

Recuerda que:

$$\int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = \frac{2\pi}{\sqrt{a^2 - 1}}, \quad a > 1$$

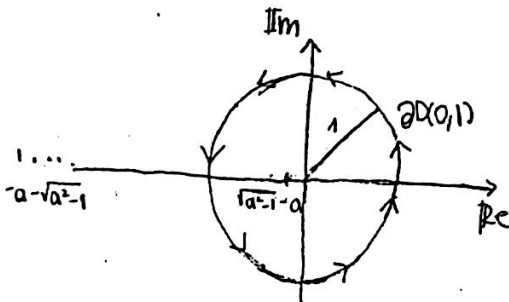
sea $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$, ahora:

$$\int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = \int_0^{2\pi} \frac{d\theta}{a + \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)} = \frac{1}{i} \oint_{|z|=1} \frac{1}{a + \left(\frac{z + z^{-1}}{2}\right)} \frac{dz}{z}$$

$$= \frac{2}{i} \oint_{\partial D(0,1)} \frac{1}{2az + z^2 + 1} dz, \text{ buscamos los ceros de } z^2 + 2az + 1 = 0$$

$$z = \frac{-2a \pm \sqrt{4a^2 - 4}}{2} = -a \pm \sqrt{a^2 - 1}$$

$$= \frac{2}{i} \oint_{\partial D(0,1)} \frac{dz}{(z + a + \sqrt{a^2 - 1})(z + a - \sqrt{a^2 - 1})}, \text{ veremos los polos dentro de } \partial D(0,1)$$



Tenemos que $\sqrt{a^2 - 1} \leq a + 1$.

$$\therefore \sqrt{a^2 - 1} - a \leq 1, \text{ ahora } \sqrt{a^2 - 1} \geq a - 1$$

$\therefore \sqrt{a^2 - 1} - a \geq -1$, luego es polo de orden 1 con indicatriz distinta de uno

$$\int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = \frac{2}{i} 2\pi i \operatorname{Res}\left(\frac{1}{z^2 + 2az + 1}, \sqrt{a^2 - 1} - a\right) \text{ calculamos este residuo}$$

$$\operatorname{Res}\left(\frac{1}{z^2 + 2az + 1}, \sqrt{a^2 - 1} - a\right) = \lim_{z \rightarrow \sqrt{a^2 - 1} - a} \frac{1}{0!} \frac{d^0}{dz^0} \left(\frac{z + a - \sqrt{a^2 - 1}}{(z + a + \sqrt{a^2 - 1})(z + a - \sqrt{a^2 - 1})} \right)$$

$$= \lim_{z \rightarrow \sqrt{a^2 - 1} - a} \frac{1}{z + a + \sqrt{a^2 - 1}} = \frac{1}{2\sqrt{a^2 - 1}}, \text{ ahora}$$

$$\int_0^{2\pi} \frac{d\theta}{a + \cos \theta} d\theta = \frac{4\pi}{2\sqrt{a^2 - 1}} = \frac{2\pi}{\sqrt{a^2 - 1}}$$

$$\int_0^{2\pi} \frac{\cos^2(2\theta)}{1 - 2a \cos(2\theta) + a^2} d\theta = \pi \frac{1 - a + a^2}{1 - a}, \quad 0 < a < 1$$

$$\text{sea } z = e^{i\theta}, dz = ie^{i\theta} d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$$

$$\begin{aligned} \int_0^{2\pi} \frac{\cos^2(3\theta)}{1-2a\cos(2\theta)+a^2} d\theta &= \int_0^{2\pi} \frac{1+\cos(6\theta)}{1-2a\cos(2\theta)+a^2} d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \frac{1}{1-2a\left(\frac{e^{2i\theta}+e^{-2i\theta}}{2}\right)+a^2} d\theta + \frac{1}{2} \int_0^{2\pi} \frac{\operatorname{Re}(e^{6i\theta})}{1-2a\left(\frac{e^{2i\theta}+e^{-2i\theta}}{2}\right)+a^2} d\theta \\ &= \frac{1}{2} \oint_{\partial D(0,1)} \frac{1}{1-az^2-az^{-2}+a^2} \frac{dz}{iz} + \frac{1}{2} \operatorname{Re} \left(\oint_{\partial D(0,1)} \frac{z^4}{1-az^2-az^{-2}+a^2} \frac{dz}{iz} \right) \\ &= \frac{1}{2i} \oint_{\partial D(0,1)} \frac{z}{z^2-az^4-a+a^2z^2} dz + \operatorname{Re} \left(\frac{1}{2i} \oint_{\partial D(0,1)} \frac{z^7}{z^2-az^4-a+a^2z^2} dz \right) \\ &= \frac{1}{2i} \oint_{\partial D(0,1)} \frac{z}{(1-az^2)(z^2-a)} dz + \operatorname{Re} \left(\frac{1}{2i} \oint_{\partial D(0,1)} \frac{z^7}{(1-az^2)(z^2-a)} dz \right) \\ &= \frac{1}{2i} \oint_{\partial D(0,1)} \frac{z}{(1-z^2)(z-\sqrt{a})(z+\sqrt{a})} dz + \operatorname{Re} \left(\frac{1}{2i} \oint_{\partial D(0,1)} \frac{z^7}{(1-az^2)(z-\sqrt{a})(z+\sqrt{a})} dz \right) \\ &= \frac{1}{2i} 2\pi i \left(\operatorname{Res}(f_1, \sqrt{a}) + \operatorname{Res}(f_1, -\sqrt{a}) \right) + \operatorname{Re} \left(\frac{1}{2i} 2\pi i \left(\operatorname{Res}(f_2, \sqrt{a}) + \operatorname{Res}(f_2, -\sqrt{a}) \right) \right) \end{aligned}$$

calculamos dichos Residuos.

$$\operatorname{Res}(f_1, \sqrt{a}) = \lim_{z \rightarrow \sqrt{a}} \frac{1}{0!} \frac{d^0}{dz^0} \left(\frac{(z-\sqrt{a})z}{(1-z^2)(z-\sqrt{a})(z+\sqrt{a})} \right) = \frac{\sqrt{a}}{(1-a^2)2\sqrt{a}} = \frac{1}{2(1-a^2)}$$

$$\operatorname{Res}(f_1, -\sqrt{a}) = \lim_{z \rightarrow -\sqrt{a}} \frac{1}{0!} \frac{d^0}{dz^0} \left(\frac{(z+\sqrt{a})z}{(1-z^2)(z+\sqrt{a})(z-\sqrt{a})} \right) = \frac{-\sqrt{a}}{(1-a^2)(-2\sqrt{a})} = \frac{1}{2(1-a^2)}$$

$$\operatorname{Res}(f_2, \sqrt{a}) = \lim_{z \rightarrow \sqrt{a}} \frac{1}{0!} \frac{d^0}{dz^0} \left(\frac{(z-\sqrt{a})z^7}{(1-z^2)(z+\sqrt{a})(z-\sqrt{a})} \right) = \frac{a^3\sqrt{a}}{(1-a^2)2\sqrt{a}} = \frac{a^3}{2(1-a^2)}$$

$$\operatorname{Res}(f_2, -\sqrt{a}) = \lim_{z \rightarrow -\sqrt{a}} \frac{1}{0!} \frac{d^0}{dz^0} \left(\frac{(z+\sqrt{a})z^7}{(1-z^2)(z+\sqrt{a})(z-\sqrt{a})} \right) = \frac{-a^3\sqrt{a}}{(1-a^2)(-2\sqrt{a})} = \frac{a^3}{2(1-a^2)}$$

$$\int_0^{2\pi} \frac{\cos^2(3\theta)}{1-2a\cos(2\theta)+a^2} d\theta = \pi \left(\frac{1}{2(1-a^4)} + \frac{1}{2(1-a^2)} + \operatorname{Re} \left(\frac{a^3}{2(1-a^4)} + \frac{a^3}{2(1-a^2)} \right) \right)$$

$$\int_0^{2\pi} \frac{\cos^2(\theta)}{1+2a\cos(\theta)+a^2} d\theta = \pi \left(\frac{1+a^2}{1-a^2} \right) = \pi \frac{(1+a)(a^2-a+1)}{(1+a)(1-a)} = \pi \frac{1-a+a^2}{1-a}$$

c) $\int_0^{2\pi} \frac{\cos n\theta}{\cosh a + \cos \theta} d\theta = \frac{2\pi(-1)^n e^{-na}}{\sinh a}, n \geq 0$ en un entero y $a > 0$

$$\int_0^{2\pi} \frac{\cos n\theta}{\cosh a + \cos \theta} d\theta = \int_0^{2\pi} \frac{\operatorname{Re}(e^{in\theta})}{\cosh a + \frac{e^{i\theta} + e^{-i\theta}}{2}} d\theta = \operatorname{Re} \left(2 \int_0^{2\pi} \frac{e^{in\theta}}{2\cosh a + e^{i\theta} + e^{-i\theta}} d\theta \right)$$

sea $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$, luego

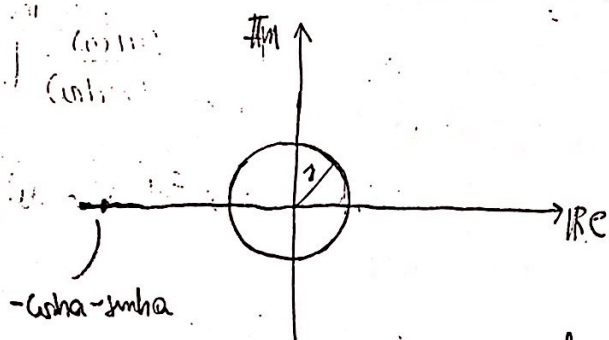
$$\int_0^{2\pi} \frac{\cos n\theta}{\cosh a + \cos \theta} d\theta = \operatorname{Re} \left(2 \oint_{\partial D(0,1)} \frac{z^n}{2\cosh a + z + z^{-1}} \frac{dz}{iz} \right)$$

$\cos(n\theta) = \operatorname{Re}(e^{in\theta})$

$$= \operatorname{Re} \left(\frac{2}{i} \oint_{\partial D(0,1)} \frac{z^n}{z^2 + 2z\cosh a + 1} dz \right), \text{ ahora encontramos los ceros de: } z^2 + 2z\cosh a + 1 = 0$$

$$\Rightarrow z = \frac{-2\cosh a \pm \sqrt{4\cosh^2 a - 4}}{2} = -\cosh a \pm \sinh a, \text{ luego:}$$

sea $f(z) = \frac{z^n}{(z + \cosh a + \sinh a)(z + \cosh a - \sinh a)}$, tenemos que



$$\cosh a + \sinh a = \frac{e^a + e^{-a} + e^a - e^{-a}}{2} = e^a$$

luego $a > 0 \Rightarrow e^a > 1$ entonces

$$-\cosh a - \sinh a \notin \partial D(0,1)$$

$$\sinh a - \cosh a = \frac{e^a - e^{-a} - e^a - e^{-a}}{2} = -e^{-a}$$

luego $|-e^{-a}| < 1$, entonces el único polo de f en $\partial D(0,1)$ es $\sinh a - \cosh a$ de orden 1.

$$\int_0^{2\pi} \frac{\cos n\theta}{\cosh a + \cos \theta} d\theta = \operatorname{Re} \left(\frac{2}{i} \oint_{\partial D(0,1)} f(z) dz \right) = \operatorname{Re} \left(\frac{2 \cdot 2\pi i}{i} \operatorname{Res} \left(f, \sinh a - \cosh a \right) \right)$$

$$= 4\pi \operatorname{Re} \left(\lim_{z \rightarrow \sinh a - \cosh a} \frac{1}{(z - (\sinh a - \cosh a))} \frac{d}{dz} \left(\frac{(z + \cosh a - \sinh a) z^n}{(z + \cosh a - \sinh a)(z + \cosh a + \sinh a)} \right) \right)$$

$$\int_0^{2\pi} \frac{\cos n\theta}{\cos n\theta + \cosh a} d\theta = 4\pi \operatorname{Re} \left(\frac{(\sinh a - \cosh a)^n}{2 \sinh a} \right), \sinh a - \cosh a = -e^{-a}$$

$$= \frac{2\pi (-e^{-a})^n}{\sinh a} = \frac{2\pi (-1)^n e^{-na}}{\sinh a}$$

$$d) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta}{5-4\sin \theta} d\theta = \frac{\pi}{6}$$

por imparidad de $\sin x$ se tiene que

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta}{5-4\sin \theta} d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin \theta}{5-4\sin \theta} d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\operatorname{Im}(e^{i\theta})}{5-4\left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right)} d\theta$$

sea $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$, entonces

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta}{5-4\sin \theta} d\theta = \operatorname{Im} \left(\frac{1}{2i} \oint_{\partial D(0,1)} \frac{z}{5-2(z-z^{-1})} \frac{dz}{z} \right) = \operatorname{Im} \left(\frac{1}{2i} \oint_{\partial D(0,1)} \frac{z}{5z-2z^2+1} dz \right)$$

buscamos los ceros de $-2z^2+5z+1=0$, $z = \frac{-5 \pm \sqrt{25+8}}{-4}$

Puede que:

$$a) \int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3}$$

tenemos que $p(z)=1$, $q(z)=1+z^6$, $gr(q)=6 > 1+2 = gr(p)+2$, venmos
 ahora los ceros de q que son los polos de $f(z) = \frac{p(z)}{q(z)}$, ahora
 los ceros son $z^6 = -1 = e^{i\pi} \Rightarrow z = e^{\frac{(2k+1)\pi}{6}i}$ $\forall k=0,1,\dots,5$, de orden 1
 luego $\text{Int}(H) \cap P = \{e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{2}}, e^{\frac{i\pi}{3}}\} = \{\frac{1}{2}(\sqrt{3}+i), i, \frac{1}{2}(\sqrt{3}-i)\}$
 adem\u00e1s $e^{\frac{i\pi}{3}} = \frac{1}{2}(1+\sqrt{3}i)$, $e^{-\frac{i\pi}{3}} = \frac{1}{2}(1-\sqrt{3}i)$, ahora:

$$\begin{aligned} \text{Res}(f, e^{\frac{i\pi}{6}}) &= \lim_{z \rightarrow e^{\frac{i\pi}{6}}} \frac{(z - e^{\frac{i\pi}{6}})}{(z - e^{\frac{i\pi}{6}})(z + e^{\frac{i\pi}{6}})(z^2 + 1)(z^2 - e^{\frac{i\pi}{3}})} = \frac{1}{2e^{\frac{i\pi}{6}}(e^{\frac{i\pi}{3}} + 1)(e^{\frac{i\pi}{3}} - e^{-\frac{i\pi}{3}})} \\ &= \frac{1}{2(\frac{1}{2}(\sqrt{3}+i))(\frac{1}{2}(3+\sqrt{3}i))(\sqrt{3}i)} = \frac{2}{(3\sqrt{3}-\sqrt{3}+i(3+3))\sqrt{3}i} \\ &= \frac{2}{6i - 6\sqrt{3}} = -\frac{\sqrt{3}+i}{12} \end{aligned}$$

$$\begin{aligned} \text{Res}(f, e^{\frac{i\pi}{2}}) &= \lim_{z \rightarrow e^{\frac{i\pi}{2}}} \frac{(z - i)}{(z - i)(z + i)(z^2 - e^{\frac{i\pi}{3}})(z^2 - e^{-\frac{i\pi}{3}})} = \frac{1}{2i(-1 - e^{\frac{i\pi}{3}})(-1 - e^{-\frac{i\pi}{3}})} \\ &= \frac{1}{2i(1 + \frac{1}{2}(1+\sqrt{3}i))(1 + \frac{1}{2}(1-\sqrt{3}i))} = \frac{2}{i(3+\sqrt{3}i)(3-\sqrt{3}i)} = \frac{-i}{3} \end{aligned}$$

$$\begin{aligned} \text{Res}(f, e^{-\frac{i\pi}{6}}) &= \lim_{z \rightarrow e^{-\frac{i\pi}{6}}} \frac{(z - e^{-\frac{i\pi}{6}})}{(z - e^{-\frac{i\pi}{6}})(z + e^{-\frac{i\pi}{6}})(z^2 + 1)(z^2 - e^{\frac{i\pi}{3}})} = \frac{1}{2e^{-\frac{i\pi}{6}}(e^{-\frac{i\pi}{3}} + 1)(e^{-\frac{i\pi}{3}} - e^{\frac{i\pi}{3}})} \\ &= \frac{1}{2(\frac{1}{2}(\sqrt{3}-i))(\frac{1}{2}(1-\sqrt{3}i)+1)(-\sqrt{3}i)} = -\frac{2}{(\sqrt{3}-i)(3-\sqrt{3}i)\sqrt{3}i} \\ &= -\frac{2}{(3\sqrt{3}-\sqrt{3}-i(3+3))\sqrt{3}i} = -\frac{2}{6i + 6\sqrt{3}} = -\frac{\sqrt{3}-i}{12} \end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \left(-\frac{\sqrt{3}+i}{12} - \frac{i}{3} - \frac{\sqrt{3}-i}{12} \right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{3}\pi i$$

$$b) \int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}, \text{ con } a > 0$$

tenemos que si $x > 0 \Rightarrow x \sin x = (-x) \cdot (\sin(-x))$ porque $\sin(-x) = -\sin(x)$
entonces

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx, \text{ luego:}$$

$p(x) = x$, $q(x) = x^2 + a^2$, luego $gr(q) = 2 > 1 + 1 = gr(p) + 1$,
luego $f(z) = \frac{p(z)}{q(z)}$, buscamos los polos de f , entonces buscamos los ceros de
 q , que son $z_1 = ai$, $z_2 = -ai$, luego solo z_1 es tal que $\text{Im}(z_1) > 0$, entonces

$$\int_{-\infty}^{\infty} f(z) e^{isz} dz = 2\pi i \text{Res}(f(z)e^{isz}, ai), \text{ calculamos este residuo}$$

$$\text{Res}(f(z)e^{isz}, ai) = \lim_{z \rightarrow ai} \frac{(z - ai) z e^{isz}}{(z - ai)(z + ai)} = \frac{ai e^{-as}}{2ai} = \frac{e^{-as}}{2}$$

$$\int_{-\infty}^{\infty} f(x) (\cos(sx) + i \sin(sx)) dx = \pi i e^{-as} \quad \begin{aligned} dz &= i e^{i\theta} d\theta \\ \text{Im}(\pi i e^{-as}) &= \pi e^{-a} \end{aligned}$$

$$\int_{-\infty}^{\infty} f(x) \sin(x) dx = \frac{\pi}{2} e^{-a} \Rightarrow \int_0^{\infty} f(x) \sin(x) dx = \frac{\pi}{2} e^{-a}$$

$$c) \int_0^{\infty} \frac{\cos(\lambda x)}{x^2 + a^2} dx = \frac{\pi}{2} e^{-\lambda a}, \text{ donde } \lambda \in \mathbb{R} \text{ y } a > 0$$

$$\text{tenemos que } \cos(\lambda x) = \cos(-\lambda x) \text{ luego } \int_0^{\infty} \frac{\cos(\lambda x)}{x^2 + a^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(\lambda x)}{x^2 + a^2} dx$$

entonces: sea $p(x) = 1$, $q(x) = x^2 + a^2$,

$gr(q) = 2 > 0 + 1 = gr(p) + 1$, luego sea $f(z) = \left(\frac{p}{q}\right)(z)$, se tiene

que los polos de z son $z_1 = ai$, $z_2 = -ai$, solo se tiene que $\text{Im}(z_1) > 0$

entonces

$$\int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx = 2\pi i \operatorname{Res}(f(z) e^{i\lambda z}, ai) \quad , \text{ donde}$$

$$\operatorname{Res}(f(z) e^{i\lambda z}, ai) = \lim_{z \rightarrow ai} \frac{(z - ai) e^{i\lambda z}}{(z - ai)(z + ai)} = \frac{e^{-\lambda a}}{2ai}$$

$$\therefore \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx = \frac{\pi}{a} e^{-\lambda a} = \int_{-\infty}^{\infty} f(x) (\cos \lambda x + i \sin \lambda x) dx$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) \cos(\lambda x) dx = \frac{\pi}{a} e^{-\lambda a} \Rightarrow \int_0^{\infty} \frac{\cos \lambda x}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-\lambda a}$$

$$d) \int_0^{\infty} \frac{\ln x}{(1+x^2)^{n+1}} dx = -n \frac{\pi}{4} \quad , \text{ para } n=0 \text{ y } n=1.$$

$$\text{tomamos que } \ln(x) = \ln|x| \Rightarrow \int_0^{\infty} \frac{\ln(x)}{(1+x^2)^{n+1}} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\ln|x|}{(x^2+1)^{n+1}} dx$$

para $n=0$

$$f(z) = \frac{\ln|z|}{z^2+1} \quad , \quad |f(z)| = \frac{|\ln|z||}{|z^2+1|} \leq \frac{\frac{|z+1|}{|z|}}{|z^2+1|} \leq \frac{1}{|z|^2}$$

luego f tiene tres polos $z_1=0$, $z_2=i$, $z_3=-i$, uno sobre el eje real y $\operatorname{Im}(z_2) < 0$, entonces

$$\int_{-\infty}^{\infty} \frac{\ln|z|}{z^2+1} = 2\pi i \operatorname{Res}(f, i) + \pi i \operatorname{Res}(f, 0)$$

$$\operatorname{Res}(f, ai) = \lim_{z \rightarrow ai} \frac{\ln|z| (z - ai)}{(z - ai)(z + ai)} = \frac{\ln 1}{2ai} = 0$$

$$\operatorname{Res}(f, 0) = \lim_{z \rightarrow 0} \frac{z \ln z}{(z^2+1)} \underset{L'H}{=} \lim_{z \rightarrow 0} \frac{\frac{1}{z}}{\frac{-1}{z^2}(z^2+1) + \frac{1}{z}(2z)}$$

$$= \lim_{z \rightarrow 0} \frac{z}{z^2-1} = 0$$

$$\therefore \int_0^{\infty} \frac{\ln(x)}{x^2+1} = 0 \cdot \frac{\pi}{2} = 0$$

• para $n=1$

$$f(z) = \frac{\ln(z)}{(z^2+1)^2}$$

$$|f(z)| = \left| \frac{\ln(z)}{(z^2+1)^2} \right| \leq \left| \frac{z}{(z^2+1)^2} \right| \leq \frac{1}{|z|^3}$$

luego $\int_0^\infty f(z) dz = \frac{1}{2} \int_{-\infty}^\infty f(|z|) dz$ f tiene tres polos: $z=0$ de orden 1
 $z=i$ de orden 2
 $z=-i$ de orden 2

$$\begin{aligned} \text{Res}(f, 0) &= \lim_{z \rightarrow 0} \frac{z \ln(z)}{(z^2+1)^2} = \lim_{z \rightarrow 0} \frac{\frac{1}{z}}{\frac{-1}{z^2}(z^2+1)^2 + \frac{2}{z}z(z^2+1)} \\ &= \lim_{z \rightarrow 0} \frac{z}{-(z^2+1)^2 + 2z(z^2+1)} = 0 \end{aligned}$$

ahora tenemos que los otros dos polos son de orden 2, luego:

$$\begin{aligned} \text{Res}(f, i) &= \lim_{z \rightarrow i} \frac{d}{dz} \left(\frac{(z-i)^2 \ln|z|}{(z-i)^2(z+i)^2} \right) = \lim_{z \rightarrow i} \frac{1}{z(z+i)^2} - \frac{2 \ln|z|}{(z+i)^3} \\ &= -\frac{1}{4i} \end{aligned}$$

$$\begin{aligned} \text{Res}(f, -i) &= \lim_{z \rightarrow -i} \frac{d}{dz} \left(\frac{(z+i)^2 \ln|z|}{(z+i)^2(z-i)^2} \right) = \lim_{z \rightarrow -i} \frac{1}{z(z-i)^2} - \frac{2 \ln|z|}{(z-i)^3} \\ &= -\frac{1}{4i} \end{aligned}$$

$$\text{luego} \int_{-\infty}^\infty f(|z|) dz = 2\pi i \left(-\frac{1}{4i} - \frac{1}{4i} \right) = -\frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin x}{x(x^2+a^2)} dx = \frac{\pi}{2a^2} (1 - e^{-a}), \quad a > 0$$

tenemos que $\frac{\sin x}{x} = \frac{\sin(-x)}{-x}$ por imparidad del seno, entonces

$$\int_0^{\infty} \frac{\sin(x)}{x(x^2+a^2)} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2+a^2)} dx, \text{ ahora tenemos que}$$

sea $p(z)=1$, $q(z)=z(z^2+a^2)$, $\text{gr}(q)=3 > 0+1 = \text{gr}(p)+1$

Entonces buscamos los polos de $f(z) = \frac{p}{q}(z)$, esto es: $z_1=0, z_2=ai, z_3=-ai$
solo z_1 y z_2 están en $\text{Im}(z) \geq 0$.

$$\int_{-\infty}^{\infty} f(x) e^{ix} dx = 2\pi i \text{Res}(fe^{iz}, ai) + \pi i \text{Res}(fe^{iz}, 0), \text{ calculamos estos residuos}$$

$$\text{Res}(fe^{iz}, ai) = \lim_{z \rightarrow ai} \frac{(z-ai)e^{iz}}{(z-ai)z(z+ai)} = \frac{e^{-a}}{ai \cdot 2ai} = -\frac{e^{-a}}{2a^2}$$

$$\text{Res}(fe^{iz}, 0) = \lim_{z \rightarrow 0} \frac{ze^{iz}}{z(z^2+a^2)} = \frac{1}{a^2}$$

$$\therefore \int_{-\infty}^{\infty} f(x) e^{ix} dx = \int_{-\infty}^{\infty} f(x) (\cos x + i \sin x) dx = \frac{\pi}{a^2} (1 - e^{-a})$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) \sin x dx = \frac{\pi}{a^2} (1 - e^{-a}) \quad \text{luego} \quad \int_0^{\infty} \frac{\sin x dx}{x(x^2+a^2)} = \frac{\pi}{2a^2} (1 - e^{-a})$$

