PAUTA AUXILIAR 1

P1 a) P.D.Q $\operatorname{div}(\varphi\nabla\varphi) = \|\nabla\varphi\|^2 + \varphi\Delta\varphi$

- en $\mathbb{R}^3$

$$\operatorname{div}(\varphi\nabla\varphi) = \operatorname{div}\left(\varphi\left[\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right]\right) \quad \text{(def. de } \nabla\varphi)$$

$$= \operatorname{div}\left(\varphi\frac{\partial\varphi}{\partial x}\hat{\imath} + \varphi\frac{\partial\varphi}{\partial y}\hat{\jmath} + \varphi\frac{\partial\varphi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial z}\right) \quad \text{(def. de divergencia)}$$

$$= \left(\frac{\partial\varphi}{\partial x}\right)^2 + \varphi\frac{\partial^2\varphi}{\partial x^2} + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \varphi\frac{\partial^2\varphi}{\partial y^2} + \left(\frac{\partial\varphi}{\partial z}\right)^2 + \varphi\frac{\partial^2\varphi}{\partial z^2} \quad \text{(regla del producto)}$$

$$= \varphi\frac{\partial^2\varphi}{\partial x^2} + \varphi\frac{\partial^2\varphi}{\partial y^2} + \varphi\frac{\partial^2\varphi}{\partial z^2} + \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2$$

$$= \underbrace{\varphi\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right)}_{\varphi\Delta\varphi} + \underbrace{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2}_{\|\nabla\varphi\|^2} \quad //$$

- en $\mathbb{R}^n$:

$$\operatorname{div}(\varphi\nabla\varphi) = \operatorname{div}\left(\varphi\sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}\hat{e}_i\right) \quad \text{(def. de } \nabla\varphi)$$

$$= \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\varphi\sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}\hat{e}_i\right)_j \quad \text{(def de divergencia)}$$

(* ¿qué es $\left(\varphi\sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}\hat{e}_i\right)_j$?

es la componente j-ésima de ese vector, que corresponde al término que acompaña a $\hat{e}_j$, es decir, el término j-ésimo de la sumatoria)

$$= \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\varphi\frac{\partial\varphi}{\partial x_j}\right)$$

$$= \sum_{j=1}^{n}\left(\frac{\partial\varphi}{\partial x_j}\right)^2 + \varphi\frac{\partial^2\varphi}{\partial x_j^2} \quad \text{(regla del producto)}$$

$$= \sum_{j=1}^{n}\left(\frac{\partial\varphi}{\partial x_j}\right)^2 + \varphi\sum_{j=1}^{n}\frac{\partial^2\varphi}{\partial x_j^2}$$

$$= \|\nabla\varphi\|^2 + \varphi\Delta\varphi \quad //$$

b) P.D.Q $\Delta G = \nabla(\operatorname{div}(G)) - \operatorname{rot}(\operatorname{rot}(G))$

(Notemos que como G es un campo vectorial,

$$\Delta G = \left(\frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_1}{\partial y^2} + \frac{\partial^2 G_1}{\partial z^2}\right)\hat{\imath} + \left(\frac{\partial^2 G_2}{\partial x^2} + \frac{\partial^2 G_2}{\partial y^2} + \frac{\partial^2 G_2}{\partial z^2}\right)\hat{\jmath} + \left(\frac{\partial^2 G_3}{\partial x^2} + \frac{\partial^2 G_3}{\partial y^2} + \frac{\partial^2 G_3}{\partial z^2}\right)\hat{k} \;)$$

$$\operatorname{rot}(G) = \left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\hat{k}$$

$$\Rightarrow \operatorname{rot}(\operatorname{rot}(G)) = \left[\frac{\partial}{\partial y}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right)\right]\hat{\imath}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\right]\hat{\jmath}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right)\right]\hat{k}$$

$$= \left(\frac{\partial^2 G_2}{\partial y \partial x} - \frac{\partial^2 G_1}{\partial y^2} - \frac{\partial^2 G_1}{\partial z^2} + \frac{\partial^2 G_3}{\partial z \partial x} \right) \hat{\imath} + \left(\frac{\partial^2 G_3}{\partial z \partial y} - \frac{\partial^2 G_2}{\partial z^2} - \frac{\partial^2 G_2}{\partial x^2} + \frac{\partial^2 G_1}{\partial x \partial y} \right) \hat{\jmath}$$

$$+ \left(\frac{\partial^2 G_1}{\partial x \partial z} - \frac{\partial^2 G_3}{\partial x^2} - \frac{\partial^2 G_3}{\partial y^2} + \frac{\partial^2 G_2}{\partial y \partial z} \right) \hat{k}$$

