PAUTA AUXILIAR 10

P1 $f(z) = \dfrac{1}{z^2+2z}$ es holomorfa salvo en $z=0$ y $z=-2$, como -2 no está en Γ, sólo nos preocupamos de $z=0$.

- $z=0$ es polo simple?

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{1}{z+2} = \frac{1}{2} \neq 0$$

$\therefore$ 0 es polo simple y $R(f,0) = \frac{1}{2}$

$$\therefore \oint_\Gamma f(z)\,dz = 2\pi i \cdot Res(f,0) = \frac{2\pi i}{2} = \pi i \;//$$

P2 $f(z) = -\dfrac{z}{2}\,\dfrac{sen(\pi z^2) + \cos(\pi z^2)}{(z-\frac{1}{2})(z-2)}$ es holomorfa salvo en $z=\frac{1}{2}$ y $z=2$ pero sólo $z=\frac{1}{2}$ está en $\partial D(0,1)$

en la pauta de la aux 9 está este desarrollo

- $z=\frac{1}{2}$ es polo simple?

$$\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2}) f(z) = \lim_{z\to\frac{1}{2}} -\frac{z}{2}\,\frac{sen(\pi z^2)+\cos(\pi z^2)}{z-2}$$

$$= -\frac{1}{4}\,\frac{sen(\pi/4)+\cos(\pi/4)}{\frac{-3}{2}} = \frac{\sqrt{2}}{6} \neq 0$$

$\therefore$ $1/2$ es polo simple y $Res(f,\frac{1}{2}) = \frac{\sqrt{2}}{6}$

$$\therefore \oint_{\partial D(0,1)} f(z)\,dz = 2\pi i \cdot Res(f,1/2) = 2\pi i \cdot \frac{\sqrt{2}}{6} = \frac{\pi i\sqrt{2}}{3} \;//$$

P3 $I_1 = \displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{5}+\cos\theta}$ $\qquad I_2 = \displaystyle\int_{\partial D(0,1)} \frac{2}{i(z^2+2\sqrt{5}z+1)}\,dz$

I_2) $\gamma(\theta) = e^{i\theta}$ $\quad \theta\in[0,2\pi]$ $\Rightarrow$ $I_2 = 2\displaystyle\int_0^{2\pi} \frac{\cancel{i}\, e^{i\theta}}{\cancel{i}(e^{2i\theta}+2\sqrt{5}e^{i\theta}+1)}\,d\theta$

$\gamma'(\theta) = ie^{i\theta}$

$$= 2\int_0^{2\pi} \frac{d\theta}{e^{-i\theta}(e^{2i\theta}+1+2\sqrt{5}e^{i\theta})} = 2\int_0^{2\pi}\frac{d\theta}{\underbrace{e^{i\theta}+e^{-i\theta}}+2\sqrt{5}}$$

$$= 2\int_0^{2\pi}\frac{d\theta}{2\cos\theta+2\sqrt{5}} = \int_0^{2\pi}\frac{d\theta}{\cos\theta+\sqrt{5}} = I_1 \;//$$

$z+\bar{z} = 2Re(z) = 2Re(e^{i\theta})$
$= 2Re(\cos\theta + i\,sen\theta)$
$= 2\cos\theta$

Entonces basta calcular I_2 para obtener I_1

$f(z) := \dfrac{2}{i(z^2+2\sqrt{5}z+1)}$ es holomorfa salvo en $z = \dfrac{-2\sqrt{5} \pm \sqrt{20-4}}{2}$

$= -\sqrt{5} \pm 2$

Pero $-\sqrt{5}-2$ no está en $\partial D(0,1)$, por lo que sólo nos preocupamos de $-\sqrt{5}+2$

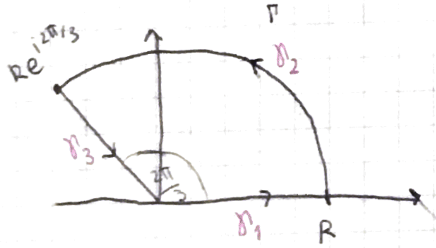
- $-\sqrt{5}+2$ es polo simple?

$$\lim_{z \to -\sqrt{5}+2} \frac{2}{i(z-(-\sqrt{5}-2))} = \frac{2}{i(4)} = \frac{1}{2i} \neq 0$$