12.4 Problemas

Calcular las siguientes integrales

$$(i) \int_0^{2\pi} \frac{\sqrt{\sin(2\theta)}}{3-5\cos\theta} d\theta$$

sea $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$

$$\begin{aligned} \int_0^{2\pi} \frac{\sqrt{\sin(2\theta)}}{3-5\cos\theta} d\theta &= \int_0^{2\pi} \frac{\sqrt{\frac{e^{2i\theta} - e^{-2i\theta}}{2}}}{3-5\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)} d\theta = \frac{\sqrt{2}}{i} \oint_{\partial D(0,1)} \frac{\sqrt{z^2 - z^{-2}}}{6-5(z+z^{-1})} \frac{dz}{z} \\ &= \frac{\sqrt{2}}{i} \oint \frac{\sqrt{z^4-1}}{6z^2-5z^3-5z} dz = -\frac{\sqrt{2}}{i} \oint \frac{\sqrt{z^4-1}}{z(5z^2-6z+5)} dz \end{aligned}$$

luego $f(z) = \frac{\sqrt{z^4-1}}{z(5z^2-6z+5)} = \frac{\sqrt{z^4-1}}{z(z-\frac{3-4i}{5})(z-\frac{3+4i}{5})}$ - tiene 3 polos dentro de $\partial D(0,1)$ de orden 1

$$\text{Res}(f, 0) = \lim_{z \rightarrow 0} \frac{z\sqrt{z^4-1}}{z(z-\frac{3-4i}{5})(z-\frac{3+4i}{5})} = \frac{\sqrt{-1}}{\frac{9+16}{25}} = i$$

$$\text{Res}(f, \frac{3-4i}{5}) = \lim_{z \rightarrow \frac{3-4i}{5}} \frac{(z-\frac{3-4i}{5})\sqrt{z^4-1}}{z(z-\frac{3+4i}{5})(z-\frac{3-4i}{5})} = \frac{\sqrt{(\frac{3-4i}{5})^4-1}}{(\frac{3-4i}{5})(-\frac{8i}{5})} = \frac{\sqrt{(3-4i)^4-5^4}}{(3-4i)(-8i)}$$

$$\text{Res}(f, \frac{3+4i}{5}) = \lim_{z \rightarrow \frac{3+4i}{5}} \frac{(z-\frac{3+4i}{5})\sqrt{z^4-1}}{z(z-\frac{3+4i}{5})(z-\frac{3-4i}{5})} = \frac{\sqrt{(3+4i)^4-5^4}}{(3+4i)(8i)}$$

$$\therefore \int_0^{2\pi} \frac{\sqrt{\sin(2\theta)}}{3-5\cos\theta} d\theta = 2\pi i \left(i - \frac{\sqrt{(3-4i)^4-5^4}}{(3-4i)8i} + \frac{\sqrt{(3+4i)^4-5^4}}{(3+4i)8i} \right)$$

$$(ii) \int_{-\infty}^{\infty} \frac{x \sin(2x)}{(x-4)(x^2+3)} dx$$

sea $p(z) = z$, $q(z) = (x-4)(x-i\sqrt{3})(x+i\sqrt{3})$ y $f(z) = \frac{p}{q}(z)$

tenemos que $\text{gr}(q) = 3 > 1+1 = \text{gr}(p)+1$, luego buscamos los polos de f cuya parte imaginaria sea mayor que 0 o que estén en $\mathbb{R}$,
están son $z_1 = 4$, $z_2 = \sqrt{3}i$, $z_3 = -i\sqrt{3}$, $z_1 \in \mathbb{R}$, $z_2 \in \text{Int}(H)$ luego

$$\int_{-\infty}^{\infty} f(z) e^{2ix} dx = 2\pi i \operatorname{Res}(f(z) e^{2iz}; \sqrt{3}i) + \pi i \operatorname{Res}(f(z) e^{2iz}; 4)$$

calculamos:

$$\operatorname{Res}(f(z) e^{2iz}; \sqrt{3}i) = \lim_{z \rightarrow \sqrt{3}i} \frac{(z - \sqrt{3}i) z e^{2iz}}{(z-4)(z-\sqrt{3}i)(z+\sqrt{3}i)} = \frac{\sqrt{3}i e^{-2\sqrt{3}}}{(\sqrt{3}i-4)2\sqrt{3}i} = \frac{-(4+\sqrt{3}i)e^{-2\sqrt{3}}}{38}$$

$$\operatorname{Res}(f(z) e^{2iz}; 4) = \lim_{z \rightarrow 4} \frac{(z-4) z e^{2iz}}{(z-4)(z^2+3)} = \frac{4e^{8i}}{19}$$

luego

$$\begin{aligned} \int_{-\infty}^{\infty} f(z) (\cos 2x + i \sin 2x) dx &= 2\pi i \left(-\frac{(4+\sqrt{3}i)e^{-2\sqrt{3}}}{38} \right) + \pi i \left(\frac{4e^{8i}}{19} \right) \\ &= \frac{\pi}{19} (4i \cos(8) - 4 \sin(8) - 4ie^{-2\sqrt{3}} + \sqrt{3}e^{-2\sqrt{3}}) \end{aligned}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x \sin 2x}{(x-4)(x^2+3)} dx = \frac{4\pi}{19} (\sin(8) - e^{-2\sqrt{3}})$$

Problema 12.2.

Calcular

$$\int_0^{2\pi} \frac{\cos nx}{1+a^2-2a \cos x} dx$$

Con $a \in (0,1)$

$$\text{Sea } z = e^{i\theta}, dz = ie^{i\theta} d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{\cos nx}{1+a^2-2a \cos x} dx = \int_0^{2\pi} \frac{\operatorname{Re}(e^{in\theta})}{1+a^2-2a(e^{i\theta}+e^{-i\theta})} d\theta = \operatorname{Re} \left(\int_0^{2\pi} \frac{e^{in\theta}}{1+a^2-a(e^{i\theta}+e^{-i\theta})} d\theta \right)$$

$$= \operatorname{Re} \left(\oint_{\partial D(0,1)} \frac{z^n}{1+a^2-a(z+z^{-1})} \frac{dz}{iz} \right) = \operatorname{Re} \left(\frac{1}{i} \oint_{\partial D(0,1)} \frac{z^n}{z+a^2z-az^2-a} dz \right)$$

$$= \operatorname{Re} \left(\frac{1}{i} \oint_{\partial D(0,1)} \frac{z^n}{az(a-z)+z-a} dz \right) = \operatorname{Re} \left(\frac{1}{ai} \oint_{\partial D(0,1)} \frac{z^n}{(\frac{1}{a}-z)(z-a)} dz \right)$$

Como $a \in (0,1)$ entonces solo hay un polo, $z_1 = a$, donde que $|z_1| = \frac{1}{a} > 1$, entonces de orden 1, luego:

$$\operatorname{Res} \left(\frac{z^n}{(\frac{1}{a}-z)(z-a)}; a \right) = \lim_{z \rightarrow a} \frac{(z-a) z^n}{(\frac{1}{a}-z)(z-a)} = \frac{a^{n+1}}{1-a^2}$$

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Entonces

$$\int_0^{2\pi} \frac{\omega \cos x}{a^2 + 1 - 2a \cos x} dx = \operatorname{Re} \left(\frac{1}{a} 2\pi i \operatorname{Res}(f, a) \right) = \operatorname{Re} \left(\frac{2\pi}{a} \frac{a^{n+1}}{1-a^2} \right) = \frac{2\pi a^n}{1-a^2} \quad \square$$

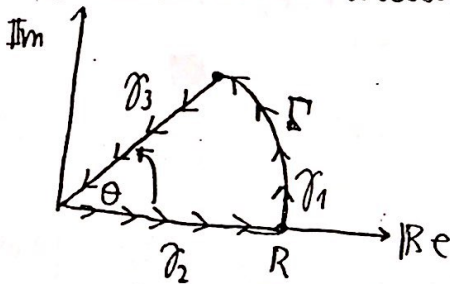
Problema 12.3

Sea $D = \{x+iy : x>0, y>0\}$ y $f: \bar{D} \subseteq \mathbb{C} \rightarrow \mathbb{C}$ continua en $\bar{D}$ y holomorfa en D . Suponga que existe una constante $M \geq 0$ tal que $|f(z)| \leq M/|z|^2$ para todo $z \in \bar{D}$, $|z| \geq 1$

1. Pruebe que $\forall \theta \in [0, \pi/2]$ se tiene:

$$\int_0^\infty f(x) dx = e^{i\theta} \int_0^\infty f(e^{i\theta} x) dx$$

consideramos el contorno:



luego como f es holomorfa en D se tiene

$$\oint_{\Gamma} f(z) dz = 0, \text{ con } \Gamma \subset D.$$

$$\oint_{\Gamma} f(z) dz = \sum_{i=1}^3 \int_{\gamma_i} f(z) dz = 0, \text{ luego}$$

• para γ_1 , sea $z = R e^{i\phi}$, $dz = R i e^{i\phi} d\phi$, $\phi \in [0, \theta]$

$$\int_{\gamma_1} f(z) dz = R i \int_0^\theta f(R e^{i\phi}) e^{i\phi} d\phi, \text{ si } R \rightarrow +\infty, \text{ entonces}$$

$$\left| \int_{\gamma_1} f(z) dz \right| \leq L(\Gamma) \sup_{z \in \Gamma} |f(z)| = \theta R \frac{M}{R^2} = \frac{\theta M}{R} \xrightarrow{R \rightarrow +\infty} 0$$

• para γ_2 , sea $z = x$, $dz = dx$, $x \in [0, R]$

$$\begin{aligned} \int_{\gamma_2} f(z) dz &= \int_0^R f(x) dx \quad \text{si } R \rightarrow +\infty \\ &= \int_0^\infty f(x) dx. \end{aligned}$$

sea $z = x e^{i\theta}$, $dz = e^{i\theta} dx$, $x \in [R, 0]$

$$\int_{\gamma_3} f(z) dz = \int_R^0 f(x e^{i\theta}) e^{i\theta} dx = -e^{i\theta} \int_0^R f(x e^{i\theta}) dx, \text{ si } R \rightarrow +\infty$$

$$= -e^{i\theta} \int_0^\infty f(e^{i\theta} x) dx, \text{ luego}$$

$$\int_{\Gamma_\infty} f(z) dz = \int_0^\infty f(x) dx - e^{i\theta} \int_0^\infty f(e^{i\theta} x) dx = 0$$

$$\Rightarrow \int_0^\infty f(x) dx = e^{i\theta} \int_0^\infty f(e^{i\theta} x) dx.$$

2. Utilice lo anterior con $f(z) = \exp(iz)/(1+z)^2$ y $\theta = \pi/2$ para demostrar que

$$\int_0^\infty \frac{\cos(x)}{(1+x)^2} dx + \int_0^\infty \frac{\exp(-x)}{1+x^2} dx = 1.$$

- en efecto:

$$\int_0^\infty f(x) dx = \int_0^\infty \frac{e^{ix}}{(x+i)^2} dx = \int_0^\infty \frac{\cos(x) + i \sin(x)}{(x+i)^2} dx$$

ahora por la parte 1. x tiene:

$$\int_0^\infty f(x) dx = i \int_0^\infty f(ix) dx = i \int_0^\infty \frac{e^{-x}}{(1+xi)^2} dx$$

$$= i \int_0^\infty \frac{e^{-x}}{(1-x^2+4x^2)} dx = \int_0^\infty \frac{e^{-x}(2x+i(1-x^2))}{(1+x^2)^2} dx$$

$$\therefore \int_0^\infty \frac{\cos(x)}{(x+i)^2} dx = \int_0^\infty \frac{2x e^{-x}}{(1+x^2)^2} dx$$

sea $u = e^{-x}$ $du = -e^{-x} dx$ $dv = \frac{2x}{(1+x^2)^2} dx$ $v = -\frac{1}{(1+x^2)}$ luego

$$\int_0^{\infty} \frac{\omega(x)}{(x+1)^2} dx = \left(\frac{-e^{-x}}{1+x^2} \right) \Big|_{x=0}^{x=+\infty} - \int_0^{\infty} (-e^{-x}) \left(-\frac{1}{x^2+1} \right) dx$$

$$\Rightarrow \int_0^{\infty} \frac{\omega(x)}{(x+1)^2} dx + \int_0^{\infty} \frac{\exp(-x)}{x^2+1} dx = 1 \quad \square$$

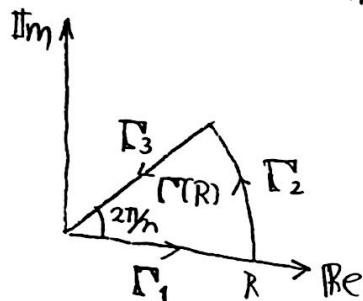
Problema 12.4

Demuestre que

$$\int_0^{\infty} \frac{x^m}{x^n+1} dx = \frac{\pi}{n \sin [\pi(m+1)/n]}$$

donde m y n son enteros positivos tales que $n-m \geq 2$

Indicacion: use el camino:



$$\text{sea } f(z) = \frac{z^m}{z^n+1}, \text{ entonces } z^n = e^{(\frac{2k+1}{n})i\pi}$$

$$\oint_{\Gamma(R)} f(z) dz = 2\pi i \cdot \text{Res}(f, e^{i\frac{\pi}{n}})$$

• Sobre Γ_1 se tiene $z=x$, $dz=dx$, $x \in [0, R]$, luego

$$\begin{aligned} \int_{\Gamma_1} f(z) dz &= \int_0^R \frac{x^m}{x^n+1} dx \quad \text{Si } R \rightarrow +\infty \text{ entonces} \\ &= \int_0^{\infty} \frac{x^m}{x^n+1} dx \end{aligned}$$

• Sobre Γ_2 se tiene $z = Re^{i\theta}$, $\theta \in [0, \frac{2\pi}{n}]$, $dz = Ri e^{i\theta} d\theta$

$$\int_{\Gamma_2} f(z) dz = \int_0^{\frac{2\pi}{n}} \frac{R^m e^{im\theta}}{R^n e^{in\theta} + 1} Ri e^{i\theta} d\theta, \text{ si } R \rightarrow +\infty \text{ entonces}$$

$$\left| \int_{\Gamma_2} f(z) dz \right| \leq L(\Gamma_2) \sup_{z \in \Gamma_2} |f(z)| \leq \frac{2\pi R}{n} \cdot \frac{R^m}{R^n} = \frac{2\pi}{n} R^{m-n+1} \rightarrow 0 \quad R \rightarrow +\infty$$

se tiene que $z = x e^{\frac{2\pi i}{n}}$, $dz = e^{\frac{2\pi i}{n}} dx$, luego $x \in [R, 0]$:

$$\int_{\Gamma_3} f(z) dz = e^{\frac{2\pi i}{n}} \int_R^0 \frac{x^m e^{\frac{2\pi i m}{n}}}{x^n e^{\frac{2\pi i m}{n}} + 1} dx, \text{ luego se tiene que:}$$

$\swarrow$
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13.4 Ejercicios

1. Encuentra la serie de Fourier de las siguientes funciones:

(a) $f(x) = |x|$ en $[-\pi, \pi]$

• Si $k=0$

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle |x|, 1 \rangle_{L^2}, \text{ con } \|1\|_{L^2}^2 = \int_{-\pi}^{\pi} 1 \cdot 1 dx = 2\pi$$

$$\langle |x|, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} |x| dx = -\int_{-\pi}^0 x dx + \int_0^{\pi} x dx = \frac{\pi^2}{2} + \frac{\pi^2}{2} = \pi^2$$

$$C_0 = \frac{\pi}{2} \Rightarrow a_0 = 2C_0 = \pi$$

• Si $k \neq 0$

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle |x|, e^{ikx} \rangle_{L^2}, \text{ con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{ikx} e^{-ikx} dx = 2\pi$$

$$\begin{aligned} \langle |x|, e^{ikx} \rangle_{L^2} &= \int_{-\pi}^{\pi} |x| e^{ikx} dx = -\int_{-\pi}^0 x e^{ikx} dx + \int_0^{\pi} x e^{ikx} dx, \text{ sea } u=x \text{ } dv=e^{ikx} dx \\ &= \left(\frac{x}{ik} e^{ikx} \right) \Big|_{-\pi}^0 - \frac{1}{ik} \int_{-\pi}^0 e^{ikx} dx + \left(\frac{x}{ik} e^{ikx} \right) \Big|_0^{\pi} + \frac{1}{ik} \int_0^{\pi} e^{ikx} dx \\ &= \frac{\pi}{ik} e^{ik\pi} + \frac{1}{(ik)^2} (1 - e^{ik\pi}) + \frac{-\pi}{ik} e^{-ik\pi} - \frac{1}{(ik)^2} (e^{-ik\pi} - 1) \\ &= \frac{2((-1)^k - 1)}{k^2}, \text{ ahora } \end{aligned}$$

$$C_k = \frac{((-1)^k - 1)}{\pi k^2} \Rightarrow a_k = 2 \operatorname{Re}(C_k) = \frac{2((-1)^k - 1)}{\pi k^2}, \text{ y } b_k = 2 \operatorname{Im}(C_k) = 0$$

entonces

$$S_f(x) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2((-1)^k - 1)}{\pi k^2} \cos(kx)$$

(b) $f(x) = \alpha x$ en $[-\pi, \pi]$

• para $k=0$, entonces

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle \alpha x, 1 \rangle_{L^2}$$

, entonces $\|1\|_{L^2}^2 = \int_{-\pi}^{\pi} dx = 2\pi$

$$\langle \alpha x, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} \alpha x dx = \alpha \left(\frac{\pi^2}{2} - \frac{\pi^2}{2} \right) = 0.$$

$$\therefore C_0 = a_0 = 0.$$

• para $k \neq 0$, entonces

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle \alpha x, e^{ikx} \rangle_{L^2}, \text{ con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{ikx} \overline{e^{ikx}} dx = 2\pi$$

$$\langle \alpha x, e^{ikx} \rangle_{L^2} = \int_{-\pi}^{\pi} \alpha x e^{ikx} dx = \alpha \int_{-\pi}^{\pi} x e^{ikx} dx, \text{ sea } u = x, \quad dv = e^{ikx} dx$$

$$du = dx, \quad v = \frac{e^{ikx}}{ik}$$

$$= \left(\frac{-\alpha x e^{-ikx}}{ik} \right) \Big|_{-\pi}^{\pi} + \frac{\alpha}{ik} \int_{-\pi}^{\pi} e^{-ikx} dx$$

$$= \frac{-\alpha \pi}{ik} e^{-ik\pi} - \frac{\alpha \pi}{ik} e^{ik\pi} + \frac{\alpha}{ik} \left(\frac{-e^{ikx}}{ik} \right) \Big|_{-\pi}^{\pi}$$

$$= \frac{-2\alpha\pi}{ik} (-1)^k + \frac{\alpha}{k^2} (e^{ik\pi} - e^{-ik\pi}) = \frac{2\alpha\pi}{k} (-1)^k i$$

luego $C_k = \frac{\alpha (-1)^k}{k} i \Rightarrow b_k = -2 \operatorname{Im}(C_k) = -2\alpha \frac{(-1)^k}{k}$

$$\wedge a_k = 2 \operatorname{Re}(C_k) = 0$$

finalmente

$$S_f(x) = -\sum_{k=1}^{\infty} \frac{2\alpha (-1)^k}{k} \sin(kx)$$

(c) $f(x) = ax^2$ en $[-\pi, \pi]$

• para $k=0$

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle ax^2, 1 \rangle_{L^2}, \text{ con } \|1\|_{L^2}^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\langle ax^2, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} ax^2 dx = a \frac{\pi^3}{3} + a \frac{\pi^3}{3} = \frac{2}{3} a \pi^3$$

$$\therefore C_0 = \frac{a\pi^2}{3} \Rightarrow a_0 = 2C_0 = \frac{2a\pi^2}{3}$$

• para $k \neq 0$

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle ax^2, e^{ikx} \rangle_{L^2}, \text{ con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{ikx} e^{-ikx} dx = 2\pi$$

$$\langle ax^2, e^{ikx} \rangle_{L^2} = \int_{-\pi}^{\pi} ax^2 e^{ikx} dx = a \int_{-\pi}^{\pi} x^2 e^{ikx} dx$$

sea $u = x^2$ $dv = e^{-ikx} dx$
 $du = 2x dx$ $v = \frac{-e^{-ikx}}{ik}$

$$= \left(\frac{-ax^2 e^{ikx}}{ik} \right) \Big|_{-\pi}^{\pi} + \frac{2a}{ik} \int_{-\pi}^{\pi} x e^{-ikx} dx$$

sea $u = x$ $dv = e^{-ikx} dx$
 $du = dx$ $v = \frac{-e^{-ikx}}{ik}$

$$= \left(\frac{-2ax e^{ikx}}{(ik)^2} \right) \Big|_{-\pi}^{\pi} + \frac{2a}{(ik)^2} \int_{-\pi}^{\pi} e^{-ikx} dx = \frac{4a\pi(-1)^k}{k^2}$$

luego $C_k = \frac{2a}{k^2} (-1)^k \Rightarrow a_k = 2\operatorname{Re}(C_k) = \frac{4a}{k^2} (-1)^k \wedge b_k = 2\operatorname{Im}(C_k) = 0$

finalmente

$$Sf(x) = \frac{a\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4a(-1)^k}{k^2} \cos(kx)$$

(d) $f(x) = ax^2$ en $[0, 2\pi]$

esto es equivalente a hacer un desarrollo en serie de Fourier entre $[-\pi, \pi]$ de $g(x) = a(x+\pi)^2$.

• para $k=0$

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle g, 1 \rangle_{L^2}, \text{ con } \|1\|_{L^2}^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\langle g, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} a(x+\pi)^2 dx = \frac{a}{3} (x+\pi)^3 \Big|_{-\pi}^{\pi} = \frac{8}{3} a\pi^3$$

$$\text{luego } G = \frac{4}{3} a\pi^2 \Rightarrow a_0 = 2G = \frac{8}{3} a\pi^2.$$

• para $k \neq 0$.

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle g, e^{ikx} \rangle_{L^2}, \text{ con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{-ikx} e^{ikx} dx = 2\pi$$

$$\begin{aligned} \langle g, e^{ikx} \rangle_{L^2} &= \int_{-\pi}^{\pi} a(x+\pi)^2 e^{-ikx} dx, \text{ sea } u = x+\pi \\ &\quad du = dx \\ &= \int_0^{2\pi} a u^2 e^{-ik(u-\pi)} du = - \int_0^{2\pi} a u^2 e^{-iku} du, \text{ sea } v = u^2 \\ &\quad dv = 2u du \quad w = \frac{e^{-iku}}{ik} \\ &= \left(\frac{u^2 e^{-iku}}{ik} \right) \Big|_0^{2\pi} - \frac{2}{ik} \int_0^{2\pi} u e^{-iku} du, \text{ sea } v = u \quad dv = du \quad w = \frac{e^{-iku}}{ik} \\ &= \frac{4\pi^2 (-1)^k}{ik} + \left(\frac{2u e^{-iku}}{(ik)^2} \right) \Big|_0^{2\pi} - \frac{2}{(ik)^2} \left(e^{-iku} \right) \Big|_0^{2\pi} = -\frac{4\pi}{k^2} - \frac{4\pi^2 (-1)^k}{k} i \end{aligned}$$

$$\text{luego } C_k = -\frac{2}{k^2} - \frac{2\pi (-1)^k}{k} i \Rightarrow a_k = 2 \operatorname{Re}(C_k) = -\frac{4}{k^2} \wedge b_k = -2 \operatorname{Im}(C_k) = \frac{4\pi (-1)^k}{k}$$

finalmente

$$S_f(x) = \frac{4}{3} a\pi^2 + \sum_{k=1}^{\infty} -\frac{4}{k^2} \cos(kx) + \frac{4\pi (-1)^k}{k} \sin(kx).$$

$$f(x) = e^{ax} \text{ en } [-\pi, \pi]$$

• para $k=0$

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle f, 1 \rangle_{L^2}, \text{ con } \|1\|_{L^2}^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\langle f, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} e^{ax} dx = \left(\frac{1}{a} e^{ax} \right) \Big|_{-\pi}^{\pi} = \frac{1}{a} (e^{a\pi} - e^{-a\pi}) = \frac{2}{a} \sinh(a\pi)$$

$$\therefore C_0 = \frac{\sinh(a\pi)}{a\pi} \Rightarrow a_0 = 2C_0 = \frac{2\sinh(a\pi)}{a\pi}$$

• para $k \neq 0$

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle f, e^{ikx} \rangle_{L^2}, \text{ con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{ikx} \overline{e^{ikx}} dx = 2\pi$$

$$\begin{aligned} \langle f, e^{ikx} \rangle_{L^2} &= \int_{-\pi}^{\pi} e^{ax} \overline{e^{ikx}} dx = \int_{-\pi}^{\pi} e^{(a-ik)x} dx = \left(\frac{1}{a-ik} e^{(a-ik)x} \right) \Big|_{-\pi}^{\pi} \\ &= \frac{1}{a-ik} (e^{(a-ik)\pi} - e^{-(a-ik)\pi}) = \frac{a+ik}{a^2+k^2} (e^a(-1)^k - (-1)^k e^{-a}) \end{aligned}$$

$$= 2 \frac{(a+ik)(-1)^k}{a^2+k^2} \sinh(a), \text{ luego}$$

$$C_k = \frac{(a+ik)(-1)^k}{\pi(a^2+k^2)} \sinh(a) \Rightarrow a_k = 2\operatorname{Re}(C_k) = \frac{2a(-1)^k}{\pi(a^2+k^2)} \sinh(a)$$

$$b_k = -2\operatorname{Im}(C_k) = \frac{-2k(-1)^k}{\pi(a^2+k^2)} \sinh(a), \text{ luego}$$

$$Sf(x) = \frac{\sinh(a\pi)}{a\pi} + \sum_{k=1}^{\infty} \frac{2a(-1)^k \sinh(a)}{\pi(a^2+k^2)} \cos(kx) - \frac{2k(-1)^k \sinh(a)}{\pi(a^2+k^2)} \sin(kx)$$

$$= \frac{\sinh(a\pi)}{a\pi} + \frac{2\sinh(a)}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{a^2+k^2} (a\cos(kx) + k\sin(kx))$$



