PAUTA AUXILIAR 8

P1 Recordemos

- criterio del cuociente : $\sum_{n=0}^{\infty} a_n$ se evalúa $\left|\frac{a_{n+1}}{a_n}\right| \to L$

  - si $L < 1$, es absolutamente convergente
  - si $L > 1$, es divergente
  - si $L = 1$, no sabemos nada

- criterio de la raíz : se evalúa $\sqrt[n]{|a_n|} \to L$ con las mismas condiciones que el del cuociente.

a) imponemos $\left|\frac{a_{n+1}}{a_n}\right| < 1$ porque queremos ver donde converge.

$$\Leftrightarrow \lim_{n\to\infty} \frac{(n+1)^2 |z+2|^{2^{n+1}}}{n^2 |z+2|^{2^n}} < 1$$

$$\Leftrightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 |z+2|^{2^n} < 1 \qquad / \; (\;)^{1/2^n}$$

$$\Leftrightarrow \lim_{n\to\infty} \cancel{\sqrt[2^n]{\left(1+\frac{1}{n}\right)^2}}^{\,1} |z+2| < 1$$

$$\Leftrightarrow |z+2| < 1$$

una serie converge en $D(a,R) = \{z \in \mathbb{C} : |z-a| < R\}$ donde R es el radio de convergencia

$$\therefore \sum_{n=1}^{\infty} n^2 (z+2)^{2^n} \text{ converge en } D(-2,1) \Rightarrow \boxed{R = 1}$$

b) imponemos $\sqrt[n]{|a_n|} < 1$ porque queremos ver donde converge

$$\boxed{\therefore\; R = \sqrt[3]{9}}$$

$$\Leftrightarrow \lim_{n\to\infty} \left(\frac{3^n n^3 (n!)^3}{(3n)!} |z|^{3n}\right)^{1/n} < 1$$

stirling (hint)

$$\Leftrightarrow \lim_{n\to\infty} \left(\frac{3^n n^3 (2\pi n)^{3/2} \left(\frac{n}{e}\right)^3 |z|^{3n}}{(6\pi n)^{1/2} \left(\frac{3n}{e}\right)^{3n}}\right)^{1/n} < 1$$

$(2^{1/2} 3^{1/2})$

$$\Leftrightarrow \lim_{n\to\infty} \left(\frac{n^4 \cdot \pi \cdot 2 \cdot |z|^{3n}}{3^{2n} \cdot 3^{1/2}}\right)^{1/n} < 1$$

$$\Leftrightarrow \lim_{n\to\infty} \left(\frac{\cancel{\sqrt[n]{n^4}}^{\,1} \cdot \cancel{\sqrt[n]{2\pi}}^{\,1}\; |z|^3}{9 \; \cancel{\sqrt[n]{3^{1/2}}}^{\,1}}\right) < 1$$

$$\Leftrightarrow |z|^3 < 9$$

$$\Leftrightarrow |z| < \sqrt[3]{9}$$

P3 a) $\gamma(\theta) = 2e^{i\theta}$ $\quad \theta \in [0, 2\pi]$
$\dot{\gamma}(\theta) = 2e^{i\theta} i$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(2e^{i\theta}) \cdot 2i e^{i\theta}\, d\theta = \int_0^{2\pi} 2e^{-i\theta} \cdot 2i e^{i\theta}\, d\theta$$

$$= 4i \int_0^{2\pi} d\theta = 8\pi i \quad //$$

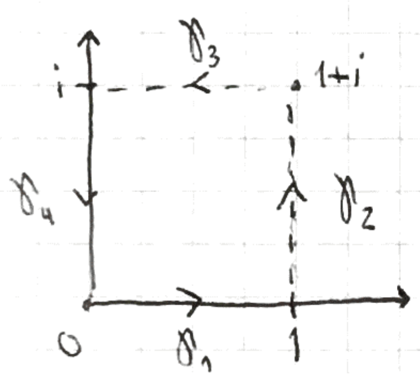
b) $\gamma(\theta) = e^{i\theta}$ $\quad \theta \in [-\pi/2, \pi/2]$
$\dot{\gamma}(\theta) = i e^{i\theta}$

