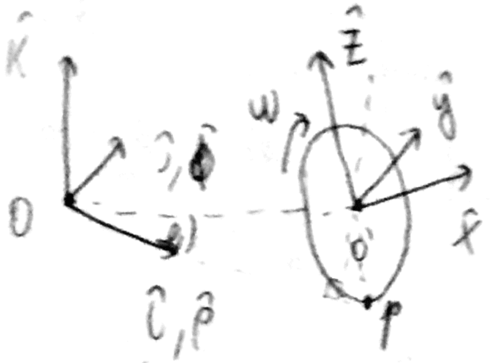
auxiliar 11

2



$\phi = 0$

- O' rota de forma NO-SOLIDARIA con la moneda, pero es tal que $\vec{R}_{cm} = \vec{R}_{O'}$, y $\hat{x}, \hat{z}, \hat{y}$ están en los ejes principales
  → rota con velocidad $\Omega \hat{K}$

$\vec{R}_{O'} = \rho \hat{\rho} + z \hat{K} =$ ~~$b\hat{\rho} + (b-a$~~

$= (b - a \operatorname{sen} \theta)\hat{\rho} + a \cos\theta \hat{K}$

$\rightarrow \vec{v}_{O'} = (b - a \operatorname{sen}\theta)\dot{\phi}\hat{\phi}$

- Condición de rodadura

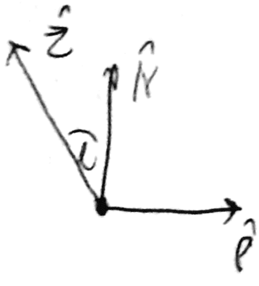
$\rightarrow \vec{v}_p = \vec{V}_{cm} + \vec{W}_{cuerpo} \times \vec{r}_{cm \to p} = 0$

- $\vec{W}_{cuerpo}$ es la suma de $\vec{w}$ y $\vec{\Omega}$ (rotación + precesión)

$\rightarrow (b - a\operatorname{sen}\theta)\dot{\phi}\hat{\phi} + (\vec{w} + \vec{\Omega}) \times (-a\hat{z}) = 0$

$\rightarrow (b - a\operatorname{sen}\theta)\dot{\phi}\hat{\phi} + (-w\hat{x} + \Omega\hat{K}) \times (-a\hat{z}) = 0$

$\hat{x} \times \hat{z} = -\hat{y} = -\hat{\phi}$, $\hat{K} \times \hat{z} = \hat{K} \times (\hat{K}\cos\theta - \hat{\rho}\operatorname{sen}\theta)$



$=$ ~~$\cos\theta$~~ $- \operatorname{sen}\theta\hat{\phi}$

$$\to (b - a \,\text{Sen}\,\theta)\dot{\phi}\hat{\phi} + a(-W\hat{\theta} + \Omega\,\text{Sen}\,\theta\,\hat{\phi}) = 0$$

$$\to b\Omega - aW = 0 \to \boxed{W = \frac{b\Omega}{a}}$$ lo guardamos

- buscamos $\vec{L}_{cm}$, para usar las ecuaciones de euler

$\vec{L}_{cm} = I_{cm}\vec{W}_{cuerpo}$ (en base de O', tenemos lo sgte)

$$= I_{cm}(-W\hat{x} + \Omega\hat{R}) = I_{cm}(-W\hat{x} + \Omega(\text{Sen}\,\theta\,\hat{x} + \text{Cos}\,\theta\,\hat{z}))$$

$$= I^{11}_{cm}(\Omega\,\text{Sen}\,\theta - W)\hat{x} + I^{33}_{cm}\Omega\,\text{Cos}\,\theta\,\hat{z}$$

$$I^{11}_{cm} = \int_\Gamma (\|\vec{r}\|^2\delta_{11} - r_1 r_1)\,dm \overset{r_1=0}{=} \int_\Gamma \|\vec{r}\|^2\,dm$$