(f) $f(x) = |\sin(x)|$ en $[-\pi, \pi]$

• para $k=0$

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle f, 1 \rangle_{L^2}, \text{ con } \|1\|_{L^2}^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\begin{aligned} \langle f, 1 \rangle_{L^2} &= \int_{-\pi}^{\pi} |\sin(x)| dx = -\int_{-\pi}^0 \sin(x) dx + \int_0^{\pi} \sin(x) dx \quad \text{propiedad del } \sin(x) \\ &= (\cos(x)) \Big|_{-\pi}^0 - (\cos(\pi)) \Big|_{\pi}^0 = 1+1+1+1 = 4 \end{aligned}$$

$$C_0 = \frac{2}{\pi} \Rightarrow a_0 = 2C_0 = \frac{4}{\pi}$$

• para $k \neq 0$

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle f, e^{ikx} \rangle_{L^2}, \text{ con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{ikx} e^{-ikx} dx = 2\pi$$

$$\langle f, e^{ikx} \rangle_{L^2} = \int_{-\pi}^{\pi} |\sin(x)| e^{ikx} dx = -\int_{-\pi}^0 \sin(x) e^{ikx} dx + \int_0^{\pi} \sin(x) e^{ikx} dx$$

calculamos dicha primitiva

$$I = \int \sin(x) e^{-ikx} dx, \text{ sea } u = \sin(x) \quad dv = e^{-ikx} dx$$

$$du = \cos(x) dx \quad v = \frac{-e^{-ikx}}{ik}$$

$$I = \frac{-\sin(x) e^{-ikx}}{ik} + \frac{1}{ik} \int \cos(x) e^{-ikx} dx, \text{ sea } u = \cos(x) \quad dv = e^{-ikx} dx$$

$$du = -\sin(x) dx \quad v = \frac{-e^{-ikx}}{ik}$$

$$I = \frac{-\sin(x) e^{-ikx}}{ik} - \frac{\cos(x) e^{-ikx}}{(ik)^2} - \underbrace{\frac{1}{(ik)^2} \int \sin(x) e^{-ikx} dx}_I$$

$$I \left(1 - \frac{1}{k^2}\right) = \frac{e^{-ikx}}{k^2} (\cos(x) + ik \sin(x)) \Rightarrow I = \frac{e^{-ikx}}{k^2 - 1} (\cos(x) + ik \sin(x))$$

$$\begin{aligned} \langle f, e^{ikx} \rangle_{L^2} &= - \left(\frac{e^{-ikx}}{k^2 - 1} (\cos(x) + ik \sin(x)) \right) \Big|_{-\pi}^0 + \left(\frac{e^{-ikx}}{k^2 - 1} (\cos(x) + ik \sin(x)) \right) \Big|_0^{\pi} \\ &= \frac{-1}{k^2 - 1} + \frac{(-1)^k}{k^2 - 1} + \frac{(-1)^k}{k^2 - 1} - \frac{1}{k^2 - 1} = \frac{2}{k^2 - 1} ((-1)^k - 1) \end{aligned}$$

$$c_k = \frac{((-1)^k - 1)}{\pi(k^2 + 1)} \Rightarrow a_k = 2\operatorname{Re}(c_k) = \frac{2((-1)^k - 1)}{\pi(k^2 + 1)} \wedge b_k = -2\operatorname{Im}(c_k) = 0$$

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k^2 + 1} \cos(kx)$$

(g)

$$f(x) = \begin{cases} a & \text{si } -\pi < x < 0 \\ b & \text{si } 0 < x < \pi \end{cases}$$

• para $k=0$

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle f, 1 \rangle_{L^2}, \quad \text{con } \|1\|_{L^2}^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\langle f, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} f(x) dx = a \int_{-\pi}^0 dx + b \int_0^{\pi} dx = (a+b)\pi$$

$$\therefore C_0 = \frac{a+b}{2} \Rightarrow a_0 = 2C_0 = a+b.$$

• para $k \neq 0$

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle f, e^{ikx} \rangle_{L^2}, \quad \text{con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{ikx} \overline{e^{ikx}} dx = 2\pi$$

$$\begin{aligned} \langle f, e^{ikx} \rangle_{L^2} &= \int_{-\pi}^{\pi} f(x) \overline{e^{ikx}} dx = a \int_{-\pi}^0 e^{-ikx} dx + b \int_0^{\pi} e^{-ikx} dx \\ &= -\frac{a}{ik} (1 - (-1)^k) - \frac{b}{ik} ((-1)^k - 1) \\ &= \frac{(b-a)i}{k} ((-1)^k - 1), \quad \text{ luego } \end{aligned}$$

$$C_k = \frac{(b-a)i}{2\pi k} ((-1)^k - 1) \Rightarrow a_k = 2\operatorname{Re}(C_k) = 0 \wedge b_k = -2\operatorname{Im}(C_k) = \frac{(a-b)((-1)^k - 1)}{\pi k}$$

luego

$$f(x) = \frac{a+b}{2} + \frac{(a-b)}{\pi} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k} \sin(kx)$$

2. Desarrolle en serie de Fourier la función

$$f(x) = \begin{cases} 0 & \text{si } -\pi < x < 0 \\ \sin(x) & \text{si } 0 < x < \pi \end{cases}$$

tenemos que

• para $k=0$

$$C_0 = \frac{1}{\|1\|_{L^2}^2} \langle f, 1 \rangle_{L^2}, \quad \text{con } \|1\|_{L^2}^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\langle f, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} f(x) dx = \int_0^{\pi} \sin(x) dx = (-\cos \pi + \cos 0) = 2$$

$$\text{luego } C_0 = \frac{1}{\pi} \Rightarrow a_0 = 2C_0 = \frac{2}{\pi}$$

• para $k \neq 0$

$$C_k = \frac{1}{\|e^{ikx}\|_{L^2}^2} \langle f, e^{ikx} \rangle_{L^2}, \quad \text{con } \|e^{ikx}\|_{L^2}^2 = \int_{-\pi}^{\pi} e^{ikx} e^{-ikx} dx = 2\pi$$

$$\langle f, e^{ikx} \rangle_{L^2} = \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \int_0^{\pi} \sin(x) e^{-ikx} dx$$

calculamos la primitiva $I = \int \sin(x) e^{-ikx} dx$, sea $u = \sin x$ $du = \cos x dx$ $v = -\frac{e^{-ikx}}{ik}$
 $I = -\frac{\sin(x)}{ik} e^{-ikx} + \frac{1}{ik} \int \cos(x) e^{-ikx} dx$, sea $u = \cos x$ $du = -\sin x dx$ $v = -\frac{e^{-ikx}}{ik}$

$$I = -\frac{\sin(x)}{ik} e^{-ikx} - \frac{\cos(x)}{(ik)^2} e^{-ikx} - \frac{1}{(ik)^2} \int \sin x e^{-ikx} dx$$

$$\therefore I \left(1 - \frac{1}{k^2}\right) = \left(\frac{\cos(x)}{k^2} + \frac{\sin(x)}{k}\right) e^{-ikx}$$

$$I = \frac{\cos(x) + ik \sin(x)}{k^2 - 1} e^{-ikx}$$

ahora:

$$\langle f, e^{ikx} \rangle = \left(\frac{\cos(x) + ik \sin(x)}{k^2 - 1} e^{-ikx} \right) \Big|_0^{\pi} = \frac{-((-1)^k + 1)}{k^2 - 1}$$

$$c_k = \frac{-((-1)^k + 1)}{2\pi(k^2 - 1)} \Rightarrow a_k = 2\operatorname{Re}(c_k) = \frac{-((-1)^k + 1)}{\pi(k^2 - 1)} \wedge b_k = -2\operatorname{Im}(c_k) = 0$$

luego

$$S_f(x) = \frac{1}{\pi} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k + 1}{k^2 - 1} \sin(kx)$$

3. Dada la función $f(x) = |x|$, encuentre:

(i) Su serie de Fourier en el intervalo $[0, 2]$

en este caso $l = 1$, entonces

$$a_0 = \int_0^2 f(x) dx = \int_0^2 x dx = \frac{4}{2} = 2.$$

ahora $\forall k \neq 0$ se tiene

$$a_k = \int_0^2 f(x) \cos(k\pi x) dx = \int_0^2 x \cos(k\pi x) dx, \text{ sea } \begin{matrix} u = x & dv = \cos(k\pi x) dx \\ du = dx & v = \frac{\sin(k\pi x)}{k\pi} \end{matrix}$$

$$= \left(\frac{x \sin(k\pi x)}{k\pi} \right) \Big|_0^2 - \frac{1}{k\pi} \int_0^2 \sin(k\pi x) dx = \frac{1}{(k\pi)^2} (\cos(k\pi x)) \Big|_0^2 = 0$$

$$b_k = \int_0^2 f(x) \sin(k\pi x) dx = \int_0^2 x \sin(k\pi x) dx, \text{ sea } \begin{matrix} u = x & dv = \sin(k\pi x) dx \\ du = dx & v = -\frac{\cos(k\pi x)}{k\pi} \end{matrix}$$

$$= \left(-\frac{x \cos(k\pi x)}{k\pi} \right) \Big|_0^2 + \frac{1}{k\pi} \int_0^2 \cos(k\pi x) dx = \frac{-2}{k\pi} + \frac{1}{(k\pi)^2} (\sin(k\pi x)) \Big|_0^2$$

$$= \frac{-2}{\pi k}, \text{ luego } S_f(x) = 2 - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k}$$

(ii) Su serie de Fourier en el intervalo $[-1, 1]$

en este caso $l = 1$, luego

$$a_0 = \int_{-1}^1 f(x) dx = -\int_{-1}^0 x dx + \int_0^1 x dx = \frac{1}{2} + \frac{1}{2} = 1$$

ahora $\forall k \neq 0$ se tiene

$$a_k = \int_{-1}^1 f(x) \cos(k\pi x) dx = \int_0^1 x \cos(k\pi x) dx - \int_{-1}^0 x \cos(k\pi x) dx \quad \text{sea } \begin{matrix} u = x & dv = \cos(k\pi x) dx \\ du = dx & v = \frac{\sin(k\pi x)}{k\pi} \end{matrix}$$

$$\begin{aligned}
 a_k &= \left(\frac{x \sin(k\pi x)}{k\pi} \right) \Big|_0^1 - \frac{1}{k\pi} \int_0^1 \sin(k\pi x) dx - \left(\frac{x \sin(k\pi x)}{k\pi} \right) \Big|_{-1}^0 + \frac{1}{k\pi} \int_{-1}^0 \sin(k\pi x) dx \\
 &= \frac{1}{(k\pi)^2} (\cos(k\pi x)) \Big|_0^1 - \frac{1}{(k\pi)^2} (\cos(k\pi x)) \Big|_{-1}^0 \\
 &= \frac{1}{(k\pi)^2} (\cos(k\pi) - 1 - 1 + \cos(k\pi)) = \frac{2}{(k\pi)^2} \left(\frac{e^{ik\pi} + e^{-ik\pi}}{2} - 1 \right) \\
 a_k &= \frac{2}{(k\pi)^2} ((-1)^k - 1) = \begin{cases} \frac{-4}{(k\pi)^2} & \text{si } k \text{ es impar} \\ 0 & \text{si } k \text{ es par.} \end{cases}
 \end{aligned}$$

ahora

$b_k = 0$, porque f es par en $[-1, 1]$, luego

$$\begin{aligned}
 S_f(x) &= \frac{1}{2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k^2} \cos(k\pi x) \\
 &= \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k+1)\pi x)}{(2k+1)^2}
 \end{aligned}$$

(iii) Si desarrollamos en serie de senos en $[0, 1]$.

extendemos f de manera impar a $[-1, 1]$ como $f(x) = x$, luego $l = 1$

$$\begin{aligned}
 b_k &= \int_{-1}^1 x \sin(k\pi x) dx, \text{ sea } u = x, \quad dv = \sin(k\pi x) dx \\
 &\quad du = dx, \quad v = -\frac{\cos(k\pi x)}{k\pi} \\
 &= \left(-\frac{x \cos(k\pi x)}{k\pi} \right) \Big|_{-1}^1 + \frac{1}{k\pi} \int_{-1}^1 \cos(k\pi x) dx = \frac{-2}{k\pi}
 \end{aligned}$$

$$\therefore S_f(x) = \frac{-2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\pi x)$$

luego para $l = \frac{1}{2}$ se tiene $([0, 1])$

$$b_k = \frac{1}{(1/2)} \left(\frac{-2}{k\pi} \right) = \frac{-4}{k\pi}, \quad S_f(x) = \frac{-4}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\pi x)$$

(iv) Si desarrollamos en serie de cosenos en $[0, 1]$

se tiene que extendemos f a $[-1, 1]$ como $f(x) = |x|$, luego

$$S_f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k+1)\pi x)}{(2k+1)^2}$$

3.5 Problemas

Problema 13.1

Calcule la serie de Fourier en $[-\pi, \pi]$ de la función:

$$f(x) = \begin{cases} 2 & -\pi < x < 0 \\ 6 & 0 \leq x < \pi \end{cases}$$

Señale el límite de la serie en cada punto de $[-\pi, \pi]$ y justifique. Deduzca que

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

Indicación: Para calcular la serie descrita arriba evalúe la serie de Fourier en un punto adecuado

Calculamos la serie de Fourier, reconociendo $l = \pi$, luego

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} 6 dx + \frac{1}{\pi} \int_{-\pi}^0 2 dx = 6 + 2 = 8$$

$\forall k \neq 0$ se tiene

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx = \frac{6}{\pi} \int_0^{\pi} \cos(kx) dx + \frac{2}{\pi} \int_{-\pi}^0 \cos(kx) dx$$

$$= \frac{6}{k\pi} (\sin k\pi - \sin 0) + \frac{2}{k\pi} (\sin(0) - \sin(-k\pi)) = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \frac{6}{\pi} \int_0^{\pi} \sin(kx) dx + \frac{2}{\pi} \int_{-\pi}^0 \sin(kx) dx$$

$$= -\frac{6}{k\pi} (\cos(k\pi) - \cos(0)) - \frac{2}{k\pi} (\cos(0) - \cos(k\pi))$$

$$= -\frac{4}{k\pi} ((-1)^k - 1) = \frac{4}{k\pi} (1 - (-1)^k) = \begin{cases} \frac{8}{k\pi} & \text{si } k \text{ impar} \\ 0 & \text{si } k \text{ par} \end{cases}$$

$$\therefore S_f(x) = 4 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin((2k-1)x)$$

$$\text{tenemos luego que } S_f(\pi) = 4 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin((2k-1)\pi) = 4$$

$$S_f(-\pi) = 4 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(-(2k-1)\pi) = 4$$

además $f(\pi) = 6 \wedge f(-\pi) = 2$, luego

$$S_f(\pi) = S_f(-\pi) = \frac{1}{2}(f(\pi) + f(-\pi)) = 4.$$

luego $f(x) = S_f(x)$, ahora en $\frac{\pi}{2}$ se tiene $f(\frac{\pi}{2}) = 6 = S_f(\frac{\pi}{2})$

$$6 = 4 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin((2k-1)\frac{\pi}{2}) = 4 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$



Problema 13.2

Desarrolle en serie de Fourier la función

$$f(x) = \begin{cases} 0 & \text{si } -\pi < x < 0 \\ x & \text{si } 0 < x < \pi \end{cases}$$

Aplique el teorema de convergencia para deducir

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

tenemos que $l = \pi$, entonces

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \frac{\pi^2}{2} = \frac{\pi}{2}$$

$\forall k \neq 0$ se tiene:

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx = \frac{1}{\pi} \int_0^{\pi} x \cos(kx) dx & \begin{aligned} \text{sea } u &= x & dv &= \cos(kx) dx \\ du &= dx & v &= \frac{1}{k} \sin(kx) dx \end{aligned} \\ &= \left(\frac{x}{k\pi} \sin(kx) \right) \Big|_0^{\pi} - \frac{1}{\pi k} \int_0^{\pi} \sin(kx) dx = \frac{1}{\pi k^2} (\cos(k\pi) - \cos(0)) \\ &= \frac{1}{\pi k^2} ((-1)^k - 1) = \begin{cases} -\frac{2}{\pi k^2} & \text{si } k \text{ es impar} \\ 0 & \text{si } k \text{ es par} \end{cases} \end{aligned}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \frac{1}{\pi} \int_0^{\pi} x \sin(kx) dx$$

Sea $u = x$ $du = dx$ $v = \frac{-\cos(kx)}{k}$

$$= \left(\frac{-x}{k\pi} \cos(kx) \right) \Big|_0^{\pi} + \frac{1}{\pi k} \int_0^{\pi} \cos(kx) dx = \frac{-\pi}{k\pi} (-1)^k = -\frac{(-1)^k}{k}$$

luego

$$S_f(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{\pi k^2} \cos(kx) - \frac{(-1)^k}{k} \sin(kx)$$

tenemos que f es continua en todo $[-\pi, \pi]$ y derivable en $(-\pi, \pi)$ con derivado por la izquierda a π y por la derecha a $-\pi$, luego

$$S_f(x^+) = S_f(x^-) = \frac{1}{2} (f(x^+) + f(x^-)) \quad \text{por el teorema de convergencia, entonces}$$

$$S_f(\pi) = S_f(-\pi) = \frac{1}{2} (f(\pi) + f(-\pi)) = \frac{\pi}{2}, \text{ luego}$$

$$S_f(\pi) = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{\pi k^2} \cos(k\pi) - \frac{(-1)^k}{k} \sin(k\pi) = \frac{\pi}{2}$$

$$\Rightarrow -\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi)}{(2k-1)^2} = \frac{\pi}{4}$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

Problema 13.3

Sea $f \in C^1$, 2π periódica, tal que $\int_0^{2\pi} f(x) dx = 0$

(a) probar la identidad de Parseval esto es:

$$\int_0^{2\pi} f^2(x) dx = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

- Como f es 2π periódica, entonces si S_f es su serie de Fourier entre 0 y 2π , se tiene que $f = S_f$.

ahora

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad , \text{luego}$$

$$\int_0^{2\pi} f(x) dx = 0 \Leftrightarrow \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right) dx = 0$$

pero $S_f(x)$ converge uniformemente (por el corolario B.2.5), luego $\int \sum = \sum \int$

$$\Rightarrow \int_0^{2\pi} \frac{a_0}{2} dx + \sum_{k=1}^{\infty} \int_0^{2\pi} (a_k \overset{0}{\cancel{\cos(kx)}} + b_k \overset{0}{\cancel{\sin(kx)}}) dx = 0$$

$$\Rightarrow a_0 = 0 \quad , \text{luego} \quad S_f(x) = \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

luego

$$\int_0^{2\pi} f^2(x) dx = \int_0^{2\pi} S_f^2(x) dx = \int_0^{2\pi} \left(\sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \right)^2 dx$$

$$\text{pero} \quad \left(\sum_{k=1}^N a_k \right)^2 = \sum_{k=1}^N \sum_{j=1}^N a_k a_j = \sum_{k=1}^N a_k^2 + \sum_{k=1}^N \sum_{\substack{j=1 \\ k \neq j}}^N a_k a_j$$

luego

$$\int_0^{2\pi} f^2(x) dx = \int_0^{2\pi} \left(\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right)^2 + \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{j=1}^{\infty} (a_k \cos kx + b_k \sin kx) (a_j \cos jx + b_j \sin jx) dx$$

calculamos por separado.

$$\begin{aligned} \int_0^{2\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)^2 dx &= \int_0^{2\pi} \left(\sum_{n=1}^{\infty} a_n^2 \cos^2 nx + 2a_n b_n \overset{0}{\cancel{\cos nx \sin nx}} + b_n^2 \sin^2 nx \right) dx \\ &= \sum_{n=1}^{\infty} \left(\int_0^{2\pi} (a_n^2 \cos^2(nx) + 2a_n b_n \overset{0}{\cancel{\cos(nx) \sin(nx)}} + b_n^2 \sin^2(nx)) dx \right) \\ &= \sum_{n=1}^{\infty} \left(\int_0^{2\pi} \left(a_n^2 \left(\frac{1 + \overset{0}{\cancel{\cos(2nx)}}}{2} \right) + b_n^2 \left(\frac{1 - \overset{0}{\cancel{\cos(2nx)}}}{2} \right) \right) dx \right) \end{aligned}$$

$$\therefore \int_0^{2\pi} \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))^2 dx = \pi \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

ahora:

$$\int_0^{2\pi} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{j=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) (a_j \cos(jx) + b_j \sin(jx)) dx = \int_0^{2\pi} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{j=1}^{\infty} (a_k a_j \cos(kx) \cos(jx) +$$

$$a_k b_j \cos(kx) \sin(jx) + a_j b_k \cos(jx) \sin(kx) + b_k b_j \sin(kx) \sin(jx)) dx$$

$$= \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \sum_{k=1}^{\infty} a_k a_j \int_0^{2\pi} \cos(kx) \cos(jx) dx + a_k b_j \int_0^{2\pi} \cos(kx) \sin(jx) dx + a_j b_k \int_0^{2\pi} \cos(jx) \sin(kx) dx + b_k b_j \int_0^{2\pi} \sin(kx) \sin(jx) dx$$

, pero $j \neq k$, luego

$$\therefore \int_0^{2\pi} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{j=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) (a_j \cos(jx) + b_j \sin(jx)) dx = 0 \quad \text{y se tiene}$$

$$\int_0^{2\pi} f^2(x) dx = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

(b) Deduzca que

$$\int_0^{2\pi} f'^2(x) dx = \pi \sum_{n=1}^{\infty} (n^2 a_n^2 + n^2 b_n^2)$$

tenemos que $f(x) = S_f(x) \Rightarrow f'(x) = S_f'(x)$

$$S_f'(x) = \left(\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right)' = \sum_{n=1}^{\infty} n b_n \cos(nx) - n a_n \sin(nx)$$

haciendo un desarrollo parecido al de (a) nos queda

$$\int_0^{2\pi} f'^2(x) dx = \int_0^{2\pi} \sum_{n=1}^{\infty} (n b_n \cos(nx) - n a_n \sin(nx))^2 dx + \int_0^{2\pi} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{j=1}^{\infty} (k b_k \cos(kx) - k a_k \sin(kx)) (j b_j \cos(jx) - j a_j \sin(jx)) dx$$

usando los argumentos de (a) decimos que:

$$\int_0^{2\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k b_k \cos(kx) - k a_k \sin(kx)) (j b_j \cos(jx) - j a_j \sin(jx)) dx = 0, \text{ finalmente calculamos}$$

$$\int_0^{2\pi} \sum_{n=1}^{\infty} (n b_n \cos(nx) - n a_n \sin(nx))^2 dx = \int_0^{2\pi} \sum_{n=1}^{\infty} (n^2 b_n^2 \cos^2(nx) - 2n^2 a_n b_n \cos(nx) \sin(nx) + n^2 a_n^2 \sin^2(nx)) dx$$

$$= \sum_{n=1}^{\infty} n^2 b_n^2 \int_0^{2\pi} \frac{1 + \cos(2nx)}{2} dx - 2n^2 a_n b_n \int_0^{2\pi} \cos(nx) \sin(nx) dx + n^2 a_n^2 \int_0^{2\pi} \frac{1 - \cos(2nx)}{2} dx$$

$$= \pi \sum_{n=1}^{\infty} (n^2 a_n^2 + n^2 b_n^2), \text{ ahora concluimos.}$$

$$\int_0^{2\pi} f'^2(x) dx = \pi \sum_{n=1}^{\infty} (n^2 a_n^2 + n^2 b_n^2)$$

(c) Pruebe la desigualdad de Wirtinger

$$\int_0^{2\pi} f'^2(x) dx \geq \int_0^{2\pi} f^2(x) dx \quad (13.2)$$

En efecto tenemos que si $n \geq 1 \Rightarrow n-1 \geq 0 \Rightarrow (n-1)(n+1) \geq 0$
por lo tanto $n^2 - 1 \geq 0$, luego

$$\sum_{n=1}^{\infty} (n^2 - 1) \geq 0 \Rightarrow \sum_{n=1}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) \geq 0, \text{ si } a_n, b_n \in \mathbb{R} \forall n \geq 1 \Rightarrow a_n^2 + b_n^2 \geq 0, \text{ luego}$$

$$\Leftrightarrow \sum_{n=1}^{\infty} n^2 a_n^2 + n^2 b_n^2 \geq \sum_{n=1}^{\infty} a_n^2 + b_n^2 \Leftrightarrow \int_0^{2\pi} f'^2(x) dx \geq \int_0^{2\pi} f^2(x) dx$$

reducirse a la igualdad en (13.2) si y solo si $f(x) = a \cos(x) + b \sin(x)$

pdq:

$$\int_0^{2\pi} f^2(x) dx = \int_0^{2\pi} f'(x)^2 dx \iff f(x) = a \cos(x) + b \sin(x)$$

$\Rightarrow$ por (a) y (b) se tiene que

$$\int_0^{2\pi} f'^2(x) dx = \int_0^{2\pi} f^2(x) dx \iff \pi \sum_{n=1}^{\infty} n^2 a_n^2 + n^2 b_n^2 = \pi \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$$\iff \sum_{n=1}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) = 0 \Rightarrow n=1 \text{ porque } a_n, b_n \in \mathbb{R} \forall n \geq 1$$

luego $\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) = \sum_{n=1}^{\infty} a_n^2 + b_n^2 = a^2 + b^2, a, b \in \mathbb{R}$

luego $f(x) = S_f(x) = \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = a \cos(x) + b \sin(x)$

es decir la serie tiene $\begin{cases} a_1 = a \\ b_1 = b \end{cases} \wedge \forall i > 1, \begin{cases} a_i = 0 \\ b_i = 0 \end{cases}$

$$\Leftarrow \begin{aligned} f(x) &= a \cos(x) + b \sin(x) \\ f'(x) &= -a \sin(x) + b \cos(x) \end{aligned}$$

luego $\int_0^{2\pi} f^2(x) dx = \int_0^{2\pi} (a \cos(x) + b \sin(x))^2 dx$

$$= a^2 \int_0^{2\pi} \cos^2(x) dx + 2ab \int_0^{2\pi} \sin(x) \cos(x) dx + b^2 \int_0^{2\pi} \sin^2(x) dx$$

$$= a^2 \int_0^{2\pi} \frac{1 + \cos(2x)}{2} dx + b^2 \int_0^{2\pi} \frac{1 - \cos(2x)}{2} dx$$

$$= \pi(a^2 + b^2)$$

$$\int_0^{2\pi} f'^2(x) dx = \int_0^{2\pi} (b \cos(x) - a \sin(x))^2 dx$$

$$= b^2 \int_0^{2\pi} \cos^2(x) dx - 2ab \int_0^{2\pi} \cos(x) \sin(x) dx + a^2 \int_0^{2\pi} \sin^2(x) dx$$

$$= b^2 \int_0^{2\pi} \frac{1 + \cos(x)}{2} dx + a^2 \int_0^{2\pi} \frac{1 - \cos(x)}{2} dx$$

$$= \pi(b^2 + a^2)$$

$$\therefore \int_0^{2\pi} f(x)^2 dx = \int_0^{2\pi} f'(x)^2 dx = \pi(a^2 + b^2)$$

