PAUTA AUXILIAR 5

P1 $f(\overbrace{x+iy}^{z}) = \sqrt{|x||y|}$ $\quad x, y \in \mathbb{R}$

$\rightarrow f = u + vi$

$u(x,y) = \sqrt{|x|\,|y|} \qquad v(x,y) = 0$

Veamos que se cumplen las condiciones de C-R:

- $\dfrac{\partial u}{\partial x}(0,0) = \lim\limits_{h \to 0} \dfrac{u(h,0) - u(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0}{h} = 0$
- $\dfrac{\partial u}{\partial y}(0,0) = \lim\limits_{h \to 0} \dfrac{u(0,h) - u(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0}{h} = 0$
- $\dfrac{\partial v}{\partial x}(0,0) = 0$
- $\dfrac{\partial v}{\partial y}(0,0) = 0$

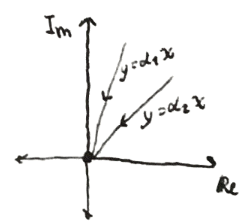
$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \wedge \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad //$$

$\therefore f$ satisface las condiciones de C-R en el origen.

Veamos que f no es diferenciable en el origen, es decir, que el siguiente límite no existe: $\lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z}$

En efecto: $\lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim\limits_{z \to 0} \dfrac{\sqrt{|x||y|}}{x + iy} \quad / \cdot \dfrac{(x - iy)}{(x - iy)}$

$$= \lim_{z \to 0} \frac{\sqrt{|x||y|}\,(x - iy)}{x^2 + y^2}$$



Para ver que este límite no existe, acerquémonos al origen por 2 caminos diferentes y veamos que da 2 límites diferentes (contradicción, luego no existe el límite).

Tomemos diferentes rectas de la forma $y = \alpha x$ (como se muestra en el dibujo más arriba).

$x = t \qquad y = \alpha t$, acerquémonos al origen por la derecha.

$$\Rightarrow \lim_{t \to 0^+} \frac{\sqrt{|t||\alpha t|}}{t^2 + \alpha^2 t^2} (t - i\alpha t)$$

notemos que como $t \to 0^+$ y $\alpha \geq 0$

$t \geq 0 \wedge \alpha t \geq 0 \Rightarrow |t| = t \wedge |\alpha t| = \alpha t$

$$= \lim_{t \to 0^+} \frac{\sqrt{t^2 \alpha}\, \cancel{t}(1 - i\alpha)}{t^{\cancel{2}}(1+\alpha^2)} = \lim_{t \to 0^+} \frac{\cancel{t}(1 - i\alpha)\sqrt{\alpha}}{\cancel{t}(1+\alpha^2)} = \frac{(1 - i\alpha)\sqrt{\alpha}}{(1+\alpha^2)}$$

Luego para diferentes valores de α el límite es distinto

$\therefore$ el límite no existe

$\Rightarrow f$ no es diferenciable en el origen.

P2 a) P.D.Q f satisface C-R $\iff \dfrac{\partial f}{\partial \bar{z}} = 0$

En efecto:

$\Rightarrow$ $\dfrac{\partial f}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) \overset{f = u + iv}{=} \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} + i\dfrac{\partial u}{\partial y} + i^2\dfrac{\partial v}{\partial y}\right)$

$\overset{\substack{C-R \\ i^2=-1}}{=} \left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} + i\left(-\dfrac{\partial v}{\partial x}\right) - \left(\dfrac{\partial u}{\partial x}\right)\right) = 0$ //

$\Leftarrow$ $\dfrac{\partial f}{\partial \bar{z}} = 0 \iff \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) = 0$

$\iff \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} + i\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial y} = 0$

$\iff \left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right) + i\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) = 0$

Para que un complejo sea 0, su parte real e imaginaria deben ser 0.