Por otro lado, $\nabla(div(G)) = \nabla\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \right) \hat{\imath} + \frac{\partial}{\partial y}\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \right) \hat{\jmath} + \frac{\partial}{\partial z}\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \right) \hat{k}$$

$$= \left(\frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_2}{\partial x \partial y} + \frac{\partial^2 G_3}{\partial x \partial z} \right) \hat{\imath} + \left(\frac{\partial^2 G_1}{\partial y \partial x} + \frac{\partial^2 G_2}{\partial y^2} + \frac{\partial^2 G_3}{\partial y \partial z} \right) \hat{\jmath} + \left(\frac{\partial^2 G_1}{\partial z \partial x} + \frac{\partial^2 G_2}{\partial z \partial y} + \frac{\partial^2 G_3}{\partial z^3} \right) \hat{k}$$

Luego, $\nabla(div(G)) - rot(rot(G)) = \left(\frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_2}{\partial x \partial y} + \frac{\partial^3 G_3}{\partial x \partial z} - \frac{\partial^2 G_2}{\partial y \partial x} + \frac{\partial^2 G_1}{\partial y^2} + \frac{\partial^2 G_1}{\partial z^2} - \frac{\partial^2 G_3}{\partial z \partial x} \right) \hat{\imath}$

$$+ \left(\frac{\partial^2 G_1}{\partial y \partial x} + \frac{\partial^2 G_2}{\partial y^2} + \frac{\partial^2 G_3}{\partial y \partial z} - \frac{\partial^2 G_3}{\partial z \partial y} + \frac{\partial^2 G_2}{\partial z^2} + \frac{\partial^2 G_2}{\partial x^2} - \frac{\partial^2 G_1}{\partial x \partial y} \right) \hat{\jmath}$$

$$+ \left(\frac{\partial^2 G_1}{\partial z \partial x} + \frac{\partial^2 G_2}{\partial z \partial y} + \frac{\partial^2 G_3}{\partial z^2} - \frac{\partial^2 G_1}{\partial x \partial z} + \frac{\partial^2 G_3}{\partial x^2} + \frac{\partial^2 G_3}{\partial y^2} - \frac{\partial^2 G_2}{\partial y \partial z} \right) \hat{k}$$

$$= \left(\frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_1}{\partial y^2} + \frac{\partial^2 G_1}{\partial z^2} \right) \hat{\imath} + \left(\frac{\partial^2 G_2}{\partial x^2} + \frac{\partial^2 G_2}{\partial y^2} + \frac{\partial^2 G_2}{\partial z^2} \right) \hat{\jmath} + \left(\frac{\partial^2 G_3}{\partial x^2} + \frac{\partial^2 G_3}{\partial y^2} + \frac{\partial^2 G_3}{\partial z^2} \right) \hat{k}$$

$$= \Delta G \quad //$$

c) P.D.Q $div(F \times G) = G \cdot rot(F) - F \cdot rot(G)$

notemos que $G \cdot rot(F) - F \cdot rot(G) = G_1\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + G_2\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + G_3\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$
$- F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} \right) - F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} \right) - F_3\left(\frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial y} \right)$

Por otro lado

$div(F \times G) = div\left((F_2 G_3 - G_2 F_3)\hat{\imath} + (F_3 G_1 - G_3 F_1)\hat{\jmath} + (F_1 G_2 - G_1 F_2)\hat{k} \right)$

$$= \frac{\partial (F_2 G_3 - G_2 F_3)}{\partial x} + \frac{\partial}{\partial y}(F_3 G_1 - G_3 F_1) + \frac{\partial}{\partial z}(F_1 G_2 - G_1 F_2)$$