Problema 13.4

(a) Demuestre que la función $f(x) = \cos(x)$, $x \in (0, \pi)$, admite el siguiente desarrollo en serie

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi} \left(\frac{n}{4n^2 - 1} \right) \sin(2nx)$$

Indicación:

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta)$$

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

tenemos que $l = \frac{\pi}{2}$, luego

$$a_0 = \frac{1}{\pi/2} \int_0^{\pi} \cos(x) dx = \frac{2}{\pi} (\sin(\pi) - \sin(0)) = 0$$

$$a_k = \frac{1}{\pi/2} \int_0^{\pi} \cos(x) \cos\left(\frac{k\pi x}{\pi/2}\right) dx = \frac{2}{\pi} \int_0^{\pi} \cos(x) \cos(2kx) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\cos((2k+1)x) + \cos((2k-1)x)) dx = \frac{1}{\pi} \left(\frac{\sin((2k+1)x)}{2k+1} + \frac{\sin((2k-1)x)}{2k-1} \right) \Big|_0^{\pi}$$

$$= 0$$

$$b_k = \frac{1}{\pi/2} \int_0^{\pi} \cos(x) \sin\left(\frac{k\pi x}{\pi/2}\right) dx = \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(2kx) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\sin((2k+1)x) - \sin((1-2k)x)) dx$$

, por imponiendo $\sin(-x) = -\sin(x)$

$$= \frac{1}{\pi} \left(-\frac{\cos((2k+1)x)}{2k+1} - \frac{\cos((2k-1)x)}{2k-1} \right) \Big|_0^{\pi}$$

$$= \frac{1}{\pi} \left(\frac{1 - \cos((2k+1)\pi)}{2k+1} + \frac{1 - \cos((2k-1)\pi)}{2k-1} \right)$$

$$= \frac{1}{\pi} \left(\frac{1 - (-1)^{2k+1}}{2k+1} + \frac{1 - (-1)^{2k-1}}{2k-1} \right) = \frac{1}{\pi} \left(\frac{1 + (-1)^{2k}}{2k+1} + \frac{1 + (-1)^{2k}}{2k-1} \right)$$

$$= \frac{2}{\pi} \left(\frac{1}{2k+1} + \frac{1}{2k-1} \right) = \frac{2}{\pi} \left(\frac{2k-1 + 2k+1}{4k^2-1} \right) = \frac{8}{\pi} \left(\frac{k}{4k^2-1} \right)$$

luego

$$S_f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi} \left(\frac{n}{4n^2-1} \right) \sin\left(\frac{n\pi x}{\pi/2}\right) = \sum_{n=1}^{\infty} \frac{8}{\pi} \left(\frac{n}{4n^2-1} \right) \sin(2nx)$$

pero $\omega(x)$ está bien definida entre 0 y π y tiene derivadas laterales hacia 0 y π . luego $f = S_f$ ▣

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi} \left(\frac{n}{4n^2-1} \right) \sin(2nx)$$

(b) Probar que:

$$\frac{\pi\sqrt{2}}{16} = \frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \dots$$

en efecto

$$f\left(\frac{\pi}{4}\right) = \sum_{n=1}^{\infty} \frac{8}{\pi} \left(\frac{n}{4n^2-1} \right) \sin\left(2n\frac{\pi}{4}\right) \quad \text{pero } \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{si } n \text{ par} \end{cases}$$

$$\Leftrightarrow \frac{\sqrt{2}}{2} = \frac{8}{\pi} \left(\frac{1}{4-1} - \frac{3}{36-1} + \frac{5}{100-1} - \frac{7}{196-1} + \dots \right)$$

$$\Leftrightarrow \frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \dots = \frac{\pi\sqrt{2}}{16} \quad \text{▣}$$

Ejercicios

1. Calcular la transformada de Fourier de $f(x) = \cos(\pi x) / [x^3 - x^2 + 4x - 4]$

$$f(x) = \frac{\cos(\pi x)}{x^2(x-1) + 4(x-1)} = \frac{\cos(\pi x)}{(x^2+4)(x-1)} \quad \text{luego}$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos(\pi y)}{(y^2+4)(y-1)} e^{-iys} dy$$

sea $u = -y$, $du = -dy$
si $y \rightarrow +\infty$, $u \rightarrow -\infty$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos(-\pi u)}{(u^2+4)(-u-1)} e^{ius} du \quad \text{pero } \cos(x) = \cos(-x)$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{\cos(\pi u)}{(u^2+4)(u+1)}}_{h(u)} e^{ius} du = -\frac{1}{\sqrt{2\pi}} \left(2\pi i \sum_{\text{Im}(z) > 0} \text{Res}(h(z), z) + \pi i \sum_{x \in \mathbb{R}} \text{Res}(h(x), x) \right)$$

h tiene 3 polos: $u_1 = 2i$, $u_2 = -2i$, $u_3 = -1$, luego tenemos un polo en el eje real y un polo con parte imaginaria mayor a 0, entonces consideramos solo dos polos ambos de orden 1

$$\hat{f}(s) = -\frac{1}{\sqrt{2\pi}} \left(2\pi i \text{Res}(h(z), z=2i) + \pi i \text{Res}(h(x), x=-1) \right)$$

$$\text{Res}(h, 2i) = \lim_{z \rightarrow 2i} \frac{1}{0!} \frac{d^0}{dz^0} \left(\frac{(z-2i) \cos(\pi z) e^{izs}}{(z-2i)(z+2i)(z+1)} \right)$$

$$= \frac{e^{-2s}}{4i(2i+1)} \cos(2\pi i) = -\frac{i e^{-2s}}{4(2i+1)} \left(\frac{e^{i(2\pi i)} + e^{-i(2\pi i)}}{2} \right)$$

$$= \frac{-i(1-2i) e^{-2s}}{4(4+1)2} (e^{2\pi} + e^{-2\pi}) = \frac{-(1+i) e^{-2s}}{20} \cosh(2\pi)$$

$$\text{Res}(h, -1) = \lim_{z \rightarrow -1} \frac{1}{0!} \frac{d^0}{dz^0} \left(\frac{(z+1) \cos(\pi z) e^{izs}}{(z^2+4)(z+1)} \right) = \frac{\cos(-\pi) e^{-s}}{1+5} = -\frac{e^{-s}}{6}$$

$$\hat{f}(s) = -\frac{1}{\sqrt{2\pi}} \left(-2\pi i (2i) e^{-2s} \cosh(2\pi) - \frac{\pi i}{6} e^{-s} \right)$$

$$= \sqrt{\frac{\pi}{2}} \left(\frac{(2i-1) e^{-2s} \cosh(2\pi)}{10} - \frac{i e^{-s}}{6} \right)$$

3. Sea $f(x) = e^{-a|x|}$, pruebe que la transformada de Fourier de f viene dada por

$$\hat{f}(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}.$$

en efecto, tenemos que

$$\begin{aligned} \hat{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-a|y|} e^{-isy} dy = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-(a+is)y} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{(a-is)y} dy \\ &= \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-(a+is)y}}{(a+is)} \right) \Big|_0^{+\infty} + \frac{1}{\sqrt{2\pi}} \left(\frac{e^{(a-is)y}}{(a-is)} \right) \Big|_{-\infty}^0 \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a+is} + \frac{1}{a-is} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{a-is}{a^2+s^2} + \frac{a+is}{a^2+s^2} \right) = \frac{2a}{\sqrt{2\pi}(a^2+s^2)} \end{aligned}$$

$$\hat{f}(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}.$$

□

14.5 Problemas

Problema 14.1

Calcular la transformada de Fourier de $f(x) = \cos(\pi x) / [x^3 - x^2 + 4x - 4]$

(Ver Ejercicio 1)

Problema 14.2

En cada caso demuestre que la transformada de Fourier es la función indicada

$$1. f(x) = e^{-x^2} \Rightarrow \hat{f}(s) = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

tenemos que:

$$\begin{aligned} \hat{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2} e^{-isy} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(y^2 + isy + (\frac{is}{2})^2) + (\frac{is}{2})^2} dy && \text{hacemos un} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(y + \frac{is}{2})^2} e^{-\frac{s^2}{4}} dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \int_{-\infty}^{+\infty} e^{-u^2} du, && \text{reemplazamos} \\ & && \text{sea } u = y + \frac{is}{2} \\ & && du = dy \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \int_{-\infty}^{+\infty} e^{-u^2} du$$

$I_x = \int_{-\infty}^{\infty} e^{-x^2} dx$, luego $I_x I_y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx$, tenemos que $x, y \in]-\infty, +\infty[$, luego integramos sobre todo el plano

para esto hacemos el cambio de variables a cilíndricas
 $\rho^2 = x^2 + y^2$, $\theta = \arctan\left(\frac{y}{x}\right)$, tal que si $x, y \in]-\infty, +\infty[\Rightarrow \rho \in [0, +\infty[$, $\theta \in [0, 2\pi[$

$$I_x I_y = \int_0^{+\infty} \int_0^{2\pi} e^{-\rho^2} \rho d\theta d\rho = 2\pi \int_0^{+\infty} \rho e^{-\rho^2} d\rho = 2\pi \left(-\frac{e^{-\rho^2}}{2} \right) \Big|_{\rho=0}^{\rho=+\infty} = \pi$$

luego, $I_x = I_y = \sqrt{\pi}$, entonces

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} I_x = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

2. $f(x) = \begin{cases} e^{-x} & \text{si } 0 \leq x \\ 0 & \text{si } 0 > x \end{cases} \Rightarrow \hat{f}(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{1+is}$

$$\begin{aligned} \hat{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-isy} dy = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-(1+is)y} dy, \text{ luego} \\ &= \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-(1+is)y}}{(1+is)} \right) \Big|_0^{+\infty} = \frac{1}{\sqrt{2\pi}} \frac{1}{1+is} \end{aligned}$$

3. $f(x) = \exp(-a|x|)$, $a > 0 \Rightarrow \hat{f}(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$

(ver ejercicio 2)

4. $f(x) = \begin{cases} -k & \text{si } |x| \leq a \\ 0 & \text{si } |x| > a \end{cases} \Rightarrow \hat{f}(s) = \sqrt{\frac{2}{\pi}} \frac{k \sin(sa)}{s}$

$$\begin{aligned} \hat{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-isy} dy = -\frac{k}{\sqrt{2\pi}} \int_{-a}^a e^{-isy} dy \\ &= -\frac{k}{\sqrt{2\pi}} \left(-\frac{e^{-isy}}{is} \right) \Big|_{y=-a}^{y=a} = \frac{k}{\sqrt{2\pi}} \left(\frac{e^{-isa}}{is} - \frac{e^{isa}}{is} \right) \\ &= \frac{-2k \sin(sa)}{\sqrt{2\pi} s} = -\sqrt{\frac{2}{\pi}} \frac{k \sin(sa)}{s} \end{aligned}$$

$$5. f(x) = \frac{1}{1+x^2} \Rightarrow \hat{f}(s) = \pi e^{-|s|}$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-isy}}{1+y^2} dy. \quad \text{sea } u = -y \quad \text{si } y \rightarrow +\infty \Rightarrow u \rightarrow -\infty$$

$$du = -dy \quad (\text{suponiendo } s > 0)$$

$$\hat{f}(s) = \frac{-1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} \frac{e^{isu}}{1+u^2} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{isu}}{(u+i)(u-i)} du = \frac{1}{\sqrt{2\pi}} \left(2\pi i \operatorname{Res} \left(\frac{e^{isu}}{1+u^2}, i \right) \right)$$

$$\operatorname{Res} \left(\frac{e^{isu}}{1+u^2}, i \right) = \lim_{z \rightarrow i} \frac{1}{0!} \frac{d^0}{dz} \left(\frac{(z-i) e^{isz}}{(u+i)(u-i)} \right) = \frac{e^{-s}}{2i}$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \left(2\pi i \frac{e^{-s}}{2i} \right) = \sqrt{\frac{\pi}{2}} e^{-|s|}$$

Problema 14.3 [Identidades de Plancherel y de Parseval]

(a) Demuestre, usando transformada de Fourier, la identidad de Plancherel

$$\int_{-\infty}^{+\infty} f(y) \overline{g(y)} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(s) \overline{\hat{g}(s)} ds$$

(b) Deduzca la identidad de Parseval:

$$\int_{-\infty}^{+\infty} |f(y)|^2 dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |\hat{f}(s)|^2 ds$$

Suponga que las funciones $f, g, \hat{f}, \hat{g}$ decaen lo suficientemente rápido en infinito de modo que todas las integrales que aparezcan sean convergentes

16.1. Ejercicios

1. Encontrar las soluciones elementales en variables separadas de

$$u_t = a u_{xx} - bu, \quad 0 < x < l, \quad t > 0$$

con condiciones de borde Dirichlet (a, b constantes positivas, variables)

tenemos que

$$\begin{cases} u_t = a u_{xx} - bu \\ u(t, 0) = u(t, l) = 0 \end{cases} \quad \begin{matrix} t > 0, & 0 < x < l. \\ \forall t > 0 \end{matrix}$$

ahora: sea $u(t, x) = T(t)X(x)$, luego

$$u_t = a u_{xx} - bu \Leftrightarrow T'(t)X(x) = a T(t)X''(x) - bT(t)X(x) \quad / (T(t)X(x))^{-1}$$

$$\Leftrightarrow \frac{T'(t)}{T(t)} = a \frac{X''(x)}{X(x)} - b, \quad \text{sea } \frac{X''(x)}{X(x)} = \lambda, \quad \lambda \text{ constante, luego}$$

$$\frac{T'(t)}{T(t)} = a\lambda - b \Rightarrow T(t) = C e^{(a\lambda - b)t}, \quad C \text{ constante}$$

$X''(x) = \lambda X(x)$, luego el polinomio característico está dado por:

$$P(\xi) = \xi^2 - \lambda = (\xi - \sqrt{\lambda})(\xi + \sqrt{\lambda}), \quad \text{ahora } X(x) = D e^{\sqrt{\lambda}x} + E e^{-\sqrt{\lambda}x}$$

$$\text{luego } u(t, x) = C e^{(a\lambda - b)t} (D e^{\sqrt{\lambda}x} + E e^{-\sqrt{\lambda}x}) = e^{(a\lambda - b)t} (\tilde{D} e^{\sqrt{\lambda}x} + \tilde{E} e^{-\sqrt{\lambda}x})$$

por condiciones de borde

$$u(t, 0) = e^{(a\lambda - b)t} (\tilde{D} + \tilde{E}) = 0 \quad \text{como } e^{(a\lambda - b)t} > 0 \Rightarrow \tilde{D} = -\tilde{E}$$

$$u(t, x) = \tilde{D} e^{(a\lambda - b)t} (e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}), \quad \text{ahora}$$

$$u(t, l) = \tilde{D} e^{(a\lambda - b)t} (e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l}) = 0, \quad \text{como no queremos soluciones$$

"triviales" suponemos $\tilde{D} \neq 0$ y $e^{(a\lambda - b)t} > 0$, entonces

$$e^{\sqrt{\lambda}l} = e^{-\sqrt{\lambda}l} \Leftrightarrow e^{2\sqrt{\lambda}l} = 1 = e^{2\pi i k} \Rightarrow 2\sqrt{\lambda}l = 2\pi i k$$

$$\lambda = -\left(\frac{\pi k}{l}\right)^2, \text{ ahora}$$

$$u_k(t, x) = D_k e^{-\left(\frac{\alpha \pi^2 k^2}{l^2} + b\right)t} \left(e^{\frac{\pi k x}{l}} - e^{-\frac{\pi k x}{l}}\right)$$

$$u_k(t, x) = \alpha_k e^{-\left(\frac{\alpha \pi^2 k^2}{l^2} + b\right)t} \operatorname{sen}\left(\frac{\pi k x}{l}\right)$$

2. Resuelva el problema de Dirichlet en el círculo

$$\begin{aligned} \Delta u &= 0 & R > r \\ u(R, \theta) &= f(\theta) & \theta \in [0, 2\pi[\end{aligned}$$

Observe que al utilizar separación de variables, si $u(r, \theta) = A(r)B(\theta)$, entonces $B(0) = B(2\pi)$ y $B'(0) = B'(2\pi)$.

tenemos que si $u = u(r, \theta)$ en cilindricas, entonces $(h_r = 1, h_\theta = r, h_z = 1)$

$$\nabla u = \frac{\partial u}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{\theta}, \quad \nabla \cdot \vec{F} = \frac{1}{h_r h_\theta h_z} \left(\frac{\partial}{\partial r} (\vec{F} \cdot \hat{r} h_\theta h_z) + \frac{\partial}{\partial \theta} (\vec{F} \cdot \hat{\theta} h_r h_z) + \frac{\partial}{\partial z} (\vec{F} \cdot \hat{k} h_r h_\theta) \right)$$

$$\Delta u = \nabla \cdot (\nabla u) = \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) \right) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$\Delta u = 0 \Leftrightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \text{ si } u(r, \theta) = A(r)B(\theta)$$

$$\Leftrightarrow A''(r)B(\theta) + \frac{1}{r} A'(r)B(\theta) + \frac{1}{r^2} A(r)B''(\theta) = 0$$

$$\Leftrightarrow \left(A''(r) + \frac{1}{r} A'(r) \right) B(\theta) = -\frac{1}{r^2} A(r) B''(\theta)$$

$$\Leftrightarrow \frac{r^2 A''(r) + r A'(r)}{A(r)} = -\frac{B''(\theta)}{B(\theta)}, \text{ sea } \frac{B''(\theta)}{B(\theta)} = \lambda$$

no quedan las E.D.O's.

$$\begin{cases} r^2 A'' + r A' + \lambda A = 0 \\ B'' = \lambda B \end{cases}$$

para $r^2 A'' + r A' + \lambda A = 0$ tomamos la ecuación de Euler, decimos $r = e^u$

$$z(u) = A(e^u), \text{ luego}$$

$$z(u) = A(e^u)$$

$$z'(u) = e^u A'(e^u)$$

$$z''(u) = e^{2u} A''(e^u) + e^u A'(e^u) \Rightarrow z'' - z' = e^{2u} A''(e^u)$$

luego $r^2 A'' + r A' + \lambda A = 0 \Leftrightarrow (z'' - z') + z' + \lambda z = 0$

$$z'' + \lambda z = 0, \text{ luego el polinomio característico es } P(\xi) = \xi^2 + \lambda = (\xi + i\sqrt{\lambda})(\xi - i\sqrt{\lambda})$$

$$z(u) = C_1 e^{i\sqrt{\lambda} u} + C_2 e^{-i\sqrt{\lambda} u} \Rightarrow A(r) = C_1 r^{i\sqrt{\lambda}} + C_2 r^{-i\sqrt{\lambda}}$$

para: $B'' = \lambda B$ tomamos el polinomio característico $P(\xi) = \xi^2 - \lambda = (\xi - \sqrt{\lambda})(\xi + \sqrt{\lambda})$

$$B(\theta) = C_3 e^{\sqrt{\lambda} \theta} + C_4 e^{-\sqrt{\lambda} \theta} \text{ ahora:}$$

$$u(r, \theta) = (C_1 r^{i\sqrt{\lambda}} + C_2 r^{-i\sqrt{\lambda}})(C_3 e^{\sqrt{\lambda} \theta} + C_4 e^{-\sqrt{\lambda} \theta})$$

tenemos que:

$$\begin{cases} u(R, \theta) = f(\theta) \\ B(0) = B(2\pi) \\ B'(0) = B'(2\pi) \end{cases}$$

ahora $B'(0) = \sqrt{\lambda}(C_3 e^{\sqrt{\lambda} \theta} - C_4 e^{-\sqrt{\lambda} \theta})$, luego

$$B(0) = C_3 + C_4, \quad B(2\pi) = C_3 e^{\sqrt{\lambda} 2\pi} + C_4 e^{-\sqrt{\lambda} 2\pi}$$

$$B'(0) = \sqrt{\lambda}(C_3 - C_4), \quad B'(2\pi) = \sqrt{\lambda}(C_3 e^{\sqrt{\lambda} 2\pi} - C_4 e^{-\sqrt{\lambda} 2\pi})$$

$$(1) \quad C_3 + C_4 = C_3 e^{\sqrt{\lambda} 2\pi} + C_4 e^{-\sqrt{\lambda} 2\pi} \Rightarrow C_4 (1 - e^{-\sqrt{\lambda} 2\pi}) = C_3 (e^{\sqrt{\lambda} 2\pi} - 1)$$

$$(2) \quad C_3 - C_4 = C_3 e^{\sqrt{\lambda} 2\pi} - C_4 e^{-\sqrt{\lambda} 2\pi} \Rightarrow C_4 (e^{-\sqrt{\lambda} 2\pi} - 1) = C_3 (e^{\sqrt{\lambda} 2\pi} - 1)$$

entonces $1 - e^{-\sqrt{\lambda} 2\pi} = e^{-\sqrt{\lambda} 2\pi} - 1 \Leftrightarrow e^{-\sqrt{\lambda} 2\pi} = 1 = e^{2\pi i k}$

$$\Rightarrow -\sqrt{\lambda} 2\pi = 2\pi i k \Leftrightarrow \lambda = -k^2 \text{ ahora}$$

$$u_k(r, \theta) = (C_1 r^{-k} + C_2 r^k)(C_3 e^{ik\theta} + C_4 e^{-ik\theta})$$

ahora $u(R, \theta) = f(\theta)$ esto es equivalente a

$$(C_1 R^{-k} + C_2 R^k)(C_3 e^{ik\theta} + C_4 e^{-ik\theta}) = f(\theta)$$

$$\Rightarrow u(r, \theta) = \frac{(C_1 r^{-k} + C_2 r^k) f(\theta)}{(C_1 R^{-k} + C_2 R^k)}$$



3. Resuelva el problema de Neumann en el anillo

$$\Delta u = 0 \quad R > r$$

$$u_r(R, \theta) = f(\theta) \quad \theta \in [0, 2\pi]$$

- tenemos de manera semejante que

$$u(r, \theta) = (C_1 r^{-k} + C_2 r^k)(C_3 e^{ik\theta} + C_4 e^{-ik\theta})$$

$$\text{luego } \frac{\partial u}{\partial r}(r, \theta) = k(C_2 r^{k-1} - C_1 r^{-k-1})(C_3 e^{ik\theta} + C_4 e^{-ik\theta})$$

$$\text{luego } \frac{\partial u}{\partial r}(R, \theta) = f(\theta)$$

$$\Rightarrow k(C_2 R^{k-1} - C_1 R^{-k-1})(C_3 e^{ik\theta} + C_4 e^{-ik\theta}) = f(\theta)$$

finalmente

$$u(r, \theta) = \frac{(C_1 r^{-k} + C_2 r^k) f(\theta)}{k(C_2 R^{k-1} - C_1 R^{-k-1})}$$



4. problema

$$\Delta u = 0$$

$$r(1, \theta) = \cos(2\theta)$$

$$1 > r$$

$$\theta \in [0, 2\pi[$$

tenemos por 2. que:

$$u(r, \theta) = \frac{(C_1 r^{-k} + C_2 r^k) f(\theta)}{C_1 r^{-k} + C_2 r^k}, \text{ si } R=1, f(\theta) = \cos(2\theta), \text{ nos queda}$$

$$u(r, \theta) = \frac{(C_1 r^{-k} + C_2 r^k) \cos(2\theta)}{C_1 + C_2}$$

||

5. Encontrar soluciones de las siguientes ecuaciones separando variables.

(a) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$

sea $u(x, y) = A(x)B(y)$, ahora:

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \Leftrightarrow A'(x)B(y) - B'(y)A(x) = 0 \Leftrightarrow \frac{A'(x)}{A(x)} = \frac{B'(y)}{B(y)} = \lambda$$

ahora $\frac{A'(x)}{A(x)} = \lambda \Rightarrow A(x) = C_1 e^{\lambda x}$

$$\frac{B'(y)}{B(y)} = \lambda \Rightarrow B(y) = C_2 e^{\lambda y}, \text{ luego:}$$

$$u(x, y) = C e^{\lambda(x+y)}$$

(b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

sea $u(x, y) = A(x)B(y)$, luego:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Leftrightarrow A''(x)B(y) + A(x)B''(y) = 0 \Leftrightarrow \frac{A''(x)}{A(x)} = -\frac{B''(y)}{B(y)} = \lambda$$

ahora $A''(x) = \lambda A(x)$ tiene como polinomios característicos:

$$P(\xi) = \xi^2 - \lambda = (\xi - \sqrt{\lambda})(\xi + \sqrt{\lambda}) \Rightarrow A(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$$

para $B'(y) = -\lambda B(y)$ tiene por polinomio característico
 $P(\xi) = \xi^2 + \lambda = (\xi - i\sqrt{\lambda})(\xi + i\sqrt{\lambda}) \Rightarrow B(y) = C_3 e^{i\sqrt{\lambda}y} + C_4 e^{-i\sqrt{\lambda}y}$

luego $u(x,y) = (C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x})(C_3 e^{i\sqrt{\lambda}y} + C_4 e^{-i\sqrt{\lambda}y})$ ▣

