PAUTA AUXILIAR 7

P1] Sea $\vec{r}_0 \in S$ fijo. Consideremos $\varepsilon > 0$ y definamos

$$S_\varepsilon = \{ \vec{r} \in S : \|\vec{r} - \vec{r}_0\| \leq \varepsilon \}$$

Por Stokes se tiene que

$$\iint_{S_\varepsilon} rot(\vec{F} - \vec{G}) \cdot \hat{n} \, dS = \int_{\partial S_\varepsilon} (\vec{F} - \vec{G}) \cdot d\vec{r}$$

Pero sabemos que en S, $\vec{F} = \vec{G}$ $\Rightarrow \displaystyle\iint_{S_\varepsilon} rot(\vec{F} - \vec{G}) \cdot \hat{n} \, dS = 0$

Ahora queremos hacer $\varepsilon \to 0$, pero para ello tenemos que normalizar para que el límite no sea 0.
Para ello, dividimos por $Area(S_\varepsilon)$

$$\Rightarrow \frac{1}{Area(S_\varepsilon)} \iint_{S_\varepsilon} rot(\vec{F} - \vec{G}) \cdot \hat{n} \, dS = 0$$

Veamos que $\displaystyle\lim_{\varepsilon \to 0} \frac{1}{Area(S_\varepsilon)} \iint_{S_\varepsilon} rot(\vec{F} - \vec{G}) \cdot \hat{n} \, dS = rot(\vec{F} - \vec{G})(\vec{r}_0) \cdot \hat{n}(\vec{r}_0)$

En efecto, como $\vec{F}$ y $\vec{G}$ coinciden en S, $rot(\vec{F} - \vec{G})$ es continua.
$\Rightarrow \forall \lambda > 0 \; \exists \delta > 0 \; \forall \vec{r} \in S \quad \|\vec{r} - \vec{r}_0\| \leq \delta \Rightarrow \| rot(\vec{F} - \vec{G})(\vec{r}) - rot(\vec{F} - \vec{G})(r_0) \| < \lambda$

Tomemos $\varepsilon \in [0, \delta)$ $\Rightarrow$

$$\left| rot(\vec{F} - \vec{G})(\vec{r}_0) - \frac{1}{Area(S_\varepsilon)} \iint_{S_\varepsilon} rot(\vec{F} - \vec{G}) \cdot d\vec{S} \right|$$

$$= \left| \frac{1}{Area(S_\varepsilon)} \iint_{S_\varepsilon} rot(\vec{F} - \vec{G})(\vec{r}_0) - rot(\vec{F} - \vec{G}) \cdot d\vec{S} \right|$$

$$\leq \frac{1}{Area(S_\varepsilon)} \iint_{S_\varepsilon} |rot(\vec{F} - \vec{G})(\vec{r}_0) - rot(\vec{F} - \vec{G})| \, dS$$

$$\leq \frac{1}{Area(S_\varepsilon)} \iint_{S_\varepsilon} \lambda \, dA = \lambda \; //.$$

$\therefore \; rot(\vec{F})(\vec{r}_0) \cdot \hat{n} = rot(\vec{G})(\vec{r}_0) \cdot \hat{n} \quad \forall \vec{r}_0 \in S$ //.

Ejemplo: $\vec{F} = (0,0,0)$ $\quad \vec{G} = (z,0,0)$

$$S = \{ (x,y,0) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \}$$

es claro que $\vec{F} = \vec{G}$ en S pero $rot(\vec{F}) = \vec{0}$ y $rot(\vec{G}) = \hat{j}$ //.

P2)

$$\vec{F}(x,y,z) = \frac{-GmM}{(x^2+y^2+z^2)^{3/2}} \underbrace{(x\hat{x}+y\hat{y}+z\hat{z})}_{\vec{v}\ \text{(Posición}}$$

en esféricas $\vec{F} = \frac{-GmM}{(r^2)^{3/2}} r\hat{r} = \frac{-GmM}{r^2}\hat{r} \Rightarrow \begin{cases} F_r = \frac{-GmM}{r^2} = \frac{1}{h_r}\frac{\partial f}{\partial r} = \frac{1}{1}\frac{\partial f}{\partial r} \\ F_\theta = 0 = \frac{1}{r\,\text{sen}\phi}\frac{\partial f}{\partial \theta} \\ F_\phi = 0 = \frac{1}{r}\frac{\partial f}{\partial \phi} \end{cases}$