$$= F_2 \frac{\partial G_3}{\partial x} + G_3 \frac{\partial F_2}{\partial x} - G_2 \frac{\partial F_3}{\partial x} - F_3 \frac{\partial G_2}{\partial x} + F_3 \frac{\partial G_1}{\partial y} + G_1 \frac{\partial F_3}{\partial y} - G_3 \frac{\partial F_1}{\partial y} - F_1 \frac{\partial G_3}{\partial y} + F_1 \frac{\partial G_2}{\partial z} + G_2 \frac{\partial F_1}{\partial z} - G_1 \frac{\partial F_2}{\partial z} - F_2 \frac{\partial G_1}{\partial z}$$

$$= G_1\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + G_2\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + G_3\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} \right) - F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} \right) - F_3\left(\frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial y} \right)$$

$$= G \cdot rot(F) - F \cdot rot(G) \quad //$$

d) P.D.Q $\int_\Omega \nabla\varphi \cdot rot(F)\, dV = 0$

(rotor de un gradiente = 0)

Sea $G = \nabla\varphi$, usando la parte (c) $\nabla\varphi \cdot rot(F) = div(F \times \nabla\varphi) + \underbrace{F \cdot rot(\nabla\varphi)}_{0}$

$$\Rightarrow \int_\Omega \nabla\varphi \cdot rot(F)\, dV = \int_\Omega div(F \times \nabla\varphi)\, dV \overset{\text{TEO. GAUSS}}{=} \int_{\partial\Omega} (F \times \nabla\varphi) \cdot \hat{n}\, dA$$

Notemos que estamos sobre la frontera de Ω, es decir, donde $\varphi(x) = 0$ en Ω $\varphi(x) \leq 0$. Recordemos que $\nabla\varphi$ es la dirección de máximo ascenso, ésta debe necesariamente apuntar hacia afuera de Ω pues $\varphi(x) \leq 0$ en Ω y estamos en la frontera.

$\Rightarrow \nabla\varphi \parallel \hat{n}$ ($\nabla\varphi$ es paralelo a $\hat{n}$ (que siempre apunta hacia afuera))

$\Rightarrow (F \times \nabla\varphi) \cdot \hat{n} = 0$ pues $(F \times \nabla\varphi) \perp \nabla\varphi \Rightarrow (F \times \nabla\varphi) \perp \hat{n} \Rightarrow (F \times \nabla\varphi) \cdot \hat{n} = 0$

Luego se concluye que $\int_\Omega \nabla\varphi \cdot rot(F)\, dV = 0 \quad //$

12| a) En coordenadas cilíndricas, $x = \rho\cos\theta$, $y = \rho \operatorname{sen}\theta$, $z = z$

$\Rightarrow \vec{r}(\rho,\theta,z) = (\rho\cos\theta, \rho\operatorname{sen}\theta, z)$

$h_\rho = \left\|\frac{\partial \vec{r}}{\partial \rho}\right\| \qquad \frac{\partial \vec{r}}{\partial \rho} = (\cos\theta, \operatorname{sen}\theta, 0) \Rightarrow \left\|\frac{\partial \vec{r}}{\partial \rho}\right\| = \sqrt{\cos^2\theta + \operatorname{sen}^2\theta} = 1 = h_\rho$

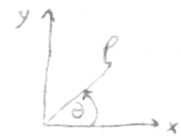
$h_\theta = \left\|\frac{\partial \vec{r}}{\partial \theta}\right\| \qquad \frac{\partial \vec{r}}{\partial \theta} = (-\rho\operatorname{sen}\theta, \rho\cos\theta, 0) \Rightarrow \left\|\frac{\partial \vec{r}}{\partial \theta}\right\| = \sqrt{\rho^2\cos^2\theta + \rho^2\operatorname{sen}^2\theta} = \rho = h_\theta$

$h_z = \left\|\frac{\partial \vec{r}}{\partial z}\right\| \qquad \frac{\partial \vec{r}}{\partial z} = (0, 0, 1) \Rightarrow \left\|\frac{\partial \vec{r}}{\partial z}\right\| = 1 = h_z$

b) $\hat{\rho} = \frac{1}{h_\rho}\frac{\partial \vec{r}}{\partial \rho} = (\cos\theta, \operatorname{sen}\theta, 0)$ $\qquad$ (*) Lo que falta está más abajo

$\hat{\theta} = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{\rho}(-\rho\operatorname{sen}\theta, \rho\cos\theta, 0) = (-\operatorname{sen}\theta, \cos\theta, 0)$