(c) $\frac{\partial^2 u}{\partial x \partial y} - u = 0$

sea $u(x,y) = A(x)B(y)$, luego

$$A'(x)B'(y) = A(x)B(y) \Leftrightarrow \frac{A'(x)}{A(x)} = \frac{B'(y)}{B(y)} = \lambda, \text{ luego}$$

de: $\frac{A'(x)}{A(x)} = \lambda \Rightarrow A(x) = C_1 e^{\lambda x}$

de: $\frac{B'(y)}{B(y)} = \frac{1}{\lambda} \Rightarrow B(y) = C_2 e^{\frac{y}{\lambda}}, \text{ luego}$

$$u(x,y) = C e^{\lambda x + \frac{y}{\lambda}}.$$
 ▣

(d) $x^2 \frac{\partial^2 u}{\partial x \partial y} + 3y^2 x = 0$

sea $u(x,y) = A(x)B(y)$, luego

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 3y^2 x = 0 \Leftrightarrow x^2 A'(x)B'(y) + 3y^2 x A(x)B(y) = 0 \Leftrightarrow x A'(x) = \frac{3y^2}{B(y)} = \lambda$$

de $x A'(x) = \lambda \Leftrightarrow dA = \frac{\lambda dx}{x} \Rightarrow A(x) = \lambda \ln(x) + C_1$

de $B'(y) = \lambda 3y^2 \Leftrightarrow dB = 3\lambda y^2 dy \Rightarrow B(y) = \lambda y^3 + C_2$

$$u(x,y) = (\lambda \ln(x) + C_1)(\lambda y^3 + C_2)$$
 ▣

16.4. Problemas

Problema 16.1

Resuelva la ecuación de Helmholtz

$$\Delta u + u = 0$$

en la región $\Omega = \{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq 2\}$ con condiciones de borde

$$u(x, 0) = u(0, y) = u(1, y) = 0 \text{ y } u(x, 2) = u(1-x), \quad \forall (x, y) \in \Omega$$

-tenemos que $u = u(x, y)$ luego $\Delta u = u_{xx} + u_{yy}$, sea $u(x, y) = A(x)B(y)$
luego la ecuación:

$$\Delta u + u = 0 \Leftrightarrow A''(x)B(y) + A(x)B''(y) + A(x)B(y) = 0 \quad / \cdot (A(x)B(y))^{-1}$$

$$\Leftrightarrow \frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} + 1 = 0 \quad , \text{ sea } \lambda = \frac{B''(y)}{B(y)}, \text{ luego}$$

$A''(x) = -(1+\lambda)A(x)$ que tiene como polinomio característico:

$$P(\xi) = \xi^2 + (1+\lambda) = (\xi - i\sqrt{1+\lambda})(\xi + i\sqrt{1+\lambda}), \text{ luego}$$

$$A(x) = C_1 e^{i\sqrt{1+\lambda}x} + C_2 e^{-i\sqrt{1+\lambda}x}$$

analogamente de $B''(y) = \lambda B(y)$ tenemos que el polinomio característico es:

$$P(\xi) = \xi^2 - \lambda = (\xi - \sqrt{\lambda})(\xi + \sqrt{\lambda}), \text{ luego}$$

$$B(y) = C_3 e^{\sqrt{\lambda}y} + C_4 e^{-\sqrt{\lambda}y}, \text{ entonces}$$

$$u(x, y) = (C_1 e^{i\sqrt{1+\lambda}x} + C_2 e^{-i\sqrt{1+\lambda}x})(C_3 e^{\sqrt{\lambda}y} + C_4 e^{-\sqrt{\lambda}y})$$

ahora, por condiciones de borde

$$u(x, 0) = (C_1 e^{i\sqrt{1+\lambda}x} + C_2 e^{-i\sqrt{1+\lambda}x})(C_3 + C_4) = 0$$

luego $C_3 = -C_4$, metemos dicha constante a:

$$u(x, y) = (e^{\sqrt{\lambda}y} - e^{-\sqrt{\lambda}y})(D e^{i\sqrt{1+\lambda}x} + E e^{-i\sqrt{1+\lambda}x})$$

$$u(0,y) = (e^{\sqrt{\lambda}y} - e^{-\sqrt{\lambda}y})(D+E) = 0 \Rightarrow D = -E, \text{ allora}$$

$$u(x,y) = D(e^{\sqrt{\lambda}y} - e^{-\sqrt{\lambda}y})(e^{i\sqrt{1+\lambda}x} - e^{-i\sqrt{1+\lambda}x})$$

allora

$$u(1,y) = D(e^{\sqrt{\lambda}y} - e^{-\sqrt{\lambda}y})(e^{i\sqrt{1+\lambda}} - e^{-i\sqrt{1+\lambda}}) = 0$$

$$\Rightarrow e^{i\sqrt{1+\lambda}} = e^{-i\sqrt{1+\lambda}} \Leftrightarrow e^{2i\sqrt{1+\lambda}} = 1 = e^{2\pi i k}$$

$$\Rightarrow 2i\sqrt{1+\lambda} = 2\pi i k \Rightarrow 1+\lambda = (\pi k)^2 \Leftrightarrow \lambda = (\pi k)^2 - 1$$

$$u(x,y) = D_k (e^{\sqrt{\pi^2 k^2 - 1}y} - e^{-\sqrt{\pi^2 k^2 - 1}y})(e^{i\pi kx} - e^{-i\pi kx})$$

$$u_k(x,y) = \alpha_k (e^{\sqrt{\pi^2 k^2 - 1}y} - e^{-\sqrt{\pi^2 k^2 - 1}y}) \sin(\pi kx)$$

allora

$$u_k(x,z) = \alpha_k (e^{\sqrt{\pi^2 k^2 - 1}z} - e^{-\sqrt{\pi^2 k^2 - 1}z}) \sin(\pi kx) = x(1-x)$$

allora

$$u_k(x,y) = \frac{(e^{\sqrt{\pi^2 k^2 - 1}y} - e^{-\sqrt{\pi^2 k^2 - 1}y})x(1-x)}{e^{2\sqrt{\pi^2 k^2 - 1}} - e^{-2\sqrt{\pi^2 k^2 - 1}}}$$

$$u(x,y) = \sum_{k=1}^{\infty} \frac{(e^{\sqrt{\pi^2 k^2 - 1}y} - e^{-\sqrt{\pi^2 k^2 - 1}y})x(1-x)}{e^{2\sqrt{\pi^2 k^2 - 1}} - e^{-2\sqrt{\pi^2 k^2 - 1}}}$$

Problema 16.2.

Resuelva el siguiente sistema:

$$y_{tt}(x,t) = y_{xx}(x,t) - \omega(x) \quad 0 < x < 2\pi \quad t > 0$$

$$y_t(x,0) = 0 \quad 0 < x < 2\pi$$

$$y(0,t) = y(2\pi,t) = 0 \quad t > 0$$

$$y(x,0) = 0 \quad 0 < x < 2\pi$$

Indicacion: Considere soluciones del tipo $y(x,t) = Y(x,t) - h(x)$ para una funcion h adecuada

Sea $h(x) = \omega(x)$, entonces: $y(x,t) = Y(x,t) - \omega(x)$

$$y_t(x,t) = Y_t(x,t) \Leftrightarrow y_{tt}(x,t) = Y_{tt}(x,t)$$

$$y_x(x,t) = Y_x(x,t) + \sin(x) \wedge y_{xx}(x,t) = Y_{xx}(x,t) + \omega(x)$$

luego:

$$y_{tt}(x,t) = y_{xx}(x,t) - \omega(x) \Leftrightarrow Y_{tt}(x,t) = Y_{xx}(x,t)$$

Sea $Y(x,t) = X(x) T(t)$, ahora

$$Y_{tt}(x,t) = Y_{xx}(x,t) \Leftrightarrow X''(x) T(t) = X(x) T''(t)$$

$$\Leftrightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda, \text{ ahora}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$T(t) = c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}$$

$$Y(x,t) = (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t})$$

$$\Rightarrow y(x,t) = (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}) - \omega(x)$$

$$y_t(x,t) = \sqrt{\lambda} (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_3 e^{\sqrt{\lambda}t} - c_4 e^{-\sqrt{\lambda}t})$$

ahora por condiciones de borde tenemos:

$$y(0,t) = (c_1 + c_2)(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}) - \omega(0) = 0$$

$$y(2\pi,t) = (c_1 e^{2\pi\sqrt{\lambda}} + c_2 e^{-2\pi\sqrt{\lambda}})(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}) - \omega(2\pi) = 0$$

$$\Rightarrow (c_1 (e^{2\pi\sqrt{\lambda}} - 1) + c_2 (e^{-2\pi\sqrt{\lambda}} - 1))(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}) = 0$$

$$\Rightarrow e^{2\pi\sqrt{\lambda}} = 1 = e^{2\pi i k} \Rightarrow 2\pi\sqrt{\lambda} = 2\pi i k \Rightarrow \lambda = -k^2$$

$$\text{ahora } y(x,t) = (c_1 e^{ikx} + c_2 e^{-ikx})(c_3 e^{ikt} + c_4 e^{-ikt}) - \omega(x)$$

$$y_t(x,0) = ik(c_1 e^{ikx} + c_2 e^{-ikx})(c_3 - c_4) = 0 \Rightarrow c_3 = c_4$$

$$y(x,t) = (E e^{ikx} + F e^{-ikx}) \omega(kt) - \omega(x)$$

$$y(x,0) = (E e^{ikx} + F e^{-ikx}) - \omega(x) = 0$$

$$\Rightarrow y(x,t) = \omega(x)(\omega(kt) - 1)$$



Problema 16.3

Sea $u = v(\sqrt{x^2 + y^2 + z^2}, t)$ una solución con simetría "radial" de la ecuación de ondas

$$u_{tt} = c^2 \Delta u \quad \text{en } \mathbb{R}^3$$

(i) Probar que v satisface la ecuación $v_{tt} = c^2 [v_{rr} + 2v_r/r]$ para $r \geq 0, t \geq 0$
 en efecto, $u(x,y,z,t) = v(r,t)$, luego tenemos que $\Delta u = \nabla \cdot (\nabla u)$, como

$$\nabla u = \frac{1}{hr} \frac{\partial u}{\partial r} \hat{r} + \frac{1}{h\theta} \frac{\partial u}{\partial \theta} \hat{\theta} + \frac{1}{h\varphi} \frac{\partial u}{\partial \varphi} \hat{\varphi}$$

$$\nabla \cdot \vec{F} = \frac{1}{hrh\theta h\varphi} \left[\frac{\partial}{\partial r} (\vec{F} \cdot \hat{r} h\theta h\varphi) + \frac{\partial}{\partial \theta} (\vec{F} \cdot \hat{\theta} hr h\varphi) + \frac{\partial}{\partial \varphi} (\vec{F} \cdot \hat{\varphi} hr h\theta) \right]$$

luego como $u = v(r,t)$, $hr = 1$, $h\theta = r \sin \varphi$, $h\varphi = r$, se tiene

$$\nabla v = \frac{\partial v}{\partial r} \hat{r}$$

$$\nabla \cdot (\nabla v) = \frac{1}{r^2 \sin \varphi} \left[\frac{\partial}{\partial r} \left(\frac{\partial v}{\partial r} r^2 \sin \varphi \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial v}{\partial r}) = \frac{1}{r^2} \left(r^2 \frac{\partial^2 v}{\partial r^2} + 2r \frac{\partial v}{\partial r} \right)$$

$$\Delta u = \Delta v = v_{rr} + \frac{2v_r}{r} \quad \text{y} \quad u_{tt} = v_{tt}, \text{ luego:}$$

$$u_{tt} = c^2 \Delta u \Leftrightarrow v_{tt} = c^2 \left(v_{rr} + \frac{2v_r}{r} \right)$$

(ii) Sea $w(r,t) = r v(r,t)$. Probar que w satisface la ecuación de ondas unidimensional

$$\text{Chequeo si } w(r,t) = r v(r,t) \Rightarrow w_{tt} = r v_{tt}$$

$$v = \frac{w}{r} \Rightarrow \frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial w}{\partial r} - \frac{1}{r^2} w$$

$$\text{ahora } \Delta v = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} - w \right) = \frac{1}{r^2} \left(r \frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r} - \frac{\partial w}{\partial r} \right)$$

$$\Delta v = \frac{1}{r} \frac{\partial^2 w}{\partial r^2}, \text{ luego } v_{tt} = c^2 \Delta v \Leftrightarrow \frac{w_{tt}}{r} = \frac{c^2}{r} w_{rr}$$

$$\Leftrightarrow w_{tt} = c^2 w_{rr}, \text{ que es la ecuación de ondas unidimensional}$$

(iii) Encontrar w para las condiciones iniciales $u(x,y,z,0) = \exp(-r^2)$ y $u_t(x,y,z,0) = 0$ tenemos que

$$u(x,y,z,0) = v(r,0) = \exp(-r^2) \Rightarrow w(r,0) = r \exp(-r^2)$$

$$u_t(x,y,z,0) = v_t(r,0) = 0 \Rightarrow w_t(r,0) = 0, \text{ ahora:}$$

sea $w(r,t) = R(r) T(t)$, luego:

$$w_{tt} = c^2 w_{rr} \Leftrightarrow R(r) T''(t) = c^2 R''(r) T(t) \Leftrightarrow \frac{T''(t)}{T(t)} = c^2 \frac{R''(r)}{R(r)}$$

$$\text{sea } \lambda = \frac{T''(t)}{T(t)}, \text{ entonces } \frac{\lambda}{c^2} = \frac{R''(r)}{R(r)}, \text{ ahora}$$

para $T''(t) = \lambda T(t)$ tenemos el polinomio característico:

$$P(\xi) = \xi^2 - \lambda = (\xi - \sqrt{\lambda})(\xi + \sqrt{\lambda}) \Rightarrow T(t) = C_1 e^{\sqrt{\lambda}t} + C_2 e^{-\sqrt{\lambda}t}$$

para $R''(r) = \frac{\lambda}{c^2} R(r)$ tenemos el polinomio característico:

$$P(\xi) = \xi^2 - \frac{\lambda}{c^2} = \left(\xi - \frac{\sqrt{\lambda}}{c} \right) \left(\xi + \frac{\sqrt{\lambda}}{c} \right) \Rightarrow R(r) = C_3 e^{\frac{\sqrt{\lambda}}{c}r} + C_4 e^{-\frac{\sqrt{\lambda}}{c}r}$$

ahora:

$$w(r,t) = (C_1 e^{\sqrt{\lambda}t} + C_2 e^{-\sqrt{\lambda}t}) (C_3 e^{\frac{\sqrt{\lambda}}{c}r} + C_4 e^{-\frac{\sqrt{\lambda}}{c}r})$$

$$w_t(r,t) = \sqrt{\lambda} (C_1 e^{\sqrt{\lambda}t} - C_2 e^{-\sqrt{\lambda}t}) (C_3 e^{\frac{\sqrt{\lambda}}{c}r} + C_4 e^{-\frac{\sqrt{\lambda}}{c}r})$$

ahora:

$$w_t(r,0) = \sqrt{\lambda} (C_1 - C_2) (C_3 e^{\frac{\sqrt{\lambda}}{c}r} + C_4 e^{-\frac{\sqrt{\lambda}}{c}r}) = 0 \Rightarrow C_1 = C_2, \text{ ahora}$$

$$w(r,t) = (e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}) (E e^{\frac{\sqrt{\lambda}}{c}r} + F e^{-\frac{\sqrt{\lambda}}{c}r})$$

si $v(r,0) = \exp(-r^2) \wedge w(r,t) = r v(r,t)$, entonces

$$w(r,0) = \tilde{E} e^{\frac{\sqrt{\lambda}}{c}r} + \tilde{F} e^{-\frac{\sqrt{\lambda}}{c}r} = r e^{-r^2} \quad \text{evaluamos en } 0$$

$$\Rightarrow \tilde{E} + \tilde{F} = 0 \Rightarrow \tilde{E} = -\tilde{F}$$

$$w(r,0) = \tilde{E} (e^{\frac{\sqrt{\lambda}}{c}r} + e^{-\frac{\sqrt{\lambda}}{c}r}) = r e^{-r^2}$$

Problema 16.5

Resolver un caso particular del problema anterior

$$u_t - Ku_{xx} = x \quad x \in [0, \pi], t > 0$$

$$u(x, 0) = u_t(x, 0) = u(0, t) = u(\pi, t) = 0$$

sea v una función tal que $Kv_{xx} = Ku_{xx} + x \Leftrightarrow v_{xx} = u_{xx} + \frac{x}{K}$

$$\Rightarrow v_x = u_x + \frac{x^2}{2K} \Rightarrow v(x, t) = u(x, t) + \frac{x^3}{6K}, \text{ ahora:}$$

v cumple que: $v_t = Kv_{xx}$, ahora tenemos que $v(x, t) = A(x)B(t)$

$$v_t = Kv_{xx} \Leftrightarrow B'(t)A(x) = K B(t)A''(x) \Leftrightarrow \frac{B'(t)}{B(t)} = K \frac{A''(x)}{A(x)}$$

sea $\frac{B'(t)}{B(t)} = \lambda$, entonces nos quedan las EDO's:

- $B'(t) = \lambda B(t) \Rightarrow B(t) = C_1 e^{\lambda t}$
- $A''(x) = \frac{\lambda}{K} A(x) \Rightarrow A(x) = C_2 e^{\sqrt{\frac{\lambda}{K}} x} + C_3 e^{-\sqrt{\frac{\lambda}{K}} x}$

$$v(x, t) = e^{\lambda t} (E e^{\sqrt{\frac{\lambda}{K}} x} + F e^{-\sqrt{\frac{\lambda}{K}} x})$$

$$u(x, t) = e^{\lambda t} (E e^{\sqrt{\frac{\lambda}{K}} x} + F e^{-\sqrt{\frac{\lambda}{K}} x}) - \frac{x^3}{6K}$$

ahora tenemos que por condiciones de borde:

$$u(0, t) = e^{\lambda t} (E + F) = 0 \Rightarrow E = -F, \text{ ahora}$$

$$u(x, t) = E e^{\lambda t} (e^{\sqrt{\frac{\lambda}{K}} x} - e^{-\sqrt{\frac{\lambda}{K}} x}) - \frac{x^3}{6K}$$

$$u_t(x, t) = E \lambda e^{\lambda t} (e^{\sqrt{\frac{\lambda}{K}} x} - e^{-\sqrt{\frac{\lambda}{K}} x}), \text{ tenemos que}$$

$$u_t(x, 0) = \lambda E (e^{\sqrt{\frac{\lambda}{K}} x} - e^{-\sqrt{\frac{\lambda}{K}} x}) = 0$$

16.2.4. Ecuación de Laplace en una banda semi-infinita

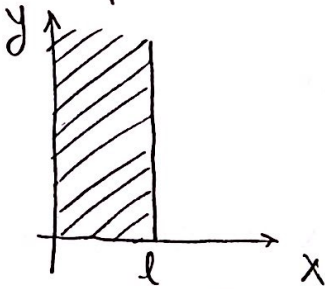
(EC) $0 = u_{xx} + u_{yy} \quad y > 0, 0 < x < l.$

(CB) $u(0, y) = u(l, y) = 0 \quad y > 0$

$u(x, 0) = f(x) \quad 0 < x < l$

$u(x, \infty) = 0 \quad 0 < x < l$

tenemos que



sea $u(x, y) = A(x)B(y)$, ahora:

$$0 = A''(x)B(y) + A(x)B''(y)$$

$$\Leftrightarrow \frac{A''(x)}{A(x)} = -\frac{B''(y)}{B(y)}, \text{ sea } \frac{B''(y)}{B(y)} = \lambda, \text{ entonces nos quedan las EDO's.}$$

• $\frac{A''(x)}{A(x)} = -\lambda \Rightarrow A(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x}$

• $\frac{B''(y)}{B(y)} = \lambda \Rightarrow B(y) = C_3 e^{\sqrt{\lambda}y} + C_4 e^{-\sqrt{\lambda}y}$

$$u(x, y) = (C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x})(C_3 e^{\sqrt{\lambda}y} + C_4 e^{-\sqrt{\lambda}y})$$

$$u(0, y) = (C_1 + C_2)(C_3 e^{\sqrt{\lambda}y} + C_4 e^{-\sqrt{\lambda}y}) = 0 \Rightarrow C_1 = -C_2$$

$$u(l, y) = C(e^{i\sqrt{\lambda}l} - e^{-i\sqrt{\lambda}l})(C_3 e^{\sqrt{\lambda}y} + C_4 e^{-\sqrt{\lambda}y}) = 0$$

$$\Rightarrow e^{i\sqrt{\lambda}l} = e^{-i\sqrt{\lambda}l} \Leftrightarrow e^{2i\sqrt{\lambda}l} = 1 = e^{2\pi i k} \Rightarrow 2i\sqrt{\lambda}l = 2\pi i k$$

$$\Rightarrow \lambda = \left(\frac{\pi k}{l}\right)^2, \text{ ahora}$$

$$u(x, y) = (e^{\frac{i\pi k}{l}x} - e^{-\frac{i\pi k}{l}x})(\tilde{C}_3 e^{\frac{\pi k y}{l}} + \tilde{C}_4 e^{-\frac{\pi k y}{l}})$$

$$= (E e^{\frac{\pi k y}{l}} + F e^{-\frac{\pi k y}{l}}) \sin\left(\frac{\pi k x}{l}\right)$$

ahora

$$u(x, \infty) = 0 \Rightarrow E = 0, \text{ luego}$$

$$u_k(x, y) = F e^{-\frac{\pi k y}{l}} \sin\left(\frac{\pi k x}{l}\right)$$

$$u(x,y) = \sum_{k=0}^{\infty} \alpha_k e^{-\frac{\pi k y}{l}} \sin\left(\frac{\pi k x}{l}\right)$$

$$u(x,0) = \sum_{k=0}^{\infty} \alpha_k \sin\left(\frac{\pi k x}{l}\right) = f(x), \text{ sea } \bar{f}(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$$

multiplicamos por $\sin\left(\frac{\pi j x}{l}\right)$ e integramos entre $-l$ y l

$$\sum_{k=0}^{\infty} \alpha_k \int_{-l}^l \sin\left(\frac{\pi k x}{l}\right) \sin\left(\frac{\pi j x}{l}\right) dx = \int_{-l}^l \bar{f}(x) \sin\left(\frac{\pi j x}{l}\right) dx$$

$$\text{pero } \int_{-l}^l \sin\left(\frac{\pi k x}{l}\right) \sin\left(\frac{\pi j x}{l}\right) dx = \begin{cases} l & \text{si } j=k \\ 0 & \text{si } j \neq k \end{cases}$$

$$\therefore l \alpha_k = \int_{-l}^l \bar{f}(x) \sin\left(\frac{\pi k x}{l}\right) dx = 2 \int_0^l f(x) \sin\left(\frac{\pi k x}{l}\right) dx$$

$$\therefore \alpha_k = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{k \pi x}{l}\right) dx, \text{ y finalmente}$$

$$u(x,t) = \sum_{k=0}^{\infty} \frac{2}{l} \int_0^l f(\tau) \sin\left(\frac{k \pi \tau}{l}\right) d\tau e^{-\frac{\pi y k}{l}} \sin\left(\frac{k \pi x}{l}\right)$$

Problema 16.2

Resuelva el siguiente sistema.

$$y_{tt}(x,t) = y_{xx}(x,t) - \cos(t) \quad 0 < x < 2\pi, t > 0$$

$$y_t(x,0) = 0 \quad 0 < x < 2\pi$$

$$y(0,t) = y(2\pi,t) = 0 \quad t > 0$$

$$y(x,0) = 0 \quad 0 < x < 2\pi$$

Indicacion: Considere soluciones del tipo $y(x,t) = Y(x,t) - h(x)$ para una función h adecuada

sea v una función, tal que

$$v_{xx} = y_{xx} - \omega(x) \Rightarrow v_x = y_x - \sin(x) \Rightarrow v = y + \cos(x)$$

luego sea $v(x,t) = y(x,t) + \omega(x)$, entonces v satisface

$$v_t = v_{xx}, \text{ luego sea } v(x,t) = A(x)B(t), \text{ no queda:}$$

$$A(x)B'(t) = A''(x)B(t) \Leftrightarrow \frac{A''(x)}{A(x)} = \frac{B'(t)}{B(t)}, \text{ sea } \frac{B'(t)}{B(t)} = \lambda$$

no queda:

$$\bullet B'(t) = \lambda B(t) \Rightarrow B(t) = C_1 e^{\lambda t}$$

$$\bullet A''(x) = \lambda A'(x) \Rightarrow A(x) = C_2 e^{\sqrt{\lambda}x} + C_3 e^{-\sqrt{\lambda}x}$$

$$\Rightarrow v(x,t) = e^{\lambda t} (C_2 e^{\sqrt{\lambda}x} + C_3 e^{-\sqrt{\lambda}x})$$

$$\Rightarrow y(x,t) = e^{\lambda t} (C_2 e^{\sqrt{\lambda}x} + C_3 e^{-\sqrt{\lambda}x}) + \omega(x)$$

ahora

$$y(0,t) = e^{\lambda t} (\tilde{C}_2 + \tilde{C}_3) + 1 = 0$$

$$y(2\pi,t) = e^{\lambda t} (\tilde{C}_2 e^{\sqrt{\lambda}\pi} + \tilde{C}_3 e^{-\sqrt{\lambda}\pi}) + 1 = 0$$

$$\Rightarrow e^{\lambda t} (\tilde{C}_2 (e^{\sqrt{\lambda}\pi} - 1) + \tilde{C}_3 (e^{-\sqrt{\lambda}\pi} - 1)) = 0$$

$$\Rightarrow e^{\sqrt{\lambda}\pi} = 1 = e^{2\pi i k} \Rightarrow \sqrt{\lambda}\pi = 2\pi i k \Rightarrow \lambda = -4k^2$$

$$y(x,t) = e^{-4k^2 t} (\tilde{C}_2 e^{2ikx} + \tilde{C}_3 e^{-2ikx}) - \omega(x)$$

$$y(x,0) = 0 \Rightarrow \tilde{C}_2 e^{2ikx} + \tilde{C}_3 e^{-2ikx} = \frac{e^{ix} + e^{-ix}}{2}$$

$$\Rightarrow \tilde{C}_2 = \tilde{C}_3, \text{ luego}$$

Problema 16.7

Resolva

$$\Delta u = 0 \quad 2 > r > 1, \pi > \theta > 0$$

$$u(1, \theta) = u(2, \theta) = u(r, 0) = 0$$

$$u_\theta(r, \pi) = r^2$$

temos que $u = u(r, \theta)$, logo $\nabla u = \frac{\partial u}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{\theta}$, $\Delta u = \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) \right)$