$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \Rightarrow \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$

b) P.D.Q f holomorfa $\Rightarrow f'(z) = \dfrac{\partial f}{\partial z}$

En efecto: como f es holomorfa se satisface C-R y

$f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ (1)

$\xrightarrow{C-R}$ $f'(z) = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$ (2)

(1) + (2) $2f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y} = \left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right) - i\left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right)$

$\Rightarrow f'(z) = \dfrac{1}{2}\left[\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right] = \dfrac{\partial f}{\partial z}$ //.

c) $\dfrac{\partial^2 f}{\partial z \partial \bar{z}} = \dfrac{\partial}{\partial z}\left(\dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right)\right) = \dfrac{1}{4}\left[\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) - i\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right)\right]$

$$= \frac{1}{4}\left[\frac{\partial^2 f}{\partial x^2} + i\frac{\partial^2 f}{\partial x \partial y} - i\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2}\right] = 0$$

$$\iff \left(\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} + i\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 v}{\partial y^2}\right) = 0$$

$$\iff \underbrace{\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2}\right)}_{=0} + i\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}\right)}_{=0}$$

En particular, si $u, v \in C^2$ se tiene que

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \wedge \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\iff \Delta u = 0 \quad \wedge \quad \Delta v = 0 \;//$$

d) P.D.Q f holomorfa $\Rightarrow$ $\Delta f = 0$

En efecto, como f es holomorfa, se satisface C-R

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Big/ \frac{\partial}{\partial x} \quad \wedge \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \wedge \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

pero como $u, v \in C^2$ $\quad \dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \iff \Delta u = 0$$

Para ver que $\Delta v = 0$ el procedimiento es análogo

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Big/ \frac{\partial}{\partial y} \quad \wedge \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial}{\partial x}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \wedge \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

como $u \in C^2$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \iff \Delta v = 0$$

$\therefore$ f holomorfa $\Rightarrow \Delta f = 0$ //.

P3 a) f holomorfa $\Rightarrow \Delta f = 0$ (por P2)

luego si $\Delta f \neq 0$, f no puede ser holomorfa

$\Delta f = 0$

$\Leftrightarrow \Delta u = 0 \wedge \Delta v = 0$

como v es "alguno adecuado", sólo tenemos que ver que $\Delta u = 0$

$\Leftrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= ay(2b^2x - 2b\,\mathrm{sen}(bx)) \\ \frac{\partial^2 u}{\partial x^2} &= 2ayb^2 - 2b^2ay\cos(bx) \\ \frac{\partial u}{\partial y} &= a(b^2x^2 + 2\cos(bx)) \\ \frac{\partial^2 u}{\partial y^2} &= 0 \end{aligned}\right\} \Rightarrow 2ayb^2(1-\cos(bx)) = 0$$

$\Rightarrow a = 0 \vee b = 0 \vee \underbrace{\cos(bx) = 1 \ \forall x}_{\Leftrightarrow b = 0}$

$\therefore f$ es holomorfa en todo $\mathbb{C}$ si $a = 0 \wedge b \in \mathbb{R}$ o $b = 0 \wedge a \in \mathbb{R}$ //.

b) si f es holomorfa, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow Re(f'(z)) = \frac{\partial u}{\partial x}$

$\frac{\partial u}{\partial x} = 0 \Rightarrow u(x,y) = u(y)$ u sólo puede depender de y

Además, como f es holomorfa, se satisface C-R $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v(x,y) = v(x)$ v sólo depende de x

y además $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Leftrightarrow u'(y) = -v'(x)$

También se tiene que $\Delta f = 0$

$$\Leftrightarrow U''(y) = 0 \quad \wedge \quad V''(x) = 0$$

$$\Rightarrow U(y) = cy + d \quad \wedge \quad V(x) = ex + f$$

Pero como $U'(y) = -V'(x) \Rightarrow c = -e$

$$\Rightarrow f(z) = cy + d + i(-cx + f)$$

$$= -ic(x + iy) + d + if$$

$$= -iz + d + if \quad //$$

P4 a) como sólo sabemos usar Stokes en $\mathbb{R}^3$, definimos

$$F_0 = (F_1(x,y),\ F_2(x,y),\ 0)$$

que es claramente equivalente a F en $\mathbb{R}^2$.