$\hat{z} = \frac{1}{h_z}\frac{\partial \vec{r}}{\partial z} = (0, 0, 1)$



c) Primero notemos que en coordenadas cilíndricas $\rho = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$

luego $F(\rho,\theta,z) = \begin{pmatrix} \rho\cos\theta - \theta\operatorname{sen}\theta \\ \rho\operatorname{sen}\theta + \theta\cos\theta \\ z \end{pmatrix} = \rho\begin{pmatrix}\cos\theta \\ \operatorname{sen}\theta \\ 0\end{pmatrix} + \theta\begin{pmatrix}-\operatorname{sen}\theta \\ \cos\theta \\ 0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix} = \rho\hat{\rho} + \theta\hat{\theta} + z\hat{z}$

$$\operatorname{div}(F) = \frac{1}{h_\rho h_\theta h_z}\left[\frac{\partial(h_\theta h_z F_\rho)}{\partial \rho} + \frac{\partial(h_\rho h_z F_\theta)}{\partial\theta} + \frac{\partial(h_\rho h_\theta F_z)}{\partial z}\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial(\rho^2)}{\partial\rho} + \frac{\partial(\theta)}{\partial\theta} + \frac{\partial(\rho z)}{\partial z}\right] = \frac{1}{\rho}\left[2\rho + 1 + \rho\right] = 3 + \frac{1}{\rho} = 3 + \frac{1}{\sqrt{x^2+y^2}} \quad //$$

$$\operatorname{rot}(F) = \frac{1}{h_\rho h_\theta h_z}\begin{vmatrix} h_\rho\hat{\rho} & h_\theta\hat{\theta} & h_z\hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_\rho h_\rho & F_\theta h_\theta & F_z h_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ \rho & \rho\theta & z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}\left(\frac{\partial z}{\partial\theta} - \frac{\partial(\theta\rho)}{\partial z}\right) - \rho\hat{\theta}\left(\frac{\partial z}{\partial\rho} - \frac{\partial\rho}{\partial z}\right) + \hat{z}\left(\frac{\partial(\theta\rho)}{\partial\rho} - \frac{\partial\rho}{\partial\theta}\right)\right] = \frac{\theta}{\rho}\hat{z} = \frac{\arctan(y/x)}{\sqrt{x^2+y^2}}\hat{z} \quad //$$

(*) continuación b

Veamos primero que $\hat{\rho}, \hat{\theta}, \hat{z}$ son ortogonales

Recordemos que v_1, v_2 son ortogonales sii $v_1 \cdot v_2 = 0$

$\hat{\rho}\cdot\hat{\theta} = \begin{pmatrix}\cos\theta\\ \operatorname{sen}\theta \\ 0\end{pmatrix}\cdot\begin{pmatrix}-\operatorname{sen}\theta \\ \cos\theta \\ 0\end{pmatrix} = -\cos\theta\operatorname{sen}\theta + \cos\theta\operatorname{sen}\theta = 0$

$\hat{\rho}\cdot\hat{z} = \begin{pmatrix}\cos\theta\\ \operatorname{sen}\theta \\ 0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

$\hat{\theta}\cdot\hat{z} = \begin{pmatrix}-\operatorname{sen}\theta\\ \cos\theta \\ 0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

el orden positivo es $(\hat{\rho}, \hat{\theta}, \hat{z})$ pues

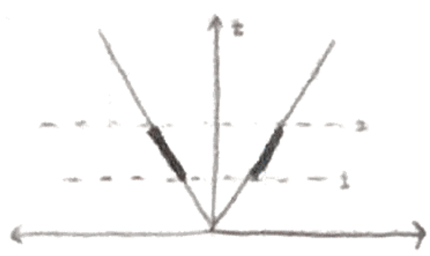
$\hat{\rho}\times\hat{\theta} = \begin{pmatrix}\cos\theta\\ \operatorname{sen}\theta \\ 0\end{pmatrix}\times\begin{pmatrix}-\operatorname{sen}\theta \\ \cos\theta \\ 0\end{pmatrix} = \begin{pmatrix}0-0\\0-0\\ \cos^2\theta + \operatorname{sen}^2\theta\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = \hat{z}$

$\hat{\theta}\times\hat{z} = \begin{pmatrix}-\operatorname{sen}\theta\\ \cos\theta \\ 0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0+\cos\theta\\ 0+\operatorname{sen}\theta \\ 0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \operatorname{sen}\theta\\ 0\end{pmatrix} = \hat{\rho}$