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

Seja $u(r, \theta) = A(r)B(\theta)$, logo

$$\Delta u = 0 \Leftrightarrow A''(r)B(\theta) + \frac{1}{r} A'(r)B(\theta) + \frac{1}{r^2} A(r)B''(\theta) = 0 \quad / \cdot (A(r)B(\theta))^{-1}$$

$$\Leftrightarrow \frac{A''(r)}{A(r)} + \frac{A'(r)}{rA(r)} + \frac{B''(\theta)}{r^2 B(\theta)} = 0 \quad / \cdot r^2$$

$$\Leftrightarrow \frac{r^2 A''(r)}{A(r)} + \frac{r A'(r)}{A(r)} + \frac{B''(\theta)}{B(\theta)} = 0, \text{ seja } \frac{B''(\theta)}{B(\theta)} = \lambda$$

entonces nos quedan las EDO's

$$\bullet B''(\theta) - \lambda B(\theta) = 0 \Rightarrow P(\xi) = \xi^2 - \lambda = (\xi - \sqrt{\lambda})(\xi + \sqrt{\lambda})$$

$$B(\theta) = C_1 e^{\sqrt{\lambda} \theta} + C_2 e^{-\sqrt{\lambda} \theta}$$

$$\bullet r^2 A''(r) + r A'(r) + \lambda A(r) = 0$$

$$\text{seja } z(u) = A(e^u), \quad z'(u) = e^u A'(e^u), \quad z''(u) = e^{2u} A''(e^u) + e^u A'(e^u)$$

$$\text{logo } r^2 A''(r) + r A'(r) + \lambda A(r) = 0 \Leftrightarrow z''(u) + \lambda z(u) = 0$$

$$\Rightarrow P(\xi) = \xi^2 + \lambda = (\xi + i\sqrt{\lambda})(\xi - i\sqrt{\lambda}) \Rightarrow z(u) = C_3 e^{i\sqrt{\lambda} u} + C_4 e^{-i\sqrt{\lambda} u}$$

$$\Rightarrow A(r) = C_3 r^{i\sqrt{\lambda}} + C_4 r^{-i\sqrt{\lambda}} \quad \text{ahora}$$

$$u(r, \theta) = (C_1 e^{\sqrt{\lambda} \theta} + C_2 e^{-\sqrt{\lambda} \theta})(C_3 r^{i\sqrt{\lambda}} + C_4 r^{-i\sqrt{\lambda}})$$

$$u_\theta(r, \theta) = \sqrt{\lambda} (C_1 e^{\sqrt{\lambda} \theta} - C_2 e^{-\sqrt{\lambda} \theta})(C_3 r^{i\sqrt{\lambda}} + C_4 r^{-i\sqrt{\lambda}})$$

Evaluamos en las condiciones de borde:

$$u(1, \theta) = (C_1 e^{\sqrt{\lambda} \theta} + C_2 e^{-\sqrt{\lambda} \theta}) (C_3 + C_4) = 0 \quad (1)$$

$$u(2, \theta) = (C_1 e^{\sqrt{\lambda} \theta} + C_2 e^{-\sqrt{\lambda} \theta}) (C_3 2^{\sqrt{\lambda}} + C_4 2^{-\sqrt{\lambda}}) = 0 \quad (2)$$

$$u(r, 0) = (C_1 + C_2) (C_3 r^{\sqrt{\lambda}} + C_4 r^{-\sqrt{\lambda}}) = 0 \quad (3)$$

Como buscamos soluciones no triviales, de (1) y (3) deducimos que $C_3 = -C_4$ y $C_1 = -C_2$, ahora de (2) decimos que

$$C_3 2^{\sqrt{\lambda}} + C_4 2^{-\sqrt{\lambda}} = 0 \quad , \text{ como } C_3 = -C_4 \text{ , entonces}$$

$$2^{\sqrt{\lambda}} = 2^{-\sqrt{\lambda}} \Leftrightarrow 2^{2\sqrt{\lambda}} = 1 \quad \text{pero } 2^{2\sqrt{\lambda}} = \exp(2\sqrt{\lambda} \ln(2))$$

$$\exp(2\sqrt{\lambda} \ln(2)) = \exp(2\pi i k) \quad , \quad k \in \mathbb{Z} \text{ , ahora}$$

$$\Rightarrow 2\sqrt{\lambda} \ln(2) = 2\pi i k \Leftrightarrow \lambda = \left(\frac{\pi k}{\ln(2)} \right)^2 \quad , \text{ sea } C = C_1 \cdot C_3$$

$$u_k(r, \theta) = C \left(e^{\frac{\pi k}{\ln(2)} \theta} - e^{-\frac{\pi k}{\ln(2)} \theta} \right) \left(r^{\frac{i\pi k}{\ln(2)}} - r^{-\frac{i\pi k}{\ln(2)}} \right)$$

$$\text{pero } e^{\frac{\pi k}{\ln(2)} \theta} - e^{-\frac{\pi k}{\ln(2)} \theta} = 2 \sinh\left(\frac{\pi k \theta}{\ln(2)}\right) \quad , \text{ adem\u00e1s: } r^{\frac{i\pi k}{\ln(2)}} = \exp\left(\frac{i\pi k \ln(r)}{\ln(2)}\right)$$

$$\text{luego } r^{\frac{i\pi k}{\ln(2)}} - r^{-\frac{i\pi k}{\ln(2)}} = \exp\left(\frac{i\pi k \ln(r)}{\ln(2)}\right) - \exp\left(-\frac{i\pi k \ln(r)}{\ln(2)}\right) = 2i \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right)$$

$$u_k(r, \theta) = \alpha_k \sinh\left(\frac{\pi k \theta}{\ln(2)}\right) \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right) \quad , \text{ finalmente}$$

$$u(r, \theta) = \sum_{k=0}^{\infty} \alpha_k \sinh\left(\frac{\pi k \theta}{\ln(2)}\right) \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right)$$

$$u_\theta(r, \theta) = \sum_{k=0}^{\infty} \frac{\alpha_k \pi k}{\ln(2)} \cosh\left(\frac{\pi k \theta}{\ln(2)}\right) \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right)$$

por condici\u00f3n de borde

$$u_\theta(r, \pi) = \sum_{k=0}^{\infty} \frac{\alpha_k \pi k}{\ln(2)} \cosh\left(\frac{\pi^2 k}{\ln(2)}\right) \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right) = r^2.$$

$$\Leftrightarrow \sum_{k=0}^{\infty} K \alpha_k \cosh\left(\frac{\pi^2 k}{\ln(2)}\right) \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right) = \frac{r^2 \ln(2)}{\pi}$$

Como $r \in L^1(1,2)$, entonces $\ln(r) \geq 0$, ahora multiplicamos a ambos lados por $\sin\left(\frac{\pi j \ln(r)}{\ln(2)}\right)$ e integramos entre 1 y 2.

$$\sum_{k=0}^{\infty} K \alpha_k \cosh\left(\frac{\pi^2 k}{\ln(2)}\right) \int_1^2 \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right) \sin\left(\frac{\pi j \ln(r)}{\ln(2)}\right) dr = \frac{\ln(2)}{\pi} \int_1^2 r^2 \sin\left(\frac{\pi j \ln(r)}{\ln(2)}\right) dr$$

$$\Rightarrow K \alpha_k \cosh\left(\frac{\pi^2 k}{\ln(2)}\right) = \frac{\ln(2)}{\pi} \int_1^2 r^2 \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right) dr.$$

$$\alpha_k = \frac{\ln(2)}{\pi k \cosh\left(\frac{\pi^2 k}{\ln(2)}\right)} \int_1^2 r^2 \sin\left(\frac{\pi k \ln(r)}{\ln(2)}\right) dr.$$

Problema 16.7

Resuelva la ecuación del calor en un cuadrado

$$u_t - K(u_{xx} + u_{yy}) = 0 \quad (x, y) \in (0, \pi)^2, t > 0$$

$$u(x, y, 0) = f(x, y)$$

$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0$$

-tenemos que, sea $u(x, y, t) = A(x)B(y)C(t)$, entonces

$$u_t - K(u_{xx} + u_{yy}) = 0 \Leftrightarrow A(x)B(y)C'(t) - K(A''(x)B(y)C(t) + A(x)B''(y)C(t)) = 0$$

$$\Leftrightarrow \frac{C'(t)}{C(t)} - K\left(\frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)}\right) = 0 \quad \text{sea } \frac{A''(x)}{A(x)} = \lambda_1, \frac{B''(y)}{B(y)} = \lambda_2$$

nos quedan las EDO's

$$\bullet A''(x) = \lambda_1 A(x) \Rightarrow A(x) = \tilde{A}_1 e^{\sqrt{\lambda_1} x} + \tilde{A}_2 e^{-\sqrt{\lambda_1} x}$$

$$\bullet B''(y) = \lambda_2 B(y) \Rightarrow B(y) = \tilde{B}_1 e^{\sqrt{\lambda_2} y} + \tilde{B}_2 e^{-\sqrt{\lambda_2} y}$$

$$\bullet C'(t) = K(\lambda_1 + \lambda_2)C(t) \Rightarrow C(t) = \tilde{C}_1 e^{K(\lambda_1 + \lambda_2)t}$$

$$\Rightarrow u(x, y, t) = e^{K(\lambda_1 + \lambda_2)t} (A_1 e^{\sqrt{\lambda_1} x} + A_2 e^{-\sqrt{\lambda_1} x}) (B_1 e^{\sqrt{\lambda_2} y} + B_2 e^{-\sqrt{\lambda_2} y})$$

ahora usamos las condiciones de borde:

$$u(0, y, t) = e^{K(\lambda_1 + \lambda_2)t} (A_1 + A_2) (B_1 e^{\sqrt{\lambda_2} y} + B_2 e^{-\sqrt{\lambda_2} y}) = 0 \Rightarrow A_1 = -A_2$$

$$u(x, 0, t) = e^{K(\lambda_1 + \lambda_2)t} (e^{\sqrt{\lambda_1} x} - e^{-\sqrt{\lambda_1} x}) (\tilde{B}_1 + \tilde{B}_2) = 0 \Rightarrow \tilde{B}_1 = -\tilde{B}_2$$

$$u(x, y, t) = D e^{K(\lambda_1 + \lambda_2)t} (e^{\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_1}x}) (e^{\sqrt{\lambda_2}y} - e^{-\sqrt{\lambda_2}y})$$

agoramos evaluando en las condiciones de borde:

$$u(\pi, y, t) = D e^{K(\lambda_1 + \lambda_2)t} (e^{\sqrt{\lambda_1}\pi} - e^{-\sqrt{\lambda_1}\pi}) (e^{\sqrt{\lambda_2}y} - e^{-\sqrt{\lambda_2}y}) = 0$$

$$\Rightarrow e^{\sqrt{\lambda_1}\pi} = e^{-\sqrt{\lambda_1}\pi} \Leftrightarrow e^{2\sqrt{\lambda_1}\pi} = 1 = e^{2\pi i k_1} \Rightarrow 2\sqrt{\lambda_1}\pi = 2\pi i k_1$$

$$\Rightarrow \lambda_1 = -k_1^2, \text{ ahora}$$

$$u(x, \pi, t) = D e^{K(-k_1^2 + \lambda_2)t} (e^{-ik_1x} - e^{ik_1x}) (e^{\sqrt{\lambda_2}\pi} - e^{-\sqrt{\lambda_2}\pi}) = 0$$

$$\Rightarrow e^{\sqrt{\lambda_2}\pi} = e^{-\sqrt{\lambda_2}\pi} \Leftrightarrow e^{2\sqrt{\lambda_2}\pi} = 1 = e^{2\pi i k_2} \Rightarrow 2\sqrt{\lambda_2}\pi = 2\pi i k_2$$

$$\Rightarrow \lambda_2 = -k_2^2, \text{ ahora}$$

$$u_{k_1 k_2}(x, y, t) = \alpha_{k_1 k_2} e^{-(k_1^2 + k_2^2)Kt} \sin(k_1 x) \sin(k_2 y)$$

ahora

$$u(x, y, t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \alpha_{k_1 k_2} e^{-(k_1^2 + k_2^2)Kt} \sin(k_1 x) \sin(k_2 y)$$

$$u(x, y, 0) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \alpha_{k_1 k_2} \sin(k_1 x) \sin(k_2 y) = f(x, y)$$

$$\text{sea } \overline{f(x, y)} = \begin{cases} f(x, y) & \text{si } x, y \geq 0 \\ -f(-x, -y) & \text{si } x, y < 0 \end{cases}$$

luego multiplicamos por $\sin(m_1 x) \sin(m_2 y)$ e integramos $x \in [-\pi, \pi]$ y $y \in [-\pi, \pi]$

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \alpha_{k_1 k_2} \int_{-\pi}^{\pi} \sin(k_1 x) \sin(m_1 x) dx \int_{-\pi}^{\pi} \sin(k_2 y) \sin(m_2 y) dy =$$

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \overline{f(x, y)} \sin(m_1 x) \sin(m_2 y) dx dy$$

pero

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sin(k_1 x) \sin(m_1 x) \sin(k_2 y) \sin(m_2 y) dy dx = \begin{cases} \pi^2 & \text{si } k_1 = m_1 \wedge k_2 = m_2 \\ 0 & \text{si } k_1 \neq m_1 \vee k_2 \neq m_2 \end{cases}$$

$$\Rightarrow \alpha_{k_1, k_2} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \sin(k_1 x) \sin(k_2 y) dx dy$$

$$= \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x, y) \sin(k_1 x) \sin(k_2 y) dx dy$$

$$\therefore u(x, y, t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(\phi, \psi) \sin(k_1 \phi) \sin(k_2 \psi) d\phi d\psi e^{-\frac{1}{2}(k_1^2 + k_2^2)t} \sin(k_1 x) \sin(k_2 y)$$

Problema 16.8

Resuelva la ecuación de calor en un cubo

$$u_t - K(u_{xx} + u_{yy} + u_{zz}) = 0 \quad (x, y, z) \in (0, \pi)^3, t > 0$$

$$u(x, y, z) = f(x, y, z)$$

$$u(0, y, z, t) = u(\pi, y, z, t) = u(x, y, 0, t) = u(x, y, \pi, t) = 0$$

$$u_y(x, 0, z, t) = u_y(x, \pi, z, t) = 0$$

sea $u(x, y, z, t) = A(x)B(y)C(z)D(t)$, luego:

$$u_t - K(u_{xx} + u_{yy} + u_{zz}) = 0 \Leftrightarrow \frac{D'(t)}{D(t)} - K \left(\frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} + \frac{C''(z)}{C(z)} \right) = 0$$

sea $\frac{A''(x)}{A(x)} = \lambda_1$, $\frac{B''(y)}{B(y)} = \lambda_2$, $\frac{C''(z)}{C(z)} = \lambda_3$, nos quedan las EDO's

$$\bullet A''(x) = \lambda_1 A(x) \Rightarrow A(x) = \tilde{A}_1 e^{\sqrt{\lambda_1} x} + \tilde{A}_2 e^{-\sqrt{\lambda_1} x}$$

$$\bullet B''(y) = \lambda_2 B(y) \Rightarrow B(y) = \tilde{B}_1 e^{\sqrt{\lambda_2} y} + \tilde{B}_2 e^{-\sqrt{\lambda_2} y}$$

$$\bullet C''(z) = \lambda_3 C(z) \Rightarrow C(z) = \tilde{C}_1 e^{\sqrt{\lambda_3} z} + \tilde{C}_2 e^{-\sqrt{\lambda_3} z}$$

$$\bullet D'(t) = K(\lambda_1 + \lambda_2 + \lambda_3) D(t) \Rightarrow D(t) = \tilde{D}_1 e^{K(\lambda_1 + \lambda_2 + \lambda_3)t}$$

$$u(x,y,z,t) = e^{K(\lambda_1 + \lambda_2 + \lambda_3)t} (A_1 e^{\sqrt{\lambda_1}x} + A_2 e^{-\sqrt{\lambda_1}x}) (B_1 e^{\sqrt{\lambda_2}y} + B_2 e^{-\sqrt{\lambda_2}y}) (C_1 e^{\sqrt{\lambda_3}z} + C_2 e^{-\sqrt{\lambda_3}z})$$

$$u(0,y,z,t) = e^{K(\lambda_1 + \lambda_2 + \lambda_3)t} (A_1 + A_2) (B_1 e^{\sqrt{\lambda_2}y} + B_2 e^{-\sqrt{\lambda_2}y}) (C_1 e^{\sqrt{\lambda_3}z} + C_2 e^{-\sqrt{\lambda_3}z}) = 0 \Rightarrow A_1 = -A_2$$

$$u(x,y,0,t) = e^{K(\lambda_1 + \lambda_2 + \lambda_3)t} (e^{\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_1}x}) (B_1 e^{\sqrt{\lambda_2}y} + B_2 e^{-\sqrt{\lambda_2}y}) (C_1 + C_2) = 0 \Rightarrow C_1 = -C_2$$

$$u(x,y,z,t) = e^{K(\lambda_1 + \lambda_2 + \lambda_3)t} (e^{\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_1}x}) (B_1 e^{\sqrt{\lambda_2}y} + B_2 e^{-\sqrt{\lambda_2}y}) (e^{\sqrt{\lambda_3}z} - e^{-\sqrt{\lambda_3}z})$$

$$u_y(x,y,z,t) = \sqrt{\lambda_2} e^{K(\lambda_1 + \lambda_2 + \lambda_3)t} (e^{\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_1}x}) (B_1 e^{\sqrt{\lambda_2}y} - B_2 e^{-\sqrt{\lambda_2}y}) (e^{\sqrt{\lambda_3}z} - e^{-\sqrt{\lambda_3}z})$$

$$u_y(x,0,z,t) = \sqrt{\lambda_2} e^{K(\lambda_1 + \lambda_2 + \lambda_3)t} (e^{\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_1}x}) (B_1 - B_2) (e^{\sqrt{\lambda_3}z} - e^{-\sqrt{\lambda_3}z}) = 0 \Rightarrow B_1 = B_2$$

$$u_y(x,0,z,t) = \tilde{B} \sqrt{\lambda_2} e^{K(\lambda_1 + \lambda_2 + \lambda_3)t} (e^{\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_1}x}) (e^{\sqrt{\lambda_2}\pi} - e^{-\sqrt{\lambda_2}\pi}) (e^{\sqrt{\lambda_3}z} - e^{-\sqrt{\lambda_3}z}) = 0$$

$$\Rightarrow e^{\sqrt{\lambda_2}\pi} = e^{-\sqrt{\lambda_2}\pi} \Leftrightarrow e^{2\sqrt{\lambda_2}\pi} = 1 = e^{2\pi i k_2} \Rightarrow 2\sqrt{\lambda_2}\pi = 2\pi i k_2 \Rightarrow \lambda_2 = -k_2^2$$

ahora

$$u(\pi,y,z,t) = \tilde{B} e^{K(\lambda_1 - k_2^2 + \lambda_3)t} (e^{\sqrt{\lambda_1}\pi} - e^{-\sqrt{\lambda_1}\pi}) (e^{i k_2 y} + e^{-i k_2 y}) (e^{\sqrt{\lambda_3}z} - e^{-\sqrt{\lambda_3}z}) = 0$$

$$\Rightarrow e^{\sqrt{\lambda_1}\pi} = e^{-\sqrt{\lambda_1}\pi} \Leftrightarrow e^{2\sqrt{\lambda_1}\pi} = 1 = e^{2\pi i k_1} \Rightarrow 2\sqrt{\lambda_1}\pi = 2\pi i k_1 \Rightarrow \lambda_1 = -k_1^2$$

$$u(x,y,\pi,t) = \tilde{B} e^{K(-k_1^2 - k_2^2 + \lambda_3)t} (e^{i k_1 x} - e^{-i k_1 x}) (e^{i k_2 y} + e^{-i k_2 y}) (e^{\sqrt{\lambda_3}\pi} - e^{-\sqrt{\lambda_3}\pi}) = 0$$

$$\Rightarrow e^{\sqrt{\lambda_3}\pi} = e^{-\sqrt{\lambda_3}\pi} \Leftrightarrow e^{2\sqrt{\lambda_3}\pi} = 1 = e^{2\pi i k_3} \Rightarrow 2\sqrt{\lambda_3}\pi = 2\pi i k_3 \Rightarrow \lambda_3 = -k_3^2$$

$$u_k(x,y,z,t) = \alpha_k e^{-(k_1^2 + k_2^2 + k_3^2)Kt} \sin(k_1 x) \sin(k_2 y) \sin(k_3 z)$$

$$u(x,y,z,t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \alpha_{k_1 k_2 k_3} e^{-(k_1^2 + k_2^2 + k_3^2)Kt} \sin(k_1 x) \sin(k_2 y) \sin(k_3 z)$$

$$\alpha_{k_1 k_2 k_3} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi f(\phi, \theta, \varphi) \sin(k_1 \phi) \sin(k_2 \theta) \sin(k_3 \varphi) d\varphi d\theta d\phi$$

□

Problema 16.9

Los gases uniformemente distribuidos en un volumen $\Omega \subset \mathbb{R}^3$ se encuentran a temperaturas $u(x,y,z,t)$ y $v(x,y,z,t)$, satisfaciendo las ecuaciones

$$(1) \quad u_t = c^2 \Delta u - k(u-v)$$

$$(2) \quad v_t = a^2 \Delta v - l(v-u)$$

donde $k > 0$ y $l > 0$ son constantes relacionadas con el intercambio de calor entre los gases. La superficie $\partial\Omega$ es adiabática ($\nabla u \cdot \hat{n} = \nabla v \cdot \hat{n} = 0$ en $\partial\Omega$) y en el instante inicial ($t=0$) las temperaturas u, v son constantes en todo Ω e iguales a u_0, v_0 respectivamente.

(a) Sea $U(t) = \iiint_{\Omega} u dx dy dz$ y $V(t) = \iiint_{\Omega} v dx dy dz$. Probar que $\dot{U} = -k(U-V)$ y análogamente $\dot{V} = -l(V-U)$

tenemos que

$$U(t) = \iiint_{\Omega} u(x,y,z,t) dx dy dz \Rightarrow \dot{U}(t) = \frac{\partial U(t)}{\partial t} = \iiint_{\Omega} \frac{\partial u}{\partial t}(x,y,z,t) dx dy dz$$

pero: $u_t = c^2 \Delta u - k(u-v)$, luego

$$\dot{U}(t) = \iiint_{\Omega} c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx dy dz - k \iiint_{\Omega} (u-v) dx dy dz$$

pero $\iiint_{\Omega} (u-v) dx dy dz = U - V$, ahora

$$\iiint_{\Omega} c^2 \Delta u dx dy dz = \iint_{\partial\Omega} c^2 \nabla u \cdot \hat{n} dS = 0$$

por el teorema de Gauss, y la condición de borde adiabática luego:

$$\dot{U}(t) = -k(U(t) - V(t)).$$

identícamente para

$$V(t) = \iiint_{\Omega} v(x,y,z,t) dx dy dz \Rightarrow \dot{V}(t) = \frac{\partial V(t)}{\partial t} = \iiint_{\Omega} \frac{\partial v}{\partial t}(x,y,z,t) dx dy dz$$

pero $v_t = a^2 \Delta v - l(v-u)$, ahora

$$\dot{V}(t) = \iiint_{\Omega} a^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) dx dy dz - l \iiint_{\Omega} (v-u) dx dy dz$$

pero $\iiint_{\Omega} (v-u) dx dy dz = V - U$, ahora

$$\iiint_{\Omega} a^2 \Delta v dx dy dz = \iint_{\partial\Omega} a^2 \nabla v \cdot \hat{n} dS = 0 \quad \text{condición de borde adiabática.}$$

$$\dot{V}(t) = -l(V(t) - U(t)).$$

□

(b) Deducir que la cantidad $K(t) = lU(t) + kV(t)$ se conserva y probar que

$$\lim_{t \rightarrow +\infty} U(t) = \lim_{t \rightarrow +\infty} V(t) = \frac{l u_0 + k v_0}{k+l} \mu(\Omega).$$

-tenemos de (a) que

$$\begin{array}{l} (*) \quad \dot{U} = -k(U-V) \\ (**) \quad \dot{V} = -l(V-U) \end{array} \quad \left. \begin{array}{l} \text{multiplicamos } (*) \text{ por } l \text{ y } (**) \text{ por } k \text{ y sumamos, no queda} \\ l\dot{U} + k\dot{V} = 0 \Leftrightarrow \frac{d}{dt} (lU(t) + kV(t)) = 0 \end{array} \right\}$$

$$\Rightarrow lU(t) + kV(t) = \text{cte} \equiv K(t) \quad , \text{ ahora en } t=0$$

$$U(0) = \iiint_{\Omega} u_0 \, dx \, dy \, dz = u_0 \mu(\Omega) \quad \text{con } \mu(\Omega) \text{ el volumen de } \Omega$$

$$V(0) = \iiint_{\Omega} v_0 \, dx \, dy \, dz = v_0 \mu(\Omega)$$

$$\Rightarrow lU(t) + kV(t) = (lu_0 + kv_0)\mu(\Omega)$$

en $t \rightarrow +\infty$, tenemos que $\dot{U}, \dot{V} \rightarrow 0$, luego $U=V$, y se tiene

$$\lim_{t \rightarrow +\infty} U(t) = \lim_{t \rightarrow +\infty} V(t) = \frac{lu_0 + kv_0}{l+k} \mu(\Omega) \quad (\text{equilibrio térmico})$$

(c) Sea $w: \Omega \rightarrow \mathbb{R}$ solución de la ecuación auxiliar

$$\begin{cases} \Delta w = \alpha w & \text{en } \Omega \\ \nabla w \cdot \hat{n} = 0 & \text{en } \partial\Omega \end{cases}$$

$$(c.1) \text{ Probar la identidad } \iint_{\partial\Omega} w \nabla w \cdot d\vec{S} = \iiint_{\Omega} \|\nabla w\|^2 + \alpha \iiint_{\Omega} w^2.$$

calculamos:

$$\nabla \cdot (w \nabla w) = \nabla \cdot \left(w \frac{\partial w}{\partial x} \hat{i} + w \frac{\partial w}{\partial y} \hat{j} + w \frac{\partial w}{\partial z} \hat{k} \right)$$

$$= \frac{\partial}{\partial x} \left(w \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(w \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left(w \frac{\partial w}{\partial z} \right)$$

$$= \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 + w \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

$$\begin{aligned} (w \nabla w) &= \|\nabla w\|^2 + w \Delta w \\ &= \|\nabla w\|^2 + \alpha w^2 \end{aligned}$$