usamos cilindricos

$dm = \frac{m}{\pi a^2}\rho\, d\phi\, d\rho$, $\|\vec{r}\| = \rho$, $\phi \in [0, 2\pi]$, $\rho \in [0, a]$

$$\to I^{11}_{cm} = \int_0^a\int_0^{2\pi} \rho^2 \cdot \frac{m}{\pi a^2}\rho\, d\phi\, d\rho = \frac{ma^2}{2}$$

$$I^{33}_{cm} = \int_\Gamma (\|\vec{r}\|^2\delta_{33} - r_3 r_3)\,dm$$

$\vec{r} = \rho\hat{\rho}$

$= \rho(\text{Cos}\,\psi\,\hat{y} + \text{Sen}\,\psi\,\hat{z})$

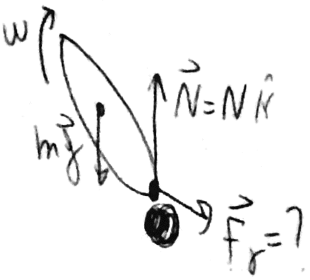
$\to r_{33} = \rho\,\text{Sen}\,\psi\,\hat{z}$

$$= \int_0^a\int_0^{2\pi} \underbrace{(\rho^2 - \rho^2\,\text{Sen}^2\psi)}_{\rho^2\,\text{Cos}^2\psi}\frac{m}{\pi a^2}\rho\, d\psi\, d\rho = \frac{ma^2}{4}$$

$$L_{cm} = ma^2\left(\frac{(\Omega \sin\theta - \omega)\hat{x}}{2} + \frac{\Omega \cos\theta\,\hat{z}}{4}\right)$$ ↑ constante para O'

- ahora calculamos el torque neto con respecto al CM

<u>DCL</u>



$$m\ddot{\vec{R}}_{cm} = \left((b-a\sin\theta)\underset{\Omega = cte}{\dot{\phi}}\hat{\phi}\right)^{\cdot} = \vec{N} + m\vec{g} + \vec{f}_r$$

$$\to -(b-a\sin\theta)\dot{\phi}^2\hat{\rho} = N\hat{k} - mg\hat{k} + \vec{f}_r$$

$$\to N = mg$$

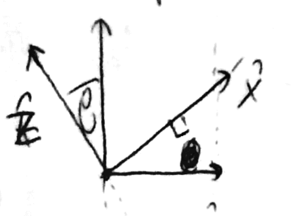
$$\vec{f}_r = -m(b-a\sin\theta)\dot{\phi}^2\hat{\rho}$$ (f_r no puede tener componente vertical)

$$\to \vec{\tau}^{\,cm}_{\vec{N}} = (-a\hat{z})\times(mg\hat{k})$$

$$= -mga(\hat{z}\times(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}))$$

$$= -mga\sin\theta\,\hat{y}$$

$$\vec{\tau}^{\,cm}_{\vec{f}_r} = (-a\hat{z})\times(-m(b-a\sin\theta)\dot{\phi}^2\hat{\rho})$$



$$= ma(b-a\sin\theta)\underset{\dot{\phi}}{\Omega^2}\,\hat{z}\times(\hat{x}\underset{\cos}{\cancel{\sin}}\theta - \hat{z}\underset{\sin}{\cancel{\cos}}\theta)$$

$$= -ma(b-a\sin\theta)\Omega^2\underset{\cos}{\cancel{\sin}}\theta\,\hat{y}$$

- ec. de euler

$$\to \underbrace{\left(\frac{d\vec{L}_{cm}}{dt}\right)_{SRNI}}_{0} + \vec{\omega}_{sis}\times\vec{L}_{cm} = \vec{\tau}_{cm}$$

$$0 + (\Omega\underbrace{(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})}_{\hat{k}})\times\frac{ma^2}{2}\left((\Omega\sin\theta - \omega)\hat{x} + \frac{\Omega\cos\theta}{2}\hat{z}\right)$$