$$\int_\gamma f(z)\,dz = \int_{-\pi/2}^{\pi/2} f(e^{i\theta})\, i e^{i\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta \; i e^{i\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right) i e^{i\theta}\, d\theta$$

$$= \frac{i}{2} \int_{-\pi/2}^{\pi/2} e^{i2\theta} + 1 \; d\theta = \frac{i}{2} \left[\frac{e^{i2\theta}}{2i} + \theta \right] \Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{i}{2} \Bigg[\pi + \underbrace{\frac{e^{\pi i} - e^{-\pi i}}{2i}}_{\operatorname{sen}(\pi) = 0} \Bigg] = \frac{\pi i}{2} \quad //$$

c)



- $\gamma_1(t) = t$ $\quad t \in [0, 1]$
  $\dot{\gamma}_1(t) = 1$

$$\int_{\gamma_1} f(z)\,dz = \int_0^1 |t|^2\, dt = \int_0^1 t^2\, dt = \frac{t^3}{3}\Bigg|_0^1 = \frac{1}{3}$$

- $\gamma_2(t) = 1 + it$ $\quad t \in [0, 1]$
  $\dot{\gamma}_2(t) = i$

$$\int_{\gamma_2} f(z)\,dz = \int_0^1 |1+it|^2 \cdot i\, dt = \int_0^1 (1+t^2)\, i\, dt = i\left[t + \frac{t^3}{3} \right]\Bigg|_0^1 = i\left(1 + \frac{1}{3}\right) = \frac{4i}{3}$$

- $\gamma_3(t) = t + i$ $\quad t \in [1, 0]$
  $\dot{\gamma}_3(t) = 1$

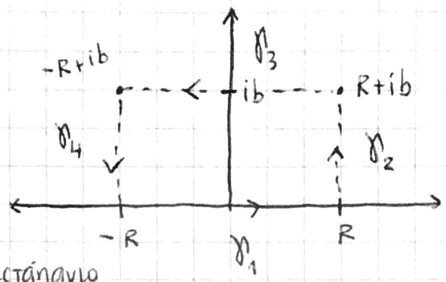
$$\int_{\gamma_3} f(z)\,dz = \int_1^0 |i+t|^2\, dt = -\int_0^1 (1+t^2)\, dt = -\left[t + \frac{t^3}{3} \right]\Bigg|_0^1 = -\frac{4}{3}$$

- $\gamma_4(t) = ti$ $\quad t \in [1, 0]$
  $\dot{\gamma}_4(t) = i$

$$\int_{\gamma_4} f(z)\,dz = \int_1^0 |ti|^2\, i\, dt = -\int_0^1 i t^2 = -i\left(\frac{t^3}{3}\right)\Bigg|_0^1 = \frac{-i}{3}$$

$$\therefore \int_\gamma f(z)\,dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = -1 + i \quad //$$

P4 Queremos calcular $\int_0^{\infty} e^{-x^2}\cos(2bx)\,dx = I$

Veamos como es la integral de $f(z)=e^{-z^2}$ en 
cuando $R\to\infty$.

Notemos primero que $f(z)$ es holomorfa y que el rectángulo es una curva cerrada, luego por Cauchy-Goursat

$$\oint_\gamma f(z)\,dz = 0$$

$$\Leftrightarrow \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)\,dz = 0$$

Veamos cuanto vale cada una de esas integrales para poder calcular I.