Por Stokes se tiene que $\displaystyle\int_R \text{rot}(F_0)\cdot d\vec{S} = \int_C F_0 \cdot d\vec{r}$

calculemos $\text{rot}(F_0) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$

notemos que $\hat{n}_R$ (siguiendo la regla de la mano derecha) es $\hat{k}$

$\Rightarrow d\vec{S} = \hat{k}\, dx\,dy$

$$\Rightarrow \int_R \text{rot}(F_0)\cdot d\vec{S} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\cdot\hat{k}\,dx\,dy = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

Por otro lado

$$\int_C F_0 \cdot d\vec{r} = \int_C \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int_C F_1\,dx + F_2\,dy$$

$$\therefore \int_C F_1\,dx + F_2\,dy = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

b) Definimos $G_0 = (G_1(x,y),\ G_2(x,y),\ 0)$

Por Gauss se tiene que

$$\int_{\Omega} \operatorname{div}(G_0)\cdot dV = \int_{\partial\Omega} G_0 \cdot d\vec{S}$$

$$\operatorname{div}(G_0) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} \Rightarrow \int_{\Omega} \operatorname{div}(G_0)\cdot dV = \int_0^h \int_R \left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\right) dx\,dy\,dz$$

$$= h \int_R \left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\right) dx\,dy$$

Notemos que $\partial\Omega = R + T$ (tapa) $+ M$ (manto)

$$\Rightarrow \int_{\partial\Omega} G_0 \cdot \hat{n}\, dS = \int_R G_0 \cdot \hat{n}_R\, dS + \int_T G_0 \cdot \hat{n}_T\, dS + \int_M G_0 \cdot \hat{n}_M\, dS$$

pero $\hat{n}_R = -\hat{k}$ y $\hat{n}_T = \hat{k}$, como G_0 es nulo en $\hat{k}$, se tiene que

$G_0 \cdot \hat{n}_R = 0 \ \wedge\ G_0 \cdot \hat{n}_T = 0$

$$\Rightarrow \int_{\partial\Omega} G_0\, d\vec{S} = \int_M G_0 \cdot \hat{n}_M\, dS$$

Parametricemos M: sabemos que existe alguna parametrización $\vec{\gamma}$ de C

$\gamma(t) = (\gamma_1(t), \gamma_2(t), 0) \quad t \in [a,b]$

$\Rightarrow \vec{r}(t,z) = (\gamma_1(t), \gamma_2(t), z) \quad t \in [a,b] \ z \in [0,h]$

$\uparrow$

Parametrización de M

$$\int_M G_0 \cdot \hat{n}_M\, dS = \int_0^h \int_a^b G_0(\vec{r}(t,z)) \cdot \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial z}\, dt\, dz$$

$\frac{\partial r}{\partial t} = (\gamma_1'(t), \gamma_2'(t), 0)$

$\frac{\partial r}{\partial z} = (0, 0, 1)$

$$\Rightarrow \frac{\partial r}{\partial t} \times \frac{\partial r}{\partial z} = \begin{pmatrix} \gamma_2'(t) \\ -\gamma_1'(t) \\ 0 \end{pmatrix}$$

$$\Rightarrow \int_M G_0 \cdot \hat{n}_M \, ds = \int_0^h \int_a^b \begin{pmatrix} G_1(\gamma(t)) \\ G_2(\gamma(t)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma_2'(t) \\ -\gamma_1'(t) \\ 0 \end{pmatrix} dt\, dz$$

$$= h \int_a^b G_1(\gamma)\gamma_2' - G_2(\gamma)\gamma_1' \, dt$$

regla de la cadena

$$= h \int_C G_1\, dy - G_2\, dx$$

$$\Rightarrow \cancel{h} \int_R \left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} \right) dx\, dy = \cancel{h} \int_C G_1\, dy - G_2\, dx$$

$$\int_C G_1\, dy - G_2\, dx = \int_R \left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} \right) dx\, dy$$

c) Basta tomar $F = (-G_2, G_1)$ y $G = (F_2, -F_1)$

y se tiene que son equivalentes. //