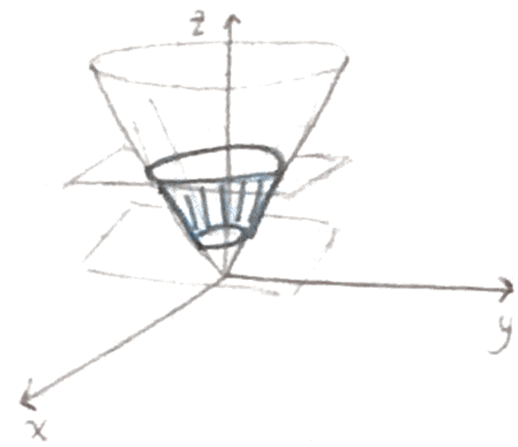
$\hat{z}\times\hat{\rho} = \begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}\cos\theta\\ \operatorname{sen}\theta \\ 0\end{pmatrix} = \begin{pmatrix}0-\operatorname{sen}\theta\\ \cos\theta - 0\\ 0\end{pmatrix} = \begin{pmatrix}-\operatorname{sen}\theta\\ \cos\theta\\ 0\end{pmatrix} = \hat{\theta}$

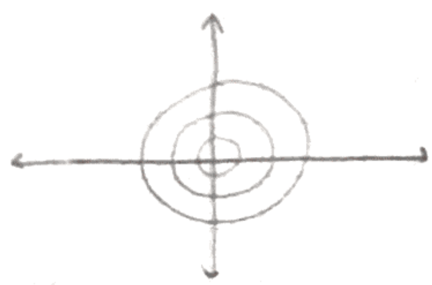
[3] a) $S = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 = x^2+y^2,\ 1 \leq z \leq 2\}$

Si fijamos z, tenemos una circunferencia de radio z.
Entonces la superficie está compuesta por circunferencias de radio entre 1 y 2.
Por otro lado, z es la altura, entonces cuando la circunferencia tiene radio z, también tiene altura z.



← vista de lado





← vista de arriba

Entonces S es un cono "cortado" entre $z=1$ y $z=2$

Parametricemos S:
la forma más fácil de hacerlo es en coordenadas cilíndricas

$x = \rho\cos\theta$
$y = \rho \operatorname{sen}\theta$ pero $z = \sqrt{x^2+y^2} = \rho$
$z = z$

$$\Rightarrow \vec{r}(\rho,\theta) = (\rho\cos\theta,\ \rho\operatorname{sen}\theta,\ \rho) \qquad \rho \in [1,2] \quad \theta \in [0,2\pi]$$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \int_0^{2\pi}\int_1^2 F(\vec{r}(\rho,\theta))\cdot\hat{n}\left\|\frac{\partial\hat{r}}{\partial\rho}\times\frac{\partial\hat{r}}{\partial\theta}\right\| d\rho\, d\theta \qquad \text{donde } \hat{n} = \frac{\frac{\partial\hat{r}}{\partial\rho}\times\frac{\partial\hat{r}}{\partial\theta}}{\left\|\frac{\partial\hat{r}}{\partial\rho}\times\frac{\partial\hat{r}}{\partial\theta}\right\|}$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dA = \int_0^{2\pi}\int_1^2 \hat{F}(\vec{r}(\rho,\theta))\cdot\left(\frac{\partial\hat{r}}{\partial\rho}\times\frac{\partial\hat{r}}{\partial\theta}\right) d\rho\, d\theta$$

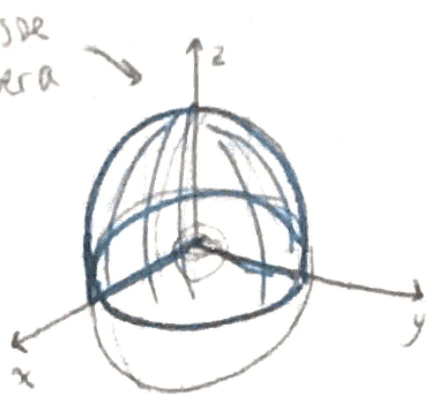
$$= \int_0^{2\pi}\int_1^2 (\rho\cos\theta\,\hat{\imath} + \rho\operatorname{sen}\theta\,\hat{\jmath} + \rho^2\hat{k})\cdot\left[\begin{pmatrix}\cos\theta\\ \operatorname{sen}\theta\\ 1\end{pmatrix}\times\begin{pmatrix}-\rho\operatorname{sen}\theta\\ \rho\cos\theta\\ 0\end{pmatrix}\right] d\rho\, d\theta \qquad \left((\hat{\rho}+\hat{k})\times\rho\hat{\theta} = \rho(\hat{k}-\hat{\rho})\right)$$