$$= -ma\hat{y}(g\sin\theta + \Omega^2(b-a\sin\theta)\cos\theta)\cancel{\hat{y}}$$

$$\rightarrow \frac{\cancel{M}a^{\cancel{2}}\Omega}{2}\left(\frac{\Omega \,\text{Cos}\,\theta\,\text{Sen}\,\theta}{2}(-\hat{y}) + (\Omega\,\text{Sen}\,\theta - w)\,\text{Cos}\,\theta\;\hat{y}\right)$$

$$= -\cancel{M}\cancel{a}\,\hat{y}\,\left(g\,\text{Sen}\,\theta + \Omega^2(b - e\,\text{Sen}\,\theta)\,\text{Cos}\,\theta\right)$$

$$\rightarrow \frac{a\Omega}{2}\left(\text{Cos}\,\theta(\Omega\,\text{Sen}\,\theta - w) - \frac{\Omega\,\text{Cos}\,\theta\,\text{Sen}\,\theta}{2}\right)$$

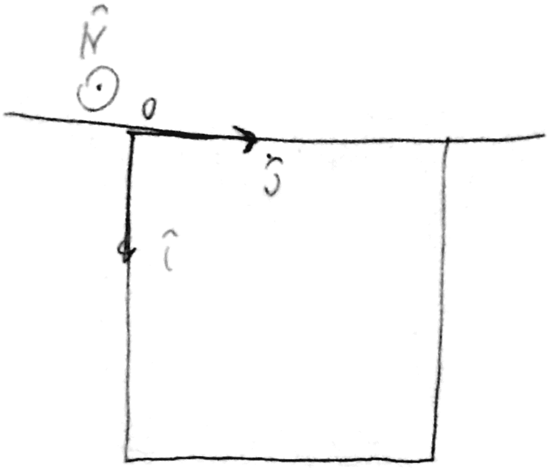
$$= -\left(g\,\text{Sen}\,\theta + \Omega^2(b - e\,\text{Sen}\,\theta)\,\text{Cos}\,\theta\right)$$

$$W = \frac{b\Omega}{a} \rightarrow \Omega^2 = \frac{4g\,\text{Sen}\,\theta}{6b\,\text{Cos}\,\theta - 5a\,\text{Sen}\,\theta\,\text{Cos}\,\theta}$$

Notemos que $6b\,\text{Cos}\,\theta - 5a\,\text{Sen}\,\theta\,\text{Cos}\,\theta > 0$

$$\rightarrow 6b > 5a\,\text{Sen}\,\theta$$

$$\rightarrow b > \frac{5a}{6}\,\text{Sen}\,\theta \;//$$



- $\vec{w} = \dot{\theta}\hat{\jmath}$
- $\hat{\jmath}$ No es un eje principal, necesitamos toda I_o
- $I_{ij} = \int_\Gamma (\|\vec{r}\|^2 \delta_{ij} - r_i r_j) dm$
- Usamos cartesianas

$$I_o = \int_\Gamma \begin{bmatrix} y^2 & -yx & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} \frac{M}{a^2} dx\,dy$$

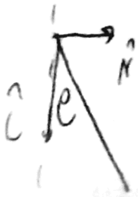
$$= \int_0^a \int_0^a \frac{M}{a^2} \begin{bmatrix} y^2 dx dy & -yx dx dy & 0 \\ -xy\, dx dy & x^2 dx dy & 0 \\ 0 & 0 & (x^2+y^2) dx dy \end{bmatrix}$$

$$= M a^2 \begin{bmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}$$

$$\therefore \vec{L}_o = I_o \vec{w} = \begin{bmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \cdot Ma^2 = Ma^2 \begin{bmatrix} -\dot{\theta}/4 \\ \dot{\theta}/3 \\ 0 \end{bmatrix}$$

