrix} \ddot{\delta}_x \\ \ddot{\delta}_y \end{pmatrix} = \omega_0^2 \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix}}$$

(e)

$$\ddot{\delta}_x = -\omega^2 \delta_x \quad ; \quad \ddot{\delta}_y = -\omega^2 \delta_y$$

$$\Rightarrow -\omega^2 \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \omega_0^2 \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{3}{2}\omega_0^2 - \omega^2 & \frac{1}{2}\omega_0^2 \\ \frac{1}{2}\omega_0^2 & \frac{1}{2}\omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = 0$$

- Polinomio característico:

$$\left(\frac{3}{2}\omega_0^2 - \omega^2\right)\left(\frac{1}{2}\omega_0^2 - \omega^2\right) - \frac{\omega_0^4}{4} = 0$$

$$\Rightarrow \omega^4 - 2\omega_0^2\omega^2 + \frac{3\omega_0^4}{4} - \frac{\omega_0^4}{4} = 0$$

$$\Rightarrow (\omega^2)^2 - 2\omega_0^2(\omega^2) + \frac{\omega_0^4}{2} = 0$$

$$\Rightarrow \omega^2 = \omega_0^2 \pm \sqrt{\omega_0^4 - \frac{\omega_0^4}{2}} \Rightarrow \boxed{\omega_\pm^2 = \frac{\sqrt{2}\,\omega_0^2 \pm \omega_0^2}{\sqrt{2}}}$$

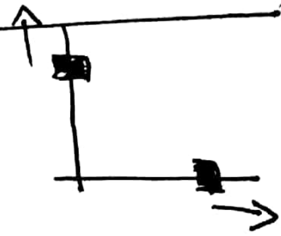
- Vectores propios:

$$\left(\frac{3}{2}\omega_0^2 - \omega_\pm^2\right)\delta_x + \frac{1}{2}\omega_0^2\delta_y = 0$$

$$\Rightarrow \delta_y = \frac{2\omega_\pm^2 - 3\omega_0^2}{\omega_0^2}\,\delta_x \quad \Rightarrow \delta_y = (2 \pm \sqrt{2} - 3)\,\delta_x$$

$$\Rightarrow \delta_y = (\pm\sqrt{2} - 1)\,\delta_x$$

$\Rightarrow$ $\omega_+^2 = \dfrac{\sqrt{2}+1}{\sqrt{2}}\,\omega_0^2$ tiene asociado $\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$



$\omega_-^2 = \dfrac{\sqrt{2}-1}{\sqrt{2}}\,\omega_0^2$ tiene asociado $\begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix}$