PAUTA AUXILIAR 16

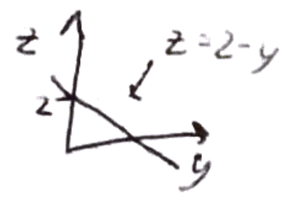
P1] Recordemos el teorema de Gauss

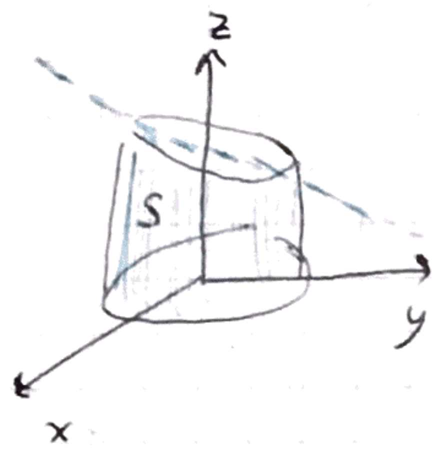
$$\iint_S F \cdot d\vec{S} = \iiint_V \operatorname{div}(F)\, dV$$

donde S es la superficie que encierra a V ($d\vec{S} = \hat{n}\, dS$ donde $\hat{n}$ es normal exterior)

$x^2+y^2 = 1 \rightarrow$ cilindro de radio 1

$0 \leq z \leq 2-y \rightarrow$ la base del cilindro está en $z=0$ y la parte de arriba tiene un corte diagonal





S es el manto del cilindro

$$\therefore \iint_S F\cdot d\vec{S} + \iint_{Base} F\cdot d\vec{S} + \iint_{Tapa} F\cdot d\vec{S} = \iiint_V \operatorname{div}(F)\, dV$$

$$\operatorname{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = z + 0 + 2z = 3z$$

Parametricemos el volumen en coordenadas cilíndricas

$$\begin{aligned} x &= r\cos\theta & r &\in [0,1] \\ y &= r\operatorname{sen}\theta & \theta &\in [0,2\pi] \\ z &= z & z &\in [0, 2-r\operatorname{sen}\theta] \end{aligned}$$

$$dV = h_r h_\theta h_z\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz$$

$$\iiint_V \operatorname{div}(F)\, dV = \int_0^{2\pi}\int_0^1\int_0^{2-r\operatorname{sen}\theta} 3z\, r\, dz\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 r\left(\frac{z^2}{2}\right)\Big|_0^{2-r\operatorname{sen}\theta} dr\, d\theta = \frac{3}{2}\int_0^{2\pi}\int_0^1 r(2-r\operatorname{sen}\theta)^2\, dr\, d\theta$$

$$= \frac{3}{2}\int_0^{2\pi}\int_0^1 4r - 4r^2\operatorname{sen}\theta + r^3\operatorname{sen}^2\theta\, dr\, d\theta = \frac{3}{2}\int_0^{2\pi}\left[\frac{4r^2}{2} - \frac{4r^3}{3}\operatorname{sen}\theta + \frac{r^4\operatorname{sen}^2\theta}{4}\right]\Big|_0^1 d\theta$$

$(\cos(2\pi) - \cos(0) = 0)$

$$= \frac{3}{2}\int_0^{2\pi} 2 - \cancel{\frac{4}{3}\operatorname{sen}\theta}^{\,0} + \frac{\operatorname{sen}^2\theta}{4}\, d\theta = \frac{3}{2}\left[4\pi + \frac{1}{4}\left(\frac{\theta}{2} - \frac{\operatorname{sen}(2\theta)}{4}\right)\Big|_0^{2\pi}\right]$$

$$= \frac{3}{2}\left[4\pi + \frac{\pi}{4}\right] = 6\pi + \frac{3\pi}{8} = \frac{51\pi}{8}$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \frac{51\pi}{8} - \underbrace{\iint_{\text{Base}} F d\vec{S}}_{I_1} - \underbrace{\iint_{\text{Tapa}} F\cdot d\vec{S}}_{I_2}$$

I_1) Parametricemos la Base

$$\begin{aligned} x &= r\cos\theta \qquad & r &\in [0,1] \\ y &= r\,\text{sen}\,\theta & \theta &\in [0, 2\pi] \\ z &= 0 \end{aligned}$$

$$\hat{n} = -\hat{k}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \left(r\cos\theta\cdot 0 + e^{0^2},\ 0-0^2-2,\ 0^2 + r\,\text{sen}\,\theta\right)\cdot(0,0,-1)\ dr\,d\theta$$

