P2 a) consideramos $f(z) = \frac{p(z)}{q(z)}$, $p(z) = z$ $q(z) = (z-4)(z^2+3)$

$\int_{-\infty}^{+\infty} f(z)e^{i2z}$ se tiene que $gr(q) = 3 \geq 1 + gr(p) = 2$

$$\therefore \int_{-\infty}^{+\infty} f(z)e^{2iz} = 2\pi i \sum \mathrm{Res}\overset{\text{de polos}}{\vee}_{\substack{\text{complejos}\\ \text{con } Im>0}} + \pi i \sum \mathrm{Res}\overset{\text{de polos}}{\vee}_{\text{reales}}$$

Polos: $z = 4$

$z^2 = -3 \Rightarrow z \pm i\sqrt{3}$, sólo consideramos $i\sqrt{3}$ pues $Im(-i\sqrt{3})<0$

$$\mathrm{Res}(f(z)e^{i2z}, 4) = \lim_{z\to 4} (z-4)f(z)e^{i2z} = \lim_{z\to 4} \frac{ze^{i2z}}{z^2+3} = \frac{4e^{i8}}{19}$$

$$\mathrm{Res}(f(z)e^{i2z}, \sqrt{3}i) = \lim_{z\to\sqrt{3}i} (z-\sqrt{3}i)f(z)e^{i2z} = \lim_{z\to\sqrt{3}i} \frac{ze^{i2z}}{(z-4)(z+i\sqrt{3})}$$

$$= \frac{\cancel{i\sqrt{3}}e^{-2\sqrt{3}}}{(i\sqrt{3}-4)2\cancel{i\sqrt{3}}} = \frac{e^{-2\sqrt{3}}}{2(i\sqrt{3}-4)} \cdot \frac{i\sqrt{3}+4}{i\sqrt{3}+4}$$

$$= \frac{e^{-2\sqrt{3}}(4+i\sqrt{3})}{2(-3-16)} = -\frac{e^{-2\sqrt{3}}(4+i\sqrt{3})}{38}$$

$$\int_{-\infty}^{+\infty} f(z)e^{i2z}dz = \cancel{2}\pi i \cdot \frac{-e^{-2\sqrt{3}}(4+i\sqrt{3})}{\cancel{38}\ 19} + \pi i \frac{4e^{i8}}{19}$$

$$= \frac{e^{-2\sqrt{3}}\sqrt{3}\pi}{19} + \frac{i\pi}{19}\left(-4e^{-2\sqrt{3}} + 4e^{i8}\right) = r$$

pero $\int_{-\infty}^{+\infty} f(z)e^{i2z}dz = \int_{-\infty}^{+\infty} f(z)[\cos(2z) + i\sin(2z)]dz = \underbrace{\int_{-\infty}^{+\infty} f(z)\cos(2z)dz + i\int_{-\infty}^{+\infty} f(z)\sin(2z)dz}_{I}$

este argumento está explicado en la pag. 166 del apunte

$\Rightarrow I = r \Rightarrow Im(I) = Im(r)$

$$\Rightarrow \int_{-\infty}^{+\infty} f(z)\sin(2z) = \frac{4\pi}{19}\left(e^{i8} - e^{-2\sqrt{3}}\right) \quad //$$

P2) c) $\displaystyle\int_0^{\infty} \frac{\ln(x)}{(1+x^2)^2}\, dx$

Consideramos $f(z) = \dfrac{\overbrace{\ln(z)}^{=\ln|z|}}{(z^2+1)^2}$

$$|f(z)| = \left|\frac{\ln(z)}{(z^2+1)^2}\right| \leq \left|\frac{z}{(z^2+1)^2}\right| \leq \left|\frac{z}{z^4}\right| = \frac{1}{|z|^3}$$

$$\therefore \int_0^{\infty} \frac{\ln(z)}{(1+z^2)^2} \underset{f \text{ par}}{=} \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\ln(z)}{\underbrace{(1+z^2)^2}_{((z-i)(z+i))^2}} = \frac{1}{2}\left[2\pi i \sum \text{Res de polos complejos con } \operatorname{Im} > 0 + \pi i \sum \text{Res de polos en } \mathbb{R}\right]$$

Polos

$z=0$ (se indefine $\log(z)$) → orden 1

$z=i$ → orden 2

$z=-i$ → orden 2 se descarta pues $\operatorname{Im}(-i) = -1 < 0$

$$\operatorname{Res}(f,i) = \lim_{z\to i} \frac{1}{1!}\frac{\partial}{\partial z}\left(\frac{(z-i)^2 \ln|z|}{\cancel{(z-i)^2}(z+i)^2}\right)$$

$$= \lim_{z\to i} \frac{\frac{1}{z}(z+i)^2 - \cancelto{0}{\ln|z|} \cdot 2(z+i)}{(z+i)^4} = \frac{\frac{1}{i}(2i)^2}{(2i)^4} = \frac{-4}{16i} = -\frac{1}{4i}$$

$$\operatorname{Res}(f,0) = \lim_{z\to 0} \frac{z\ln(z)}{(z^2+1)^2} = \lim_{z\to 0} \frac{\ln(z)}{\frac{(z^2+1)^2}{z}} \overset{L'H}{=} \lim_{z\to 0} \frac{\frac{1}{z}}{\frac{2(z^2+1)\cdot 2z\cdot z - (z^2+1)^2}{z^2}}$$

$$= \lim_{z\to 0} \frac{z}{4z^2(z^2+1) - (z^2+1)^2} = \frac{0}{-1} = 0$$

$$\therefore \int_0^{\infty} f(z) = \frac{1}{2}\left[2\pi \cancel{i} \cdot \frac{-1}{{}_2\cancel{4}\cancel{i}} + \pi i \cdot 0\right] = \frac{-\pi}{4}$$ //