- $\int_{\gamma_1} f(z)\,dz$: $\gamma_1(t)=t$ $\quad t\in[-R,R]$ $\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} f(t)\cdot 1\,dt$
  $\dot\gamma_1(t)=1$

$$= \int_{-R}^{R} e^{-t^2}\,dt \quad / \lim_{R\to\infty} \leftarrow \text{nos dicen que tomemos } \lim_{R\to\infty} \text{ en el enunciado}$$

$$= \int_{-\infty}^{+\infty} e^{-t^2} = \underbrace{\sqrt{\pi}}_{\text{conocido}}$$

- $\int_{\gamma_2} f(z)\,dz$: $\gamma_2(t) = R+it$ $\quad t\in[0,b]$
  $\dot\gamma_2(t) = i$

Veamos que $\int_{\gamma_2} f(z)\,dz \xrightarrow{R\to\infty} 0$

En efecto $\left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^b f(R+it)\,i\,dt\right| = \left|\int_0^b e^{-(R+it)^2} i\,dt\right|$

$$= \left|\int_0^b e^{-R^2} e^{t^2} e^{-2Rti}\, i\,dt\right| \leq e^{-R^2}\int_0^b \underbrace{|e^{t^2} e^{-2Rti}|}_{Re^{i\theta}}\,|i|\,dt$$

$Re^{i\theta}$: $R = e^{t^2}$, $\theta = -2Rt$; $|e^{t^2}e^{-2Rti}| = e^{t^2}$, $|i| = 1$

$$= e^{-R}\int_0^b e^{t^2}\,dt \quad / \lim_{R\to\infty}$$

$$= 0 \,//.$$

• $\int_{\gamma_3} f(z)\,dz$: $\quad \gamma_3(t) = ib + t \quad t \in [R,-R]$

$\dot{\gamma}_3(t) = 1$

$$\int_{R}^{-R} e^{-(ib+t)^2} \cdot 1\,dt = -\int_{-R}^{R} e^{-t^2} e^{b^2} e^{-2ibt}\,dt$$

$$= -e^{b^2} \int_{-R}^{R} e^{-t^2} \left[\cos(-2bt) + i\,\text{sen}(-2bt)\right] dt$$

pero e^{-t^2} es par y $\text{sen}(-2bt)$ es impar

$\Rightarrow \underbrace{e^{-t^2}\,\text{sen}(-2bt)}_{g(t)}$ es impar

$$\Rightarrow \int_{0}^{R} g(t)\,dt = -\int_{-R}^{0} g(t)\,dt$$

$$\Rightarrow \int_{-R}^{R} g(t)\,dt = 0$$

pq $e^{-t^2}\cos(2bt)$ es par

$$\Rightarrow \int_{\gamma_3} f(z)\,dz = -e^{b^2} \int_{-R}^{R} e^{-t^2}\cos(2bt)\,dt = -e^{b^2} \cdot 2 \int_{0}^{R} e^{-t^2}\cos(2bt)\,dt \Big/ \lim_{R\to\infty}$$

$$-2e^{b^2} \int_{0}^{R} e^{-t^2}\cos(2bt)\,dt = -2e^{b^2}\, I$$

• $\int_{\gamma_4} f(z)\,dz$: $\quad \gamma_4(t) = -R + it \quad t \in [b,0]$

$\dot{\gamma}_4(t) = i$

$$\left| \int_{b}^{0} e^{-(-R+it)^2}\, i\,dt \right| = \left| -\int_{0}^{b} e^{-R^2} e^{t^2} e^{2Rit}\, i\,dt \right| = \left| e^{-R^2} \int_{0}^{b} e^{t^2} e^{2Rit}\, i\,dt \right|$$

$$\leq e^{-R^2} \int_{0}^{b} \overset{e^{t^2}}{\cancel{|e^{t^2} e^{2Rit}|}}\, \overset{1}{\cancel{|i|}}\,dt = e^{-R^2} \int_{0}^{b} e^{t^2}\,dt \Big/ \lim_{R\to\infty} = 0 //$$

$$\therefore \quad \sqrt{\pi} + 0 + -2e^{b^2} I + 0 = 0 \Rightarrow I = \frac{\sqrt{\pi}}{2e^{b^2}} \quad //$$