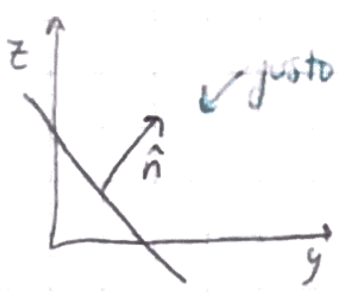
$(\cos(2\pi)-\cos(0) = 0)$

$$= \int_0^{2\pi}\int_0^1 -r\,\text{sen}\,\theta\ dr\,d\theta = -\int_0^1 r \int_0^{2\pi} \underbrace{\cancel{\text{sen}\,\theta\ d\theta}}_{0}\ dr = 0$$

I_2)

$$\begin{aligned} x &= x \\ y &= y \\ z &= 2-y \end{aligned} \qquad \hat{n} = \frac{(0,1,1)}{\sqrt{2}}$$

Para que tenga norma 1



justo la diagonal

$$\Rightarrow F\cdot\hat{n} = \left(z - \cancel{z^2} - 2 \cancel{+z^2} + y\right)\cdot\frac{1}{\sqrt{2}}$$

$$= \left(z - \underbrace{(2-y)}_{z}\right)\cdot\frac{1}{\sqrt{2}} = 0$$

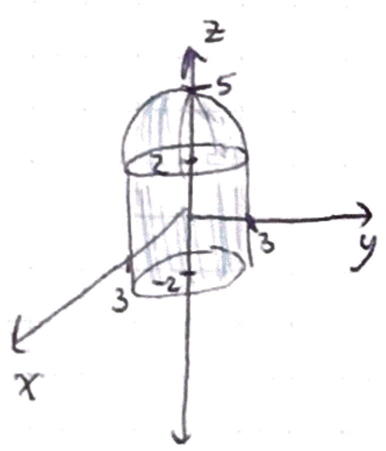
$$\therefore \iint_S F\cdot d\vec{S} = \frac{51\pi}{8} \quad //$$

P2) Stokes: $\oint_{\partial S} F\cdot d\vec{r} = \iint_S rot(F)\cdot d\vec{S}$

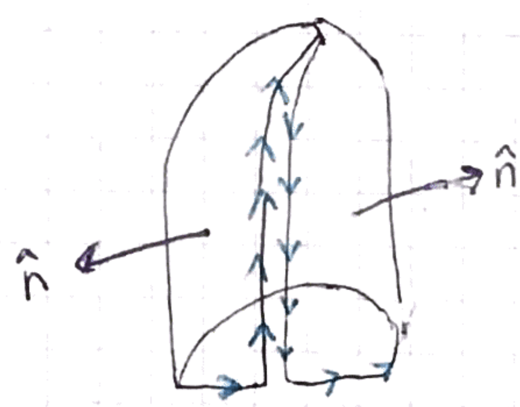
Sea $F = yz^2\,\hat{\imath}$ (Por indicación)

$$\Rightarrow rot(F) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz^2 & 0 & 0 \end{vmatrix} = 0\hat{\imath} - (-2yz)\hat{\jmath} + (-z^2)\hat{k} = 2yz\hat{\jmath} - z^2\hat{k}$$

$$\therefore \iint_S 2yz\hat{\jmath} - z^2\hat{k}\ d\vec{S} = \iint_S rot(F) = \oint_{\partial S} yz^2\,\hat{\imath}\cdot d\vec{r}$$



$\Rightarrow$ ∂S es la circunferencia de abajo (en $z=-2$ y $r=3$) en sentido antihorario para que en el cilindro la normal apunte hacia afuera (regla de la mano derecha



∂S) $x = 3\cos\theta$ $\quad \gamma(\theta) = (3\cos\theta,\ 3\,sen\theta,\ -2)$
$y = 3\,sen\theta$ $\quad \gamma'(\theta) = (-3\,sen\theta,\ 3\cos\theta,\ 0)$
$z = -2$
$\theta \in [0, 2\pi]$

$$\oint_{\partial S} F\cdot d\vec{r} = \int_0^{2\pi} F(\gamma(\theta))\cdot\gamma'(\theta)\,d\theta = \int_0^{2\pi} (3\,sen\theta\cdot(-2)^2, 0, 0)\cdot(-3\,sen\theta, 3\cos\theta, 0)\,d\theta$$