, pero $\Delta w = \alpha w$ en Ω , luego
ahora

$$\begin{aligned} \iint_{\partial\Omega} w \nabla w \cdot d\vec{S} &= \iiint_{\Omega} \nabla \cdot (w \nabla w) dV \quad \text{, por el teorema de la divergencia} \\ &= \iiint_{\Omega} \|\nabla w\|^2 dV + \alpha \iiint_{\Omega} w^2 dV. \quad \square \end{aligned}$$

(c2) Deducan que " $\alpha > 0 \Rightarrow w \equiv 0$ " y " $\alpha = 0 \Rightarrow w \equiv \text{constante}$ "

en efecto tenemos que por condición de borde: $\nabla w \cdot \hat{n} = 0$, luego

$$\iint_{\partial\Omega} w \nabla w \cdot d\vec{S} = \iint_{\partial\Omega} w \nabla w \cdot \hat{n} dS = 0 \quad \text{, ahora en (C1) digamos que:}$$

$$\iint_{\partial\Omega} w \nabla w \cdot d\vec{S} = \iiint_{\Omega} \|\nabla w\|^2 dV + \alpha \iiint_{\Omega} w^2 dV = 0$$

• luego si $\alpha > 0$, entonces

$$\iiint_{\Omega} (\|\nabla w\|^2 + \alpha w^2) dV = 0 \Rightarrow \|\nabla w\|^2 + \alpha w^2 = 0$$

pero $\|\nabla w\| \geq 0$ y $w^2 \geq 0$ entonces si $\alpha > 0 \Rightarrow w \wedge \nabla w = 0$

$$\therefore w \equiv 0$$

• ahora si $\alpha = 0$, entonces

$$\iiint_{\Omega} \|\nabla w\|^2 dV = 0 \Rightarrow \nabla w = 0 \Rightarrow w = \text{constante} \quad \square$$

(d) Considere ahora las ecuaciones (1) y (2) en régimen estacionario y sean $\bar{u}$, $\bar{v}$ las soluciones. Probar que $\bar{u} - \bar{v} \equiv 0$ y luego que $\bar{u} \equiv c_1 \equiv \bar{v}$

en efecto tenemos que

$$c^2 \Delta \bar{u} = k(\bar{u} - \bar{v})$$

$$\alpha^2 \Delta \bar{v} = \lambda(\bar{v} - \bar{u})$$

ahora tenemos que

$$c^2 \alpha^2 (\Delta \bar{u} - \Delta \bar{v}) = (k \alpha^2 + \lambda c^2) (\bar{u} - \bar{v})$$

luego por la linealidad de la derivada $\Delta \bar{u} - \Delta \bar{v} = \Delta(\bar{u} - \bar{v})$ ahora:
 $\Delta(\bar{u} - \bar{v}) = \frac{k\alpha^2 + lc^2}{a^2c^2} (\bar{u} - \bar{v})$, sea $\alpha = \frac{k\alpha^2 + lc^2}{a^2c^2}$, $w = \bar{u} - \bar{v}$

Entonces nos queda:

$\Delta w = \alpha w$, además

$\nabla w \cdot \hat{n} = \nabla(\bar{u} - \bar{v}) \cdot \hat{n} = \nabla \bar{u} \cdot \hat{n} - \nabla \bar{v} \cdot \hat{n} = 0$ por condiciones de borde

∴ w es tal que $\begin{cases} \Delta w = \alpha w \\ \nabla w \cdot \hat{n} = 0 \end{cases}$

entonces por (c2) si $\alpha = \frac{k\alpha^2 + lc^2}{a^2c^2} > 0 \Rightarrow w = 0 \Rightarrow \bar{u} - \bar{v} \equiv 0$

$\Rightarrow \bar{u} = \bar{v}$, ahora nos queda

$\begin{cases} (1) & c^2 \Delta \bar{u} = 0 \\ (2) & a^2 \Delta \bar{v} = 0 \end{cases}$

porque $\bar{u} - \bar{v} = \bar{v} - \bar{u} = 0$, ahora esto equivale a:

$\Delta w = \alpha w$ con $\alpha = 0 \Rightarrow w = \text{constante}$, luego

$\bar{u} = \bar{v} \equiv cte.$

□

(e) Calcular, usando la parte (b) la temperatura en el region esta como

$U = \iiint_{\Omega} \bar{u} dx dy dz = \bar{u} \mu(\Omega)$, $V = \iiint_{\Omega} \bar{v} dx dy dz = \bar{v} \mu(\Omega)$

ahora

$K(t) = LU(t) + KV(t) = L\bar{u} \mu(\Omega) + K\bar{v} \mu(\Omega) = (L\bar{u} + K\bar{v}) \mu(\Omega)$

$\bar{u} = u_0$ y $\bar{v} = v_0$

□

12/02/2017
 a) Determine la serie de Fourier de la función $f(x) = 1 - |x|$ en el intervalo $[-2, 2]$ y deduzca el valor de la serie de números reales:

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

tenemos que $f(x) = 1 - |x|$, $S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right)$
 para $l = 2$, ahora

$$a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \int_0^2 (1-x) dx + \frac{1}{2} \int_{-2}^0 (1+x) dx$$

$$= \frac{1}{2} \left(x - \frac{x^2}{2} \right) \Big|_0^2 + \frac{1}{2} \left(x + \frac{x^2}{2} \right) \Big|_{-2}^0 = 0$$

$$a_k = \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{k\pi x}{2}\right) dx = \frac{1}{2} \int_0^2 (1-x) \cos\left(\frac{k\pi x}{2}\right) dx + \frac{1}{2} \int_{-2}^0 (1+u) \cos\left(\frac{k\pi u}{2}\right) du$$

$$= \frac{1}{2} \int_0^2 (1-x) \cos\left(\frac{k\pi x}{2}\right) dx - \frac{1}{2} \int_0^{-2} (1+u) \cos\left(\frac{k\pi u}{2}\right) du \quad \text{sea } u = -x$$

$$= \frac{1}{2} \int_0^2 \left\{ (1-x) \cos\left(\frac{k\pi x}{2}\right) + (1-x) \cos\left(\frac{k\pi x}{2}\right) \right\} dx \quad \text{paridad del coseno}$$

$$= \int_0^2 (1-x) \cos\left(\frac{k\pi x}{2}\right) dx = \left(\frac{2 \sin\left(\frac{k\pi x}{2}\right)}{k\pi} \right) \Big|_0^2 - \int_0^2 x \cos\left(\frac{k\pi x}{2}\right) dx$$

$$= \left(\frac{-2x}{k\pi} \sin\left(\frac{k\pi x}{2}\right) \right) \Big|_0^2 + \frac{2}{k\pi} \int_0^2 \sin\left(\frac{k\pi x}{2}\right) dx$$

sea $u = x$ $dv = \cos\left(\frac{k\pi x}{2}\right) dx$
 $du = dx$ $v = \frac{2 \sin\left(\frac{k\pi x}{2}\right)}{k\pi}$

$$= \left(\frac{2}{k\pi} \right)^2 \left(-\cos\left(\frac{k\pi x}{2}\right) \right) \Big|_0^2 = \left(\frac{2}{k\pi} \right)^2 (1 - (-1)^k) = \begin{cases} \frac{8}{k^2 \pi^2} & \text{si } k \text{ impar} \\ 0 & \text{si } k \text{ par} \end{cases}$$

$$b_k = \frac{1}{2} \int_{-2}^2 f(x) \sin\left(\frac{k\pi x}{2}\right) dx$$

$$= \frac{1}{2} \int_0^2 (1-x) \sin\left(\frac{k\pi x}{2}\right) dx - \frac{1}{2} \int_0^{-2} (1+u) \sin\left(\frac{k\pi u}{2}\right) du \quad \text{sea } u = -x$$

$$du = -dx$$

$$b_k = \frac{1}{2} \int_0^2 \left\{ (1-x) \sin\left(\frac{k\pi x}{2}\right) - (1-x) \sin\left(\frac{k\pi x}{2}\right) \right\} dx = 0$$

Imparidad del seno

$$\therefore S_f(x) = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi x}{2}\right)$$

luego como $f(x)$ es continua en todo $\mathbb{R}$, además por derivadas por la derecha y por la izquierda de $x=1$, $x=-1$, ahora

$$f(x) = S_f(x), \text{ evaluamos } f(0) = 1 = S_f(0)$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

(ii) Pasando el lado derecho a la condición inicial. Resuelva el siguiente problema de valores en derivadas parciales

$$y_{tt} - y_{xx} = 4 \sin(2x), \quad 0 < x < \pi, \quad t > 0.$$

$$y(0, t) = y(\pi, t) = 0, \quad t > 0.$$

$$y(x, 0) = 0, \quad 0 < x < \pi$$

$$y_t(x, 0) = \pi - x, \quad 0 < x < \pi$$

Indicación: Considere el cambio de variables $u(x, t) = y(x, t) - 2 \sin(2x)$

-Consideramos el cambio propuesto y nos queda

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u(0, t) = u(\pi, t) = 0 & t > 0 \end{cases}$$

$$\begin{cases} u(x, 0) = y(x, 0) - 2 \sin(2x) = -2 \sin(2x) & 0 < x < \pi \\ u_t(x, 0) = \pi - x & 0 < x < \pi \end{cases}$$

Sea $u(x, t) = A(x)B(t)$, nos queda:

$$u_{tt} = u_{xx} \Leftrightarrow A''(x)B(t) = A(x)B''(t) \Leftrightarrow \frac{A''(x)}{A(x)} = \frac{B''(t)}{B(t)} = \lambda.$$

nos quedan los E.D.O.'s.

$$A''(x) = \lambda A(x) \Rightarrow A(x) = A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x}$$

$$B''(t) = \lambda B(t) \Rightarrow B(t) = B_1 e^{\sqrt{\lambda}t} + B_2 e^{-\sqrt{\lambda}t}$$

$$u(x,t) = (A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x})(B_1 e^{\sqrt{\lambda}t} + B_2 e^{-\sqrt{\lambda}t}) - 2 \sin(2x)$$

$$u(0,t) = (A_1 + A_2)(B_1 e^{\sqrt{\lambda}t} + B_2 e^{-\sqrt{\lambda}t}) = 0 \Rightarrow A_1 = -A_2$$

$$u(\pi,t) = (e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi})(\tilde{B}_1 e^{\sqrt{\lambda}t} + \tilde{B}_2 e^{-\sqrt{\lambda}t}) = 0$$

$$\Rightarrow e^{\sqrt{\lambda}\pi} = e^{-\sqrt{\lambda}\pi} \Leftrightarrow e^{2\sqrt{\lambda}\pi} = 1 = e^{2\pi i k} \Rightarrow 2\pi\sqrt{\lambda} = 2\pi i k \Rightarrow \lambda = -k^2$$

$$u(x,t) = (C_1 e^{ikt} + C_2 e^{-ikt}) \sin(kx) - 2 \sin(2x)$$

$$\Rightarrow y(x,t) = (C_1 e^{ikt} + C_2 e^{-ikt}) \sin(kx)$$

por condiciones iniciales se tiene que

$$y(x,0) = (C_1 + C_2) \sin(kx) = 0 \Rightarrow C_1 = -C_2$$

$$y_k(x,t) = \alpha_k \sin(kx) \sin(kt), \text{ ahora por superposici3n}$$

$$u(x,t) = \sum_k \alpha_k \sin(kx) \sin(kt)$$

$$y_t(x,t) = \sum_{k=0}^{\infty} k \alpha_k \sin(kx) \cos(kt)$$

$$y_t(x,0) = \sum_{k=0}^{\infty} k \alpha_k \sin(kx) = \pi - x \quad \text{, extendamos de manera par } x \text{ a } [-\pi, \pi], \text{ si } f(x) = \pi - x$$

$$\tilde{f}(x) = \begin{cases} f(x) = \pi - x & \text{si } x > 0 \\ -f(-x) = -(\pi + x) & \text{si } x < 0 \end{cases}$$

$$\sum_{k=-\infty}^{\infty} k \alpha_k \sin(kx) = \tilde{f}(x) \quad \text{, multiplicamos por } \sin(\ell x) \text{ e integramos entre } [-\pi, \pi]$$

$$\sum_{k=-\infty}^{\infty} k \alpha_k \int_{-\pi}^{\pi} \sin(kx) \sin(\ell x) dx = \int_{-\pi}^{\pi} \tilde{f}(x) \sin(\ell x) dx$$

pero $\int_{-\pi}^{\pi} \sin(kx) \sin(lx) dx = \begin{cases} 0 & \text{si } l \neq k \\ \pi & \text{si } l = k \end{cases}$, ahora

$$K\pi\alpha_k = \int_{-\pi}^{\pi} \tilde{f}(x) \sin(kx) dx = 2 \int_0^{\pi} f(x) \sin(kx) dx$$

$$\begin{aligned} \alpha_k &= -\frac{2}{K\pi} \int_0^{\pi} (\pi-x) \sin(kx) dx \\ &= \frac{2}{K} \left(-\cos(kx) \right) \Big|_0^{\pi} + \frac{2}{K\pi} \int_0^{\pi} x \sin(kx) dx, \text{ sea } \begin{matrix} u=x & du=dx \\ dv=\sin(kx) & \\ v=-\frac{\cos(kx)}{k} \end{matrix} \\ &= \frac{2}{K} (1 - (-1)^k) + \left(\frac{2x \cos(kx)}{K\pi} \right) \Big|_0^{\pi} - \frac{2}{K\pi} \int_0^{\pi} \cos(kx) dx \\ &= \frac{2}{K} (1 - (-1)^k) + \frac{2(-1)^k}{K} = \frac{2}{K} \left(1 - (-1)^k + \frac{(-1)^k}{K} \right) \end{aligned}$$

$$y(x,t) = \sum_{k=0}^{\infty} \frac{2(K(1-(-1)^k) + (-1)^k)}{K^2} \sin(Kx) \sin(Kt)$$

Problema 16.16:

1. Desarrolle la serie de Fourier de senos para la función $f(x) = x$ en $(0, \pi)$
 tenemos que $f(x) = x$ es impar, luego extendemos el intervalo a $(-\pi, \pi)$
 ahora:

$$S_f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{l}\right) \text{ con } l=\pi, \text{ ahora:}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx, \text{ sea } \begin{matrix} u=x & du=dx \\ dv=\sin(kx) & \\ v=-\frac{\cos(kx)}{k} \end{matrix}$$

$$b_k = \frac{1}{\pi} \left(\left(\frac{-x \cos(kx)}{k} \right) \Big|_{-\pi}^{\pi} + \frac{1}{k} \int_{-\pi}^{\pi} \cos(kx) dx \right)$$

$$= \frac{1}{\pi} \left(\frac{-\pi(-1)^k - \pi(-1)^k}{k} + \left(\frac{\sin(kx)}{k^2} \right) \Big|_{-\pi}^{\pi} \right)$$

$$= \frac{-2(-1)^k}{k}, \text{ ahora } f(x) = -2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(kx) \quad , x \in [-\pi, \pi]$$

1. Halla: $f(x) = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(kx)$, $x \in [0, \pi]$

2. Resolver el sistema de la ecuación del calor dado por

$$\begin{cases} u_t - u_{xx} + 6u_x = 0, & x \in (0, \pi), t > 0 \\ u(0, t) = u(\pi, t) = 0 & t > 0 \\ u(x, 0) = e^{3x} & x \in (0, \pi) \end{cases}$$

siguiendo las siguientes indicaciones:

a) Aplicando el método de separación de variables e imponiendo las condiciones de borde, demuestre que las soluciones elementales son de la forma

$$u_k = C_k e^{(-k^2 - 9)t + 3x} \sin(kx).$$

- tenemos que, sea $u(x, t) = A(x)B(t)$, entonces

$$u_t - u_{xx} + 6u_x = 0 \Leftrightarrow A(x)B'(t) - A''(x)B(t) + 6A'(x)B(t) = 0 \quad / \cdot (A(x)B(t))^{-1}$$

$$\Leftrightarrow \frac{B'(t)}{B(t)} = \frac{A''(x) - 6A'(x)}{A(x)}, \text{ sea } \frac{B'(t)}{B(t)} = \lambda, \text{ nos quedan las}$$

EDO's.

$$\bullet B'(t) = \lambda B(t) \Rightarrow B(t) = B_0 e^{\lambda t}$$

$$\bullet A''(x) - 6A'(x) - \lambda A(x) = 0, \text{ usamos polinomios característicos}$$

$$P(\xi) = \xi^2 - 6\xi - \lambda = \left(\xi + \frac{-6 + \sqrt{36 + 4\lambda}}{2}\right) \left(\xi + \frac{-6 - \sqrt{36 + 4\lambda}}{2}\right)$$

$$= (\xi - 3 + \sqrt{9 + \lambda})(\xi - 3 - \sqrt{9 + \lambda})$$

$$A(x) = A_1 \exp(-(-3 + \sqrt{9 + \lambda})x) + A_2 \exp(-(-3 - \sqrt{9 + \lambda})x)$$

$$u(x, t) = e^{\lambda t} (C_1 \exp(-(-3 + \sqrt{9 + \lambda})x) + C_2 \exp(-(-3 - \sqrt{9 + \lambda})x))$$

$$u(0, t) = e^{\lambda t} (C_1 + C_2) = 0 \Rightarrow C_1 = -C_2$$

$$u(\pi, t) = C e^{\lambda t} (\exp(-(-3 + \sqrt{9 + \lambda})\pi) - \exp(-(-3 - \sqrt{9 + \lambda})\pi)) = 0$$

$$\Rightarrow \exp(-(-3 + \sqrt{9 + \lambda})\pi) = \exp(-(-3 - \sqrt{9 + \lambda})\pi)$$

$$\Rightarrow \exp((-3 + \sqrt{9 + \lambda}) - (-3 - \sqrt{9 + \lambda}))\pi = 1 = \exp(2\pi i k)$$

$$\Rightarrow 2\sqrt{9 + \lambda}\pi = 2\pi i k \Rightarrow \lambda = -k^2 - 9, \text{ luego}$$

$$u_k = C e^{(k^2 - 9)t} (\exp(-(3 + ki)x) - \exp(-(3 - ki)x))$$

$$= C_k e^{(-k^2 - 9)t + 3x} \sin(kx)$$

b) Imponiendo la condición inicial y usando la parte (a), encuentre la solución del sistema

Capítulo 17.

17.4 Problemas

Problema 17.1

La mezcla de dos fluidos en un tubo infinitamente largo puede modelarse a través de la ecuación de convección-difusión: $u_t = u_x + \alpha u_{xx}$, con condición inicial: $u(x, 0) = f(x)$.

(i) Probar que la transformada de Fourier de $\hat{u}(s, t) = \widehat{u(\cdot, t)}(s)$ es igual a:

$$\hat{u}(s, t) = \hat{f}(s) \exp(is - \alpha s^2)t$$

tenemos la EDP:

$$\begin{cases} u_t = u_x + \alpha u_{xx} \\ u(x, 0) = f(x) \end{cases}, \quad x \in \mathbb{R} \quad t > 0$$

$$\forall x \in \mathbb{R}$$

luego usamos transformada de Fourier con respecto a la variable x

$$\widehat{u_t} = \widehat{u_x + \alpha u_{xx}} \Leftrightarrow \frac{\partial \hat{u}}{\partial t}(s, t) = is \hat{u}(s, t) + \alpha (is)^2 \hat{u}(s, t)$$

$$\Leftrightarrow \frac{\partial \hat{u}}{\partial t}(s, t) = (is - \alpha s^2) \hat{u}(s, t) \Leftrightarrow \frac{d\hat{u}}{\hat{u}}(s, t) = (is - \alpha s^2) dt$$

$$\Rightarrow \hat{u}(s, t) = A(s) \exp((is - \alpha s^2)t), \text{ pero tenemos que por}$$

$$\text{condición inicial: } u(x, 0) = f(x) \Leftrightarrow \hat{u}(s, 0) = \hat{f}(s), \text{ ahora:}$$

$$\hat{u}(s, 0) = A(s) = \hat{f}(s), \text{ luego:}$$

$$\hat{u}(s, t) = \hat{f}(s) \exp((is - \alpha s^2)t).$$

(ii) Deducir que la solución puede expresarse como:

$$u(x, t) = \int_{-\infty}^{+\infty} f(y) G(x-y, t) dy$$

en efecto, tenemos que

$$\hat{u}(s, t) = \hat{f}(s) \exp((is - \alpha s^2)t), \text{ tenemos que } \hat{f}(s) \hat{g}(s) = \frac{1}{\sqrt{2\pi}} \widehat{(f * g)}(x)$$

$$\Leftrightarrow \hat{u}(s, t) = \hat{f}(s) \hat{g}(s, t)$$

$$\text{sea } \hat{g}(s, t) = \exp((is - \alpha s^2)t)$$

$$\Rightarrow u(x, t) = \frac{1}{\sqrt{2\pi}} f * g(x, t) = \int_{-\infty}^{+\infty} f(x-y) \frac{1}{\sqrt{2\pi}} g(y) dy, \text{ luego}$$

sea $G(x,t) = \frac{1}{\sqrt{2\pi}} g(x,t) = \frac{1}{\sqrt{2\pi}} T^{-1}(\exp((is - \alpha s^2)t))(x,t)$

$$u(x,t) = \int_{-\infty}^{+\infty} f(y) G(x-y, t) dy.$$

(iii) Determinar el nucleo de Green $G(x,t)$

$$G(x,t) = \frac{1}{\sqrt{2\pi}} T^{-1}(\exp((is - \alpha s^2)t))(x,t)$$

calculamos

$$T^{-1}(\exp((is - \alpha s^2)t))(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{isx} e^{ist - \alpha s^2 t} ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{is(x+t) - \alpha s^2 t} ds$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y) e^{is(x-y+t) - \alpha s^2 t} ds dy.$$

Problema 17.2

Resolver por transformada de Fourier:

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} & -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) & -\infty < x < \infty \\ u_t(x,0) = 0 & -\infty < x < \infty \end{cases}$$

- hacemos transformada sobre el eje x

$$\widehat{u_{tt}} = \alpha^2 \widehat{u_{xx}} \Leftrightarrow \frac{\partial^2 \hat{u}(s,t)}{\partial t^2} = \alpha^2 (is)^2 \hat{u}(s,t)$$

$$\Leftrightarrow \frac{\partial^2 \hat{u}(s,t)}{\partial t^2} + \alpha^2 s^2 \hat{u}(s,t) = 0 \Rightarrow \hat{u}(s,t) = A(s) \cos(\alpha |s| t) + B(s) \sin(\alpha |s| t)$$

por condiciones de borde $u_t(x,0) = 0 \Rightarrow \hat{u}_t(s,0) = 0$

$$\Rightarrow \hat{u}_t(s,0) = \alpha |s| (B(s) \cos(0) - A(s) \sin(0)) = 0 \Rightarrow B(s) = 0$$

$$\hat{u}(s,t) = A(s) \cos(\alpha |s| t)$$

ahora la otra condición $u(x,0) = f(x) \Rightarrow \hat{u}(s,0) = \hat{f}(s)$

$$\hat{u}(s,0) = A(s) = \hat{f}(s) \Rightarrow \hat{u}(s,t) = \hat{f}(s) \cos(\alpha |s| t)$$

ahora el núcleo de Green viene dado por:

$$G(x, y) = \frac{1}{\sqrt{2\pi}} T^{-1}(\cos(alsit))(x)$$

Problema 17.3

Se desea determinar la evolución de la temperatura $u(t, x)$ en una barra semi-infinita cuando hay transferencia de calor al medio circundante según el modelo.

$$u_t = Ku_{xx} - \gamma u, \quad t > 0, \quad 0 < x < \infty$$

donde $K > 0$ es el coeficiente de difusividad térmica y $\gamma > 0$ es el coeficiente de transferencia de calor. Suponga que la barra está anclada en $x=0$, esto es:

$$u_x(t, 0) = 0, \quad t > 0$$

y que la distribución inicial de temperaturas está dada por una función $f(x)$, es decir

$$u(0, x) = f(x), \quad 0 < x < +\infty$$

Puede que u se puede expresar como $u(t, x) = [f(\cdot) * G(t, \cdot)](x)$ para una función $G(t, x)$, la cual se puede determinar usando el método de la transformada de Fourier.