$\therefore$ $-\sqrt{5}+2$ es polo simple y $\mathrm{Res}(f, -\sqrt{5}+2) = \dfrac{1}{2i}$

$$\therefore \oint_{\partial D(0,1)} f(z) = 2\pi i \ \mathrm{Res}(f, -\sqrt{5}+2) = \pi = I_1 \ //.$$

P4) Calculemos $\displaystyle\oint_{\Gamma} f(z)dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$



Notemos primero que $f(z)$ es holomorfa salvo cuando $z^3 = -8$

$$(Re^{i\theta})^3 = 8e^{i(\pi+2k\pi)}$$

$$R^3 e^{i3\theta} = 8e^{i(\pi+2k\pi)}$$

$R = \sqrt[3]{8} = 2$

$\theta = \dfrac{\pi + 2k\pi}{3}$ $\quad k \in \{0,1,2\}$

$z_1 = 2e^{i\frac{\pi}{3}}$ $\quad z_2 = 2e^{i\pi} = -2$ $\quad z_3 = 2e^{-i\frac{\pi}{3}}$

Si asumimos $R > 2$, sólo z_1 está en Γ

- $2e^{i\frac{\pi}{3}} = 2(\cos(\pi/3) + i\,\mathrm{sen}(\pi/3)) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1+i\sqrt{3}$ es polo simple?

$$\lim_{z \to 1+i\sqrt{3}} (z-(1+i\sqrt{3}))f(z) = \lim_{z\to 1+i\sqrt{3}} \frac{1}{(z+2)(z-(1-i\sqrt{3}))} = \frac{1}{(3+i\sqrt{3})2i\sqrt{3}} \neq 0$$

$\therefore$ $1+i\sqrt{3}$ es polo simple

$$\therefore \oint_{\Gamma} f(z)\,dz = 2\pi i \cdot \frac{1}{(3+i\sqrt{3})2i\sqrt{3}} = \frac{\pi}{(3+i\sqrt{3})\sqrt{3}}$$

Pero todavía no sabemos cuanto vale $\int_0^\infty \frac{dx}{x^3+8} = I$

calculemos $\int_{\gamma_1} f(t)\,dz$ $\quad r(t) = t \quad t \in [0,R]$
$\dot r(t) = 1$

$$\int_0^R \frac{1}{t^3+8}\,dt \underset{R\to\infty}{=} I$$

calculemos $\int_{\gamma_3} f(z)\,dz$ $\quad r(t) = t e^{i\frac{2\pi}{3}} \quad t \in [R,0]$
$\dot r(t) = e^{i2\pi/3}$

$$-\int_0^R \frac{e^{i2\pi/3}}{t^3+8}\,dt \underset{R\to\infty}{=} -e^{i2\pi/3}\, I$$

veamos que $\int_{\gamma_2} f(z)\,dz \xrightarrow{R\to\infty} 0$ (en general los arcos se van a 0)

$r(\theta) = Re^{i\theta} \quad \theta \in [0, 2\pi/3]$

$\dot r(\theta) = Ri\, e^{i\theta}$

$$0 \leq \left| \int_0^{\frac{2\pi}{3}} \frac{Ri\, e^{i\theta}}{R^3 e^{i3\theta}+8}\,d\theta \right| \leq \int_0^{\frac{2\pi}{3}} \frac{\overset{1}{|i|}\,\overset{R}{|Re^{i\theta}|}}{|R^3 e^{i3\theta} - -8|}\,d\theta \leq \int_0^{\frac{2\pi}{3}} \frac{R}{\underbrace{|R^3 e^{i\theta}|}_{R^3} - \underbrace{|-8|}_{8}} = \int_0^{\frac{2\pi}{3}} \frac{R}{R^3-8}\,d\theta$$

$$= \frac{2\pi}{3}\left(\frac{R}{R^3+8}\right) \xrightarrow{R\to\infty} 0$$

entonces $I + -e^{i\frac{2\pi}{3}} I = \frac{\pi}{(3+i\sqrt{3})\sqrt{3}} \Rightarrow I = \frac{\pi}{\sqrt{3}(3+i\sqrt{3})(1-e^{i2\pi/3})}$

$1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$
$= \frac{3}{2} - i\frac{\sqrt{3}}{2}$

$$= \frac{\pi}{\sqrt{3}(3+i\sqrt{3})\frac{(3-i\sqrt{3})}{2}}$$

$$= \frac{2\pi}{\sqrt{3}\,(3^2+\sqrt{3}^2)} = \frac{2\pi}{\sqrt{3}(12)} = \frac{\pi}{6\sqrt{3}} \;//$$