$$= \int_0^{2\pi}\int_1^2 (\rho\cos\theta\,\hat{\imath} + \rho\operatorname{sen}\theta\,\hat{\jmath} + \rho^2\hat{k})\cdot\rho(\hat{k}-\hat{\rho})\,d\rho\, d\theta = \int_0^{2\pi}\int_1^2 (\rho\hat{\rho} + \rho^2\hat{k})\cdot\rho(\hat{k}-\hat{\rho})\,d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 \rho^3-\rho^2\,d\rho\, d\theta = 2\pi\int_1^2 \rho^3-\rho^2\,d\rho = 2\pi\left[\frac{\rho^4}{4}-\frac{\rho^3}{3}\right]\Bigg|_1^2 = 2\pi\left[\frac{2^4-1}{4} - \frac{8-1}{3}\right] = \frac{17\pi}{6} \;//$$

b) $\Omega = \{(x,y,z)\in\mathbb{R}^3 \mid \underbrace{\overset{2^2}{4} \leq x^2+y^2+z^2 \leq \overset{3^2}{9}}_{\text{esfera con radio entre 2 y 3}},\ z \geq 0\}$

desde afuera



vista de lado

Parametricemos Ω
lo más cómodo es hacerlo en coordenadas esféricas

$x = r\operatorname{sen}\phi\cos\theta \qquad r \in [2,3]$
$y = r\operatorname{sen}\phi\operatorname{sen}\theta \qquad \theta \in [0,2\pi]$
$z = r\cos\phi \qquad \phi \in [0,\pi]$

$z \geq 0 \Rightarrow r\cos\phi \geq 0 \Rightarrow \phi \in [0,\pi/2]$

$\vec{r}(r,\theta,\phi) = (r\operatorname{sen}\phi\cos\theta,\ r\operatorname{sen}\phi\operatorname{sen}\theta,\ r\cos\phi) \qquad r\in[2,3],\ \theta\in[0,2\pi],\ \phi\in[0,\pi/2]$

$$\iint_{\partial\Omega} \vec{F}\cdot\hat{n}\, dA \underset{\substack{\text{Teo.}\\ \text{Gauss}}}{=} \iiint_{\Omega} \operatorname{div}(F)\, dV$$

$$F = x^2y\,\hat{j} + y^2z\,\hat{k} \Rightarrow \operatorname{div}(F) = \frac{\partial(x^2y)}{\partial y} + \frac{\partial(y^2z)}{\partial z} = x^2+y^2$$

$$\iiint x^2+y^2 \, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_2^3 (r\,sen\phi\cos\theta)^2 + (r\,sen\phi\,sen\theta)^2 \, dr\, d\phi\, d\theta \cdot h_r h_\theta h_\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_2^3 r^2 sen^2\phi \cdot r^2 sen\phi \, d\theta\, d\phi\, dr = 2\pi \int_0^{\pi/2}\int_2^3 r^4 sen^3\phi \, d\phi\, dr$$

$$= 2\pi \underbrace{\left(\frac{r^5}{5}\right)\Big|_2^3}_{\frac{211}{5}} \underbrace{\int_0^{\pi/2} sen^3\phi\, d\phi}$$

$$= \int_0^{\pi/2} sen\phi\, sen^2\phi\, d\phi = \int_0^{\pi/2} (1-\cos^2\phi)\, sen\phi \, d\phi = sen\phi - \cos^2\phi\, sen\phi$$

$$= -\cos\phi + \frac{\cos^3\phi}{3}\Big|_0^{\pi/2} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow \iiint_{\Omega} \operatorname{div}(F)\, dV = \frac{2\pi}{5} \cdot 211 \cdot \frac{2}{3} = \frac{844\pi}{15} \quad //$$