así $K = \frac{\vec{w} \cdot \vec{L}_o}{2} = \frac{Ma^2}{2} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\dot{\theta}/4 \\ \dot{\theta}/3 \\ 0 \end{bmatrix} = \frac{Ma^2\dot{\theta}^2}{6}$

y además $U = -Mg\frac{a}{2}\cos\theta$

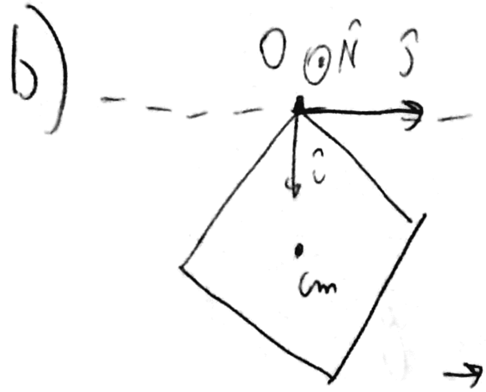


$\Rightarrow \mathcal{L} = \frac{Ma^2\dot{\theta}^2}{6} + \frac{Mga\cos\theta}{2}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \rightarrow \frac{Ma^2}{3}\ddot{\theta} + \frac{Mga}{2}\operatorname{sen}\theta = 0$$

$$\rightarrow \ddot{\theta} = -\frac{3g}{2a}\operatorname{sen}\theta$$

b)

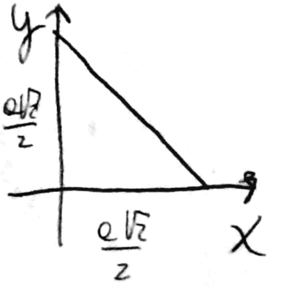


• En este caso, $\hat{j}$ si es un eje principal

$$\rightarrow I_0 \vec{\omega} = I^0_{22}\dot{\theta}\,\hat{j}$$

Por steiner, $I^0_{22} = I^{cm}_{22} + M\left(\frac{a\sqrt{2}}{2}\right)^2$

• Como I es aditivo, podemos calcular I para un cuarto, y sumarlos (por simetría, todos son iguales)



$$\therefore 4 I^{cm}_{22(cuarto)} = \frac{I^{cm}_{22}}{\cancel{4}} = \frac{4}{4}\int_0^{\frac{a\sqrt{2}}{2}}\int_0^{\frac{a\sqrt{2}}{2}-x}(x^2+y^2-x^2)\frac{M}{a^2}\,dy\,dx$$

$$y = -x + \frac{a\sqrt{2}}{2}$$

$$= \frac{4M}{\cancel{4}a^2}\int_0^{\frac{a\sqrt{2}}{2}} \frac{\left(\frac{a\sqrt{2}}{2}-x\right)^3}{3}\,dx = \frac{4Ma^2}{\cancel{4}48} = \frac{Ma^2}{12}$$

$$K = \frac{\vec{w}\cdot\left[\left(\frac{M\ell^2}{12}+\frac{M\ell^2}{2}\right)\vec{w}\right]}{2}$$

$$= \frac{7M\ell^2}{24}\dot{\theta}^2$$

$$U = -mg\frac{\ell\sqrt{2}}{2}\text{Cos}\,\theta$$

$$\rightarrow \mathcal{L} = K-U = \frac{7M\ell^2}{24}\dot{\theta}^2 + mg\frac{\ell\sqrt{2}}{2}\text{Cos}\,\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)-\frac{\partial\mathcal{L}}{\partial\theta} = 0 \rightarrow \frac{7M\ell^2}{12}\ddot{\theta} + \frac{mg\ell\sqrt{2}}{2}\text{Sen}\,\theta = 0$$

$$\ddot{\theta} = -\frac{g6\sqrt{2}}{\ell 7}\text{Sen}\,\theta$$

$$= -\frac{12}{7\sqrt{2}}\frac{g\,\text{Sen}\,\theta}{\ell} \quad //$$