Indicacum: extienda apropiadamente el problema a todo $-\infty < x < \infty$

-Tendremos que como $u_x(t, 0) = 0$, entonces hacemos la expansión par del problema es decir

$$v(t, x) = \begin{cases} u(t, x) & \text{si } x > 0 \\ u(t, -x) & \text{si } x < 0 \end{cases}, \quad \overline{f(x)} = \begin{cases} f(x) & \text{si } x > 0 \\ f(-x) & \text{si } x < 0 \end{cases}$$

ahora tenemos que

$$\begin{cases} v_t = Kv_{xx} - \gamma v & t > 0, \quad -\infty < x < +\infty \\ v(0, x) = \overline{f(x)} & -\infty < x < \infty \end{cases}$$

ahora usamos la transformada de Fourier con respecto a x

$$\widehat{v_t} = \widehat{Kv_{xx} - \gamma v} \Leftrightarrow \frac{\partial \hat{v}}{\partial t}(t, s) = K(is)^2 \hat{v}(t, s) - \gamma \hat{v}(t, s)$$

$$\Leftrightarrow \frac{\partial \hat{v}}{\partial t}(t, s) = -(Ks^2 + \gamma) \hat{v}(t, s) \Rightarrow \hat{v}(t, s) = A(s) e^{-(Ks^2 + \gamma)t}$$

ahora por condiciones iniciales tenemos $v(0, x) = \overline{f(x)} \Rightarrow \hat{v}(0, s) = \hat{\overline{f(s)}}$

luego $\hat{v}(0, s) = A(s) = \hat{\overline{f(s)}} \Rightarrow \hat{v}(t, s) = \hat{\overline{f(s)}} e^{-(ks^2 + \gamma)t}$

$$v(t, x) = \overline{f(x)} * \left(\frac{1}{\sqrt{2\pi}} T^{-1}(e^{-(ks^2 + \gamma)t}) \right)(x) = \overline{f(x)} * G(x, t)$$

$$\text{con } G(x, t) = \frac{1}{\sqrt{2\pi}} T^{-1}(e^{-(ks^2 + \gamma)t}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(ks^2 + \gamma)t} e^{isx} ds$$

$$\begin{aligned} v(t, x) &= \int_{-\infty}^{+\infty} \overline{f(y)} G(x-y, t) dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f(y)} e^{is(x-y) - (ks^2 + \gamma)t} ds dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f(y)} e^{is(x-y) - (ks^2 + \gamma)t} ds dy + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f(-y)} e^{is(x+y) - (ks^2 + \gamma)t} ds dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f(y)} e^{isx - ks^2 t} ds dy + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f(y)} e^{isx - ks^2 t} ds dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f(y)} e^{isx - ks^2 t} (e^{is y} + e^{-is y}) ds dy \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f(y)} e^{isx - ks^2 t} \cos(sy) ds dy \end{aligned}$$

Problema 17.4

Para $h > 0$ constante, considere la ecuación:

$$u_t = u_{xx} - hu \quad , \quad -\infty < x < \infty, t > 0$$

con condiciones iniciales

$$u(x, 0) = f(x) \quad -\infty < x < \infty$$

Use el método de la transformada de Fourier suponiendo que las transformadas existen para demostrar que:

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x - th + 2y\sqrt{t}) \exp(-y^2) dy$$

Indicación: Pruebe que $\int_{-\infty}^{+\infty} \exp((\alpha + is)^2) ds = \sqrt{\pi}$ para todo $\alpha \in \mathbb{R}$

para probar por medio que

$$\int_{-\infty}^{+\infty} \exp((\alpha + is)^2) ds = \int_{-\infty}^{+\infty} \exp(-(i(\frac{\alpha}{i} + s))^2) ds = \int_{-\infty}^{+\infty} \exp(-(s - \alpha i)^2) ds$$

sea $s - \alpha i = u$, $ds = du$, luego $s \rightarrow +\infty \Rightarrow u \rightarrow +\infty$, entonces

$$\int_{-\infty}^{+\infty} \exp((\alpha + is)^2) ds = \int_{-\infty}^{+\infty} \exp(-u^2) du = \sqrt{\pi}$$

ahora aplicamos transformada de Fourier a la variable x en: $u_t = u_{xx} - hu_x$

$$\widehat{u}_t = \widehat{u_{xx} - hu_x} \Leftrightarrow \frac{\partial \widehat{u}(s,t)}{\partial t} = (is)^2 \widehat{u}(s,t) - h(is) \widehat{u}(s,t)$$

$$\Leftrightarrow \frac{\partial \widehat{u}}{\partial t}(s,t) = ((is)^2 - h(is)) \widehat{u}(s,t), \text{ ahora}$$

$$\Rightarrow \widehat{u}(s,t) = A(s) \exp(((is)^2 - h(is))t)$$

ahora $u(x,0) = f(x) \Rightarrow \widehat{u}(s,0) = \widehat{f}(s)$, luego

$$\widehat{u}(s,0) = A(s) = \widehat{f}(s) \Rightarrow \widehat{u}(s,t) = \widehat{f}(s) \exp((is)^2 - h(is)t)$$

sea $G(x,t) = \frac{1}{\sqrt{2\pi}} T^{-1}(\exp((is)^2 - h(is)t))(x)$, calculamos

$$T^{-1}(\exp((is)^2 - h(is)t))(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(((is)^2 - h(is)t + isx)) ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp((is\sqrt{t})^2 - 2is\sqrt{t}(\frac{ht-x}{2\sqrt{t}}) + (\frac{ht-x}{2\sqrt{t}})^2 - (\frac{ht-x}{2\sqrt{t}})^2) ds$$

$$= \frac{1}{\sqrt{2\pi}} \exp(-(\frac{ht-x}{2\sqrt{t}})^2) \int_{-\infty}^{+\infty} \exp((is\sqrt{t} - \frac{ht-x}{2\sqrt{t}})^2) ds, \text{ sea } y = s\sqrt{t}$$

$$dy = \sqrt{t} ds$$

$$= \frac{1}{\sqrt{2\pi t}} \exp(-(\frac{ht-x}{2\sqrt{t}})^2) \int_{-\infty}^{+\infty} \exp((iy - \frac{ht-x}{2\sqrt{t}})^2) dy = \frac{1}{\sqrt{2t}} \exp(-(\frac{ht-x}{2\sqrt{t}})^2)$$

$$\text{luego } G(x,t) = \frac{1}{\sqrt{2\pi t}} \exp(-(\frac{ht-x}{2\sqrt{t}})^2)$$

$$u(x,t) = f(x) * G(x,t) = \int_{-\infty}^{+\infty} f(x-y) G(y,t) dy$$

$$= \frac{1}{\sqrt{2\sqrt{\pi t}}} \int_{-\infty}^{+\infty} f(x-y) \exp\left(-\left(\frac{y}{\sqrt{2\sqrt{t}}}\right)^2\right) dy \quad \text{sea } z = \frac{y - ht}{\sqrt{2\sqrt{t}}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x - ht - 2z\sqrt{t}) \exp(-z^2) dz$$

$$dz = \frac{dy}{\sqrt{2\sqrt{t}}}$$

□

Problema 17.5

(a) Sea $f(x) = (\mathcal{F}^{-1} \varphi)(x)$, $x \in \mathbb{R}$, la omnitransformada de Fourier de una función $\varphi = \varphi(s)$, $s \in \mathbb{R}$. Muestre que $f(x-x_0) = \mathcal{F}^{-1}[e^{-isx_0} \varphi(s)](x)$. Deduzca que

$$\mathcal{F}^{-1}[\varphi(s) \cos(sx_0)](x) = \frac{1}{2}[f(x-x_0) + f(x+x_0)]$$

tenemos que:

$$f(x) = (\mathcal{F}^{-1} \varphi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(s) e^{isx} ds, \text{ ahora}$$

$$\begin{aligned} f(x-x_0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(s) e^{is(x-x_0)} ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(s) e^{-isx_0} e^{isx} ds \\ &= \mathcal{F}^{-1}[e^{-isx_0} \varphi(s)](x) \end{aligned}$$

□

ahora

$$\begin{aligned} \mathcal{F}^{-1}[\varphi(s) \cos(sx_0)](x) &= \mathcal{F}^{-1}\left[\frac{\varphi(s)}{2} (e^{isx_0} + e^{-isx_0})\right](x) \\ &= \frac{1}{2} (\mathcal{F}^{-1}[e^{isx_0} \varphi(s)](x) + \mathcal{F}^{-1}[e^{-isx_0} \varphi(s)](x)) \\ &= \frac{1}{2} (f(x+x_0) + f(x-x_0)) \end{aligned}$$

□

(b) Comprobar la ecuación de ondas

$$(E0) \begin{cases} u_{tt} = 16u_{xx} & x \in \mathbb{R}; t > 0 \\ u(0, x) = \frac{1}{x^2 - 9} & x \in \mathbb{R} \\ u_t(0, x) = 0 & x \in \mathbb{R} \end{cases}$$

(b.i) Probar que la transformada de Fourier de $u(t, \cdot)$ está dada por

$$\hat{u}(t, s) = -\frac{1}{3} \sqrt{\frac{\pi}{2}} \sin(3|s|) \cos(4st)$$

tenemos que, aplicando transformada de Fourier con respecto a la variable x en (E0), nos queda

$$\begin{aligned} \widehat{u_{tt}} &= 16 \widehat{u_{xx}} \Leftrightarrow \frac{\partial^2}{\partial t^2} \hat{u}(t, s) = 16(is)^2 \hat{u}(t, s) \\ \Leftrightarrow \frac{\partial^2}{\partial t^2} \hat{u}(t, s) &= -16s^2 \hat{u}(t, s) \Rightarrow \hat{u}(t, s) = A(s) \cos(4st) + B(s) \sin(4st) \end{aligned}$$

$$\Rightarrow \hat{u}_t(t, s) = 4s B(s) \cos(4st) - 4s A(s) \sin(4st)$$

ahora por condición inicial, $u_t(0, x) = 0 \Rightarrow \hat{u}_t(0, s) = 0$

$$\Rightarrow \hat{u}_t(0, s) = 4s B(s) = 0 \Rightarrow B(s) \equiv 0, \text{ nos queda}$$

$$\hat{u}(t, s) = A(s) \cos(4st)$$

ahora nuevamente $u(0, x) = \frac{1}{x^2 - 9} \Rightarrow \hat{u}(0, s) = \left(\frac{1}{x^2 - 9} \right) (s)$, calculamos

$$\left(\frac{1}{x^2 - 9} \right) (s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-isx}}{(x-3)(x+3)} dx$$

- si $s > 0$, tenemos los residuos de la parte inferior del plano complejo

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-isx}}{(x-3)(x+3)} dx = \frac{1}{\sqrt{2\pi}} \left(-\pi i \left(\text{Res} \left(\frac{e^{-isx}}{x^2 - 9}, -3 \right) + \text{Res} \left(\frac{e^{-isx}}{x^2 - 9}, 3 \right) \right) \right)$$

calculamos

$$\text{Res} \left(\frac{e^{-isx}}{x^2 - 9}, -3 \right) = \lim_{z \rightarrow -3} \frac{e^{-isz} (z+3)}{(z+3)(z-3)} = -\frac{1}{6} e^{3is}$$

$$\text{Res} \left(\frac{e^{-isx}}{x^2 - 9}, 3 \right) = \lim_{z \rightarrow 3} \frac{e^{-isz} (z-3)}{(z+3)(z-3)} = \frac{1}{6} e^{-3is}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-isx}}{x^2-9} dx = \sqrt{\frac{\pi}{2}} \frac{1}{6} (e^{3is} - e^{-3is}) = -\frac{1}{3} \sqrt{\frac{\pi}{2}} \sin(3s) \quad , s > 0$$

para $s < 0$, tenemos $-s > 0$, luego son los residuos sobre el plano complejo positivo

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{i|s|x}}{x^2-9} dx = \frac{1}{\sqrt{2\pi}} \left(\pi i \left(\operatorname{Res}\left(\frac{e^{i|s|x}}{x^2-9}, 3\right) + \operatorname{Res}\left(\frac{e^{i|s|x}}{x^2-9}, -3\right) \right) \right)$$

$$\operatorname{Res}\left(\frac{e^{i|s|x}}{x^2-9}, 3\right) = \lim_{z \rightarrow 3} \frac{e^{i|s|z}}{z+3} = \frac{1}{6} e^{3i|s|}$$

$$\operatorname{Res}\left(\frac{e^{i|s|x}}{x^2-9}, -3\right) = \lim_{z \rightarrow -3} \frac{e^{i|s|z}}{z-3} = -\frac{1}{6} e^{-3i|s|} \quad , \text{ahora:}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-isx}}{x^2-9} dx = -\frac{1}{3} \sqrt{\frac{\pi}{2}} \sin(3|s|) \quad , s < 0$$

$$\therefore \forall s \in \mathbb{R}, \quad \widehat{\left(\frac{1}{x^2-9}\right)}(s) = -\frac{1}{3} \sqrt{\frac{\pi}{2}} \sin(3|s|), \text{ formalmente.}$$

$$\hat{u}(0, s) = A(s) = -\frac{1}{3} \sqrt{\frac{\pi}{2}} \sin(3|s|)$$

$$\Rightarrow \hat{u}(t, s) = -\frac{1}{3} \sqrt{\frac{\pi}{2}} \sin(3|s|) \cos(4st)$$

□

(b.ii) Utilizando (a), calcule explícitamente la solución $u(t, x)$ de (E0)

identificamos $f(x) = \frac{1}{x^2-9}$, $\varphi(s) = -\frac{1}{3} \sqrt{\frac{\pi}{2}} \sin(3|s|)$, luego

$$(\mathcal{F}^{-1}\varphi)(x) = f(x), \text{ ahora}$$

$$u(t, x) = \mathcal{F}^{-1}[\varphi(s) \cos(4st)](x) = \frac{1}{2} (f(x-4t) + f(x+4t))$$

$$u(t, x) = \frac{1}{2} \left[\frac{1}{(x-4t)^2-9} + \frac{1}{(x+4t)^2-9} \right]$$

Calcule la serie de Fourier en $[-\pi, \pi]$ de la función

$$f(x) = \begin{cases} 2 & -\pi < x < 0 \\ 6 & 0 \leq x < \pi \end{cases}$$

Señale el límite en cada punto de $[-\pi, \pi]$ y justifique. Deduzca que:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

tenemos que

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right), \quad \begin{cases} a_k = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{k\pi x}{l}\right) dx \\ b_k = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{k\pi x}{l}\right) dx \end{cases}$$

en este caso $l = \pi$, y tenemos que:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left(2 \int_{-\pi}^0 dx + 6 \int_0^{\pi} dx \right) = 8$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx = \frac{1}{\pi} \left(2 \int_{-\pi}^0 \cos(kx) dx + 6 \int_0^{\pi} \cos(kx) dx \right)$$

$$= \frac{1}{\pi} \left(2 \left(\frac{\sin(kx)}{k} \right) \Big|_{-\pi}^0 + 6 \left(\frac{\sin(kx)}{k} \right) \Big|_0^{\pi} \right) = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \frac{1}{\pi} \left(2 \int_{-\pi}^0 \sin(kx) dx + 6 \int_0^{\pi} \sin(kx) dx \right)$$

$$= -\frac{1}{\pi} \left(2 \left(\frac{\cos(kx)}{k} \right) \Big|_{-\pi}^0 + 6 \left(\frac{\cos(kx)}{k} \right) \Big|_0^{\pi} \right) = -\frac{1}{\pi k} (2(1 - (-1)^k) + 6((-1)^k - 1))$$

$$= \frac{4(1 - (-1)^k)}{\pi k} = \begin{cases} 0 & \text{si } k \text{ par} \\ \frac{8}{\pi k} & \text{si } k \text{ impar} \end{cases}$$

$$S(x) = 4 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{2k-1}$$

Como f es continua por trozos en $[-\pi, \pi]$ y tiene derivada por la derecha de $-\pi$ y por la izquierda de π , entonces $S(x) = f(x)$, ahora

$$f\left(\frac{\pi}{2}\right) = 6 = S\left(\frac{\pi}{2}\right)$$

$$6 = 4 + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\frac{\pi}{2})}{2k-1} \Leftrightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

Problema 17.6

(a) Demuestre que la transformada de Fourier de la función $f(x) = \frac{x}{(x^2+1)^2}$ viene dada por

$$\hat{f}(s) = -\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|}$$

Indicacion: Recuerde que $F\left(\frac{1}{x^2+1}\right)(s) = \sqrt{\frac{\pi}{2}} e^{-|s|}$

tenemos que:

$$\left(\frac{1}{x^2+1}\right)' = \frac{-2x}{(x^2+1)^2} = -2f(x) \Leftrightarrow f(x) = -\frac{1}{2} \left(\frac{1}{1+x^2}\right)'$$

$$\begin{aligned} \therefore \hat{f}(s) &= \widehat{-\frac{1}{2} \left(\frac{1}{1+x^2}\right)'}(s) = -\frac{1}{2} (is) \widehat{\left(\frac{1}{1+x^2}\right)}(s) \\ &= -\frac{1}{2} is \sqrt{\frac{\pi}{2}} e^{-|s|} = -\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|} \end{aligned}$$

(b) Las vibraciones $u = u(t, x)$ de una varilla infinita satisfacen la ecuación de elasticidad siguiente

$$u_{tt} + \alpha^2 u_{xxxx} = 0 \quad t > 0, -\infty < x < \infty$$

Suponga que $\int_{-\infty}^{\infty} |u(t, x)| dx < \infty$ para $t > 0$, y que la varilla inicialmente se encuentra en reposo (i.e. $u_t(0, x) = 0$ para $-\infty < x < \infty$) en la posición $u(0, x) = \frac{x}{(x^2+1)^2}$.

(1) Demuestre que la transformada de Fourier con respecto a x de la solución es:

$$\hat{u}(t, s) = -\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|} \cos(\alpha s^2 t)$$

tenemos que, aplicando transformada de Fourier con respecto a la variable x en:

$$u_{tt} + \alpha^2 u_{xxxx} = 0 \Rightarrow \frac{\partial^2 \hat{u}(t, s)}{\partial t^2} + \alpha^2 (is)^4 \hat{u}(t, s) = 0$$

$$\Leftrightarrow \frac{\partial^2 \hat{u}(t, s)}{\partial t^2} + \alpha^2 s^4 \hat{u}(t, s) = 0$$

$$\Rightarrow \hat{u}(t, s) = A(s) \sin(\alpha s^2 t) + B(s) \cos(\alpha s^2 t)$$

$$u(t,s) = \alpha s^2 A(s) \cos(\alpha s^2 t) - \alpha s^3 B(s) \sin(\alpha s^2 t)$$

luego por condición inicial se tiene que $u_t(0,x) = 0 \Rightarrow \hat{u}_t(0,s) = 0$, ahora:

$$\hat{u}_t(0,s) = \alpha s^2 A(s) = 0 \Rightarrow A(s) = 0, \text{ y nos queda}$$

$\hat{u}(t,s) = B(s) \cos(\alpha s^2 t)$, ahora nuevamente por condición inicial, tenemos que $u(0,x) = f(x) \Rightarrow \hat{u}(0,s) = \hat{f}(s) = -\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|}$, por parte (a), luego

$$\hat{u}(0,s) = B(s) = -\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|}, \text{ finalmente}$$

$$\hat{u}(t,s) = -\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|} \cos(\alpha s^2 t)$$

(ii) Concluya que la solución corresponde a:

$$u(t,x) = \frac{1}{2} \int_0^{\infty} s e^{-s} \sin(sx) \cos(\alpha s^2 t) ds$$

tenemos que

$$u(t,x) = \mathcal{F}^{-1} \left(-\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|} \cos(\alpha s^2 t) \right) (x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(-\frac{i}{2} \sqrt{\frac{\pi}{2}} s e^{-|s|} \cos(\alpha s^2 t) \right) e^{isx} ds, \text{ pero } -\frac{i}{2} = \frac{1}{2i}$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{2i} s e^{-|s|} \cos(\alpha s^2 t) e^{isx} ds$$

$$= \frac{1}{2} \int_0^{+\infty} \frac{1}{2i} s e^{-s} \cos(\alpha s^2 t) e^{isx} ds + \frac{1}{2} \int_{-\infty}^0 \frac{1}{2i} u e^u \cos(\alpha u^2 t) e^{iux} du$$

$$= \frac{1}{2} \int_0^{+\infty} \frac{1}{2i} s e^{-s} \cos(\alpha s^2 t) e^{isx} ds + \frac{1}{2} \int_{+\infty}^0 \frac{1}{2i} (-s) e^{-s} \cos(\alpha s^2 t) e^{-isx} (-ds)$$

aquí haremos un cambio de variable en la segunda integral, $u = -s, du = -ds$

$$u(t,x) = \frac{1}{2} \int_0^{+\infty} s e^{-s} \cos(\alpha s^2 t) \frac{e^{isx} - e^{-isx}}{2i} ds = \frac{1}{2} \int_0^{+\infty} s e^{-s} \cos(\alpha s^2 t) \sin(sx) ds$$

Problema 17.7

Resuelva el problema diferencial.

$$u_{xx}(x,y) + u_{yy}(x,y) - hu(x,y) = 0, \quad 0 < x < \infty, \quad 0 < y < 1$$

donde $h > 0$, con condiciones de borde dadas por

$$u_x(0,y) = 0 \quad 0 < y < 1$$

$$y \quad u(x,0) = 0, \quad u(x,1) = f(x) \quad 0 < x < \infty$$

donde $f(x) = \begin{cases} 1 & 0 < x < c \\ 0 & x \geq c \end{cases}$, con $c > 0$.

- tenemos el problema.

$$\begin{cases} u_{xx} + u_{yy} - hu = 0, & 0 < x < +\infty, \quad 0 < y < 1 \\ u_x(0,y) = 0, & 0 < y < 1 \\ u(x,0) = 0, & 0 < x, +\infty \\ u(x,1) = f(x), & 0 < x, +\infty \end{cases}$$

Como $u_x(0,y) = 0$, entonces vale la pena hacer una extensión par de u , esta es:

$$v(x,y) = \begin{cases} u(x,y) & x > 0 \\ u(-x,y) & x < 0 \end{cases}$$

además de una extensión par de f :

$$\tilde{f}(x) = \begin{cases} f(x) & \text{si } x > 0 \\ f(-x) & \text{si } x < 0 \end{cases}$$

ahora nos queda el problema

$$\begin{cases} v_{xx} + v_{yy} - hv = 0 & , x \in \mathbb{R}, \quad 0 < y < 1 \\ v(x,0) = 0 & , x \in \mathbb{R} \\ v(x,1) = \tilde{f}(x) & , x \in \mathbb{R} \end{cases}$$

hacemos transformada de Fourier con respecto a la variable x , y nos queda

$$v_{xx} + v_{yy} - hv = 0 \Leftrightarrow (s)^2 \hat{v}(s,y) + \frac{\partial^2}{\partial y^2} \hat{v}(s,y) - h \hat{v}(s,y) = 0$$

$$\Leftrightarrow \frac{\partial^2}{\partial y^2} \hat{v}(s,y) - (h+s^2) \hat{v}(s,y) = 0$$

$$\hat{v}(s, y) = A(s) \exp(\sqrt{h+s^2} y) + B(s) \exp(-\sqrt{h+s^2} y)$$

ahora por condiciones de borde tenemos que $v(x, 0) = 0 \Rightarrow \hat{v}(s, 0) = 0$

$$\Rightarrow \hat{v}(s, 0) = A(s) + B(s) = 0 \Rightarrow A(s) = -B(s), \text{ como } h > 0, s^2 > 0$$

entonces $\sqrt{h+s^2} \in \mathbb{R}, \forall s \in \mathbb{R}$ luego tenemos un 2 a la constante y nos queda

$$\hat{v}(s, y) = A(s) \sinh(\sqrt{h+s^2} y), \text{ ahora, nuevamente por condiciones de borde tenemos que: } v(x, 1) = f(x) \Rightarrow \hat{v}(s, 1) = \hat{f}(s)$$

Calculamos

$$\begin{aligned} \hat{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(x) e^{-isx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(x) e^{-isx} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(-u) e^{-is(-u)} du \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(x) e^{-isx} dx - \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(x) e^{isx} dx \quad \left(\begin{array}{l} \text{sea } u = -x \\ du = -dx \end{array} \right) \\ &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos(sx) dx = \sqrt{\frac{2}{\pi}} \int_0^c \cos(sx) dx \\ &= \frac{1}{s} \sqrt{\frac{2}{\pi}} (\sin(sx)) \Big|_0^c = \sqrt{\frac{2}{\pi}} \frac{\sin(sc)}{s} \end{aligned}$$

$$\text{ahora } \hat{v}(s, 1) = A(s) \sinh(\sqrt{h+s^2}) = \sqrt{\frac{2}{\pi}} \frac{\sin(sc)}{s}$$

$$\Rightarrow A(s) = \sqrt{\frac{2}{\pi}} \frac{\sin(sc)}{s \sinh(\sqrt{h+s^2})}$$

$$\hat{v}(s, y) = \sqrt{\frac{2}{\pi}} \frac{\sin(sc) \sinh(\sqrt{h+s^2} y)}{s \sinh(\sqrt{h+s^2})}$$

$$\text{sea } G(x, y) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}(\sinh(\sqrt{h+s^2} y))(x)$$

$$v(x, y) = \int_{-\infty}^{+\infty} f(s) G(x-s, y) ds$$

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Problema 17.8

(a) Sea $f(x) = \exp(-a|x|)$. Pruebe que $\hat{f}(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$

$$\begin{aligned}\hat{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-isx} e^{-a|x|} dx = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-(is+a)x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{(a-is)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{-e^{-(is+a)x}}{is+a} \right) \Big|_0^{+\infty} + \frac{1}{\sqrt{2\pi}} \left(\frac{e^{(a-is)x}}{a-is} \right) \Big|_{-\infty}^0 \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a+is} + \frac{1}{a-is} \right) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + s^2} = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}\end{aligned}$$

(b) Considere la ecuación de ondas

$$(E_0) \begin{cases} u_{tt} = 16u_{xx} & x \in \mathbb{R}; t > 0 \\ u(0, x) = \exp(-|x|) & x \in \mathbb{R} \\ u_t(0, x) = 0 & x \in \mathbb{R} \end{cases}$$

tenemos que, aplicando transformada de Fourier en la variable x .

$$\begin{aligned}\widehat{u_{tt}} = 16\widehat{u_{xx}} &\Leftrightarrow \frac{\partial^2}{\partial t^2} \hat{u}(t, s) = 16(is)^2 \hat{u}(t, s) \\ \Leftrightarrow \frac{\partial^2 \hat{u}(t, s)}{\partial t^2} &= -16s^2 \hat{u}(t, s) \Rightarrow \hat{u}(t, s) = A(s) \cos(4|s|t) + B(s) \sin(4|s|t)\end{aligned}$$

$$\hat{u}_t(t, s) = -4|s| A(s) \sin(4|s|t) + 4|s| B(s) \cos(4|s|t)$$

$$\hat{u}_t(0, s) = 4|s| B(s) = 0 \Rightarrow B(s) = 0, \text{ luego}$$

$$\hat{u}(t, s) = A(s) \cos(4|s|t), \text{ por condición de borde } u(0, x) = \exp(-|x|)$$

$$\hat{u}(t, s) = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2} \cos(4|s|t) \quad \hat{u}(0, s) = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$$

(c) Calcule la solución $u(t, x)$ de (E_0) . Indicación: puede usar el teorema de residuos de Cauchy, o bien propiedades de la Transformada de Fourier

por periodicidad del coseno: $\cos(4|s|t) = \cos(4st) \quad \forall s \in \mathbb{R}$, luego

$$\hat{u}(t, s) = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2} \cos(4st) = \mathcal{F}^{-1}(\exp(-|x|))(s) \cos(4st)$$

$$= \frac{1}{2} \mathcal{F}^{-1}(\exp(-|x+4t|) + \exp(-|x-4t|))$$

$$u(t, x) = \frac{1}{2} (e^{-|x+4t|} + e^{-|x-4t|})$$