PAUTA AUXILIAR 11

P1) queremos que u sea la parte real de una función holomorfa f,

es decir, $f = u + iv$

$\Rightarrow$ por Cauchy - Riemann $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \Delta u$$

como $u(x,y) = g(x+y)$ se tiene que $2g''(x+y) = 0$

$\Rightarrow g''(x+y) = 0$

$\Rightarrow g'' = 0$ (x, y eran arbitrarios)

$\Rightarrow g'(t) = a$, a constante

$\Rightarrow g(t) = at + b$

entonces $u(x,y) = a(x+y) + b \overset{C-R}{\Rightarrow} v = a(y-x) + c$

$\therefore f = a(x+y) + b + ia(y-x) + ic$

$\Leftrightarrow f(z) = za(1-i) + z_0$ con $a \in \mathbb{R}$, $z_0 \in \mathbb{C}$.

P2) Fórmula de Cauchy para las derivadas:

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{\partial D(z_0,\varepsilon)} \frac{f(z)}{(z-z_0)^{n+1}}$$

nos piden demostrar que $\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \oint_{\partial D(0,1)} \frac{x^n e^{xz}}{\cancel{n!}\, z^{n+1}} dz$

$$\Leftrightarrow \left(\frac{x^n}{n!}\right) = \frac{1}{2\pi i} \oint_{\partial D(0,1)} \frac{e^{xz}}{z^{n+1}} dz$$

entonces sea $f(z) = e^{xz} \Rightarrow f^{(n)}(z) = x^n e^{xz} \Rightarrow f^{(n)}(0) = x^n$

$$\Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{x^n}{n!} = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{xz}}{z^{n+1}} dz \quad //$$

se tiene entonces que $\sum_{n=0}^{N} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \sum_{n=0}^{N} \oint_{|z|=1} \frac{e^{xz}}{z^{n+1}} \frac{x^n}{n!} dz$

$$= \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{xz}}{z} \sum_{n=0}^{N} \frac{(x/z)^n}{n!} \, dz$$

pero $$\sum_{n=0}^{N} \frac{(x/z)^n}{n!} \xrightarrow{N\to\infty} e^{x/z}$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{xz}}{z} e^{x/z} \, dz = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{x(z+1/z)}}{z} \, dz$$

$r(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$

$\dot r(\theta) = i e^{i\theta}$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{\cancel{i e^{i\theta}} \, e^{x\overbrace{(e^{i\theta}+e^{-i\theta})}^{z+\bar z = 2\mathrm{Re}(z) = 2\cos\theta}}}{\cancel{e^{i\theta}}} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{2x\cos\theta} \, d\theta$$

$$\therefore \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{2x\cos\theta} \, d\theta \quad //$$