$$= \int_0^{2\pi} -36\,sen^2\theta\,d\theta = -36\left(\frac{\theta}{2} - \frac{sen(2\theta)}{4}\right)\Bigg|_0^{2\pi} = -36\pi\ //$$

$$\therefore \iint_S 2yz\hat{\jmath} - z^2\hat{k}\ d\vec{S} = -36\pi\ //$$

P3) $f(z) = 2\,Re(z)\,Im(z) + i\,Mod(z)$

sea $z = x + iy$

$f(x,y) = 2xy + i\sqrt{x^2+y^2}$

$f(x,y) = U(x,y) + i\,V(x,y)$

$U(x,y) = 2xy \qquad V(x,y) = \sqrt{x^2+y^2}$

Imponemos C-R

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \Rightarrow 2y = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow 2y = \frac{y}{\sqrt{x^2+y^2}} \quad (1)$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \Rightarrow 2x = \frac{-x}{\sqrt{x^2+y^2}} \quad (2)$$

$$(1)\cdot x + (2)\cdot y \Rightarrow 4xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0 \ \underline{\vee}\ y = 0$$

(no pueden ser ambos 0 pues se indefine $\frac{\partial V}{\partial y}$ y $\frac{\partial V}{\partial x}$)

- caso 1: $x \neq 0,\ y = 0$

  (1) se cumple pues $0 = 0$

  (2) $\Rightarrow 2\sqrt{x^2+y^2} = -1 \rightarrow\leftarrow \quad 2 > 0,\ \underbrace{\sqrt{x^2+y^2}}_{=|x|} > 0$

  $\therefore$ este caso no es posible

- caso 2: $x = 0,\ y \neq 0$

  (2) se cumple pues $0 = 0$

  (1) $\Rightarrow 2\sqrt{x^2+y^2} = 2\sqrt{y^2} = 2|y| = 1 \Rightarrow y = \pm\frac{1}{2}$

$\therefore$ f es holomorfa en $z = \pm\frac{i}{2}$ //

P4) Primero demostremos la indicación

P.D.Q: $\left(\frac{\alpha^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_{|z|=1} \frac{\alpha^n \exp(\alpha z)}{n!\ z^{n+1}}\, dz$

$\Leftrightarrow$ P.D.Q $2\pi i \cdot \frac{\alpha^n}{n!} = \oint_{|z|=1} \frac{\exp(\alpha z)}{z^{n+1}}\, dz$

En efecto: $\oint_{|z|=1} \frac{\exp(\alpha z)}{z^{n+1}}\, dz$ polos: $z=0$ (orden $n+1$)

$\therefore \oint_{|z|=1} \frac{\exp(\alpha z)}{z^{n+1}}\, dx = 2\pi i \operatorname{Res}\left(\frac{\exp(\alpha z)}{z^{n+1}}, 0\right)$

$\operatorname{Res}(f, 0) = \lim_{z\to 0} \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left(\cancel{z^{n+1}}\frac{\exp(\alpha z)}{\cancel{z^{n+1}}}\right) = \lim_{z\to 0}\frac{1}{n!}\alpha^n \exp(\alpha z) = \frac{\alpha^n}{n!}$

$\therefore \oint_{|z|=1} \frac{\exp(\alpha z)}{z^{n+1}}\, dz = \frac{\alpha^n}{n!}\cdot 2\pi i \Rightarrow \left(\frac{\alpha^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_{|z|=1}\frac{\alpha^n}{n!}\frac{\exp(\alpha z)}{z^{n+1}}\, dz$

$\Rightarrow \sum_{n=0}^{\infty}\left(\frac{\alpha^n}{n!}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{|z|=1}\frac{\alpha^n \exp(\alpha z)}{z^{n+1}\ n!}\, dz$

$= \frac{1}{2\pi i}\oint_{|z|=1}\frac{\exp(\alpha z)}{z}\underbrace{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha}{z}\right)^n}_{\exp\left(\frac{\alpha}{z}\right)}$

$= \frac{1}{2\pi i}\oint_{|z|=1}\frac{\exp(\alpha(z+z^{-1}))}{z}\, dz$ $\qquad z = e^{i\theta}$, $dz = i e^{i\theta}\, d\theta$

$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi}\frac{\exp(\alpha(e^{i\theta}+e^{-i\theta}))}{\cancel{e^{i\theta}}}\, \cancel{i}\cancel{e^{i\theta}}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\exp(\alpha\cdot 2\cos(\theta))\, d\theta$ //

P5) P2 C3-2014-3 Hernandez