Primera ecuación

$$-\frac{GmM}{r^2} = \frac{\partial f}{\partial r} \Big/ \int dr \Rightarrow \frac{GmM}{r} + c(\theta,\phi) = f$$ ← ahora usamos esta f

segunda ecuación

$$0 = \frac{\partial f}{\partial \theta} \Rightarrow 0 = \frac{\partial c(\theta,\phi)}{\partial \theta} \Rightarrow c(\theta,\ ) = c(\phi) \Rightarrow f = \frac{GmM}{r} + c(\phi)$$

ahora usamos esta f

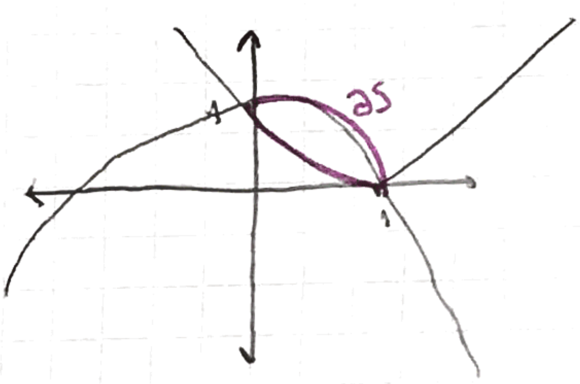
Tercera ecuación

$$0 = \frac{\partial f}{\partial \phi} \Rightarrow 0 = \frac{\partial c(\phi)}{\partial \phi} \Rightarrow c(\phi) = c \Rightarrow f = \frac{GmM}{r} + c$$

$\Rightarrow \int \vec{F}\cdot d\vec{r} = f(\text{radio final}) - f(\text{radio inicial}) = 0 \Rightarrow$ existen zonas equipotenciales

P3)

$$\iint_S \text{rot}(F)\cdot d\vec{s} = \oint_{\partial S} \vec{F}\cdot d\vec{r}$$ ← calcularemos esta



∂S: $(x-1)^2 + (y-1)^2 = z = 1-y^2$

$(x-1)^2 + y^2 - 2y + 1 = 1 - y^2$

$$(x-1)^2 + 2\underbrace{\left(y^2 - y + \tfrac{1}{4}\right)}_{(y-\frac{1}{2})^2} = 0 + 2\cdot\frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow \frac{(x-1)^2}{1/2} + \frac{(y-1/2)^2}{1/4} = 1$$ ← elipse!

* una elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ se parametriza en $\mathbb{R}^2$ como $\vec{r}(t) = (a\cos(t), b\,\text{sen}(t))$ $t \in [0,\pi]$

$$\Rightarrow \vec{r}(t) = (1, \tfrac{1}{2}, 0) + \left(\tfrac{1}{\sqrt{2}}\cos(t), \tfrac{1}{2}\text{sen}(t), 0\right) + \left(0,0,-\left(\tfrac{1}{2}+\tfrac{1}{2}\text{sen}(t)\right)^2\right)$$

$$= \Big(1 + \tfrac{1}{\sqrt{2}}\cos(t),\ \tfrac{1}{2}+\tfrac{1}{2}\text{sen}(t),\ \underbrace{1 - \left(\tfrac{1}{2}\text{sen}(t) + \tfrac{1}{2}\right)^2}_{z = 1-y^2}\Big) \qquad t \in [0,2\pi]$$

- $\frac{\partial \vec{r}}{\partial t} = \left(-\frac{1}{\sqrt{2}}\text{sen}(t), \frac{1}{2}\cos(t), -\frac{2}{2}(\text{sen}(t)+1)\frac{1}{2}\cos(t)\right)$

- $\vec{F}(\vec{r}(t)) = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cos(t) - \frac{1}{2}\text{sen}(t),\ 1 - \frac{1}{4}(\text{sen}(t)+1)^2,\ \frac{1}{2}+\frac{1}{2}\text{sen}(t)\right)$

$$\int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}\,dt = 0 + 0 + \frac{1}{2\sqrt{2}}\underbrace{\int_0^{2\pi}\text{sen}^2(t)\,dt}_{\int_0^{2\pi} \frac{1-\cos(2t)}{2} dt} + 0 - \frac{1}{8}\int_0^{2\pi}\underbrace{(\text{sen}(t)+1)^2\cos(t)\,dt}_{\frac{1}{3}\frac{d}{dt}(\text{sen}(t)+1)^3} + 0 + 0$$

$$= \frac{1}{2\sqrt{2}} \left[\int_0^{2\pi} \frac{1}{2}\, dt - \frac{1}{2} \int_0^{2\pi} \cos(2t) \right] - \frac{1}{24} \left(\sin(t) + 1 \right)^3 \Big|_0^{2\pi}$$

$$= \frac{1}{2\sqrt{2}} \left[\pi - 0 \right] - \frac{1}{24} \left[0 \right] = \frac{\pi}{2\sqrt{2}}$$