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Facultad de Ciencias Físicas y Matemáticas
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CONTROL 2: MA2002 Cálculo Avanzado y Aplicaciones

Problema 1.

a) Encuentre el radio de convergencia de la serie $\displaystyle\sum_{n=0}^{\infty} \frac{e^{in}}{2n+1}\left(\frac{2}{3i}\right)^n (z+4i)^n$.

b) Encuentre la serie de Taylor de $f(z) = z\log(z)$ en torno a $z_0 = 1+i$ y determine su radio de convergencia.

c) Sea $f : \mathbb{C} \to \mathbb{C}$ holomorfa tal que $f''(z) = 2f(z) + 1$ con $f(0) = 1, f'(0) = 0$. Encuentre la serie de potencias de f en torno a 0 y determine su radio de convergencia.

d) Sea f holomorfa en $\mathbb{C}$. Pruebe que si $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, o $|f|$ es constante, entonces f es constante.

Problema 2.

a) Calcule $\displaystyle\oint_{|z|=4} \frac{1}{z^2 \operatorname{senh}(z)} dz$, con la circunferencia $|z| = 4$ recorrida en sentido antihorario.

b) Pruebe que $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\alpha^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} \exp(2\alpha\cos\theta)d\theta$

Indicación: Comience probando que $\displaystyle\left(\frac{\alpha^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_{|z|=1} \frac{\alpha^n \exp(\alpha z)}{n! z^{n+1}} dz$.

Problema 3.

a) Calcule $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{a + \operatorname{sen}^2\theta} d\theta$ con $a > 0$.

b) Calcule $\displaystyle\int_0^{\infty} \exp(-x^2)\cos(2\beta x)dx$ donde $\beta > 0$.

Indicación: Integre $f(z) = \exp(-z^2)$ sobre el rectángulo de vértices $R, R+i\beta, -R, -R+i\beta$ y considere el límite $R \to \infty$ probando que las integrales sobre los lados verticales tienden a cero.

Pauta Control 2

P1. **a)** $\sum\limits_{n\to 0} a_m(z+4i)^n$ con $a_n = \frac{e^{in}}{2n+1}\left(\frac{2}{3i}\right)^n$

$R = 1/\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$

$|a_n| = \frac{|e^{in}|}{2n+1}|\frac{2}{3i}|^n = (\frac{2}{3})^n\frac{1}{2n+1} \Rightarrow \sqrt[n]{|a_n|} = \frac{2}{3}\frac{1}{\sqrt[n]{2n+1}} \to \frac{2}{3} \Rightarrow R = \frac{3}{2}$ (1.5 pts.)

b) $f(z) = z\log z \Rightarrow f'(z) = \log z + 1 \Rightarrow f''(z) = \frac{1}{2} \Rightarrow f'''(z) = -\frac{1}{z^2} \Rightarrow f^{(iv)}(z) = \frac{2}{z^3}\cdots$ etc

En general $f^{(k)}(z) = (-1)^k\frac{(k-2)!}{z^k-1} \qquad \forall k \leq 2$ (0.5 pts.)

$$\begin{cases} a_0 &= f(1+i) = (1+i)\log(1+i) \\ a_1 &= \log(1+i)+1 \\ a_k &= \frac{(-1)^k}{k!}\frac{(k-2)!}{(1+i)^{k-i}} = \frac{(-1)^k}{k(k-1)}\cdot\frac{1}{(1+i)^{k-1}} \end{cases}$$

La serie queda

$S(z) = (1+i)\log(1+i) + [\log(1+i)+1](z-1-i) + \sum\limits_{k=2}^{\infty}\frac{(-1)^k}{k(k+1)}\cdot\frac{1}{(1+i)^{k-1}}(z-1-i)^k$ (0.5 pts)

El radio de convergencia es:

$R = 1/\limsup\sqrt[k]{\left|\frac{(-1^k}{k(k-i)}\cdot\frac{1}{(1+i)^k}\right|} = 1/\limsup\sqrt[k]{\frac{1}{k(k-1)\sqrt{2}^k}} = \sqrt{2}$ (0.5 pts)

Alternativamente: el radio de convergencia puede calcularse como la distancia desde el punto $(1+i)$ al complemento del dominio donde f es holomorfo: $\text{dist}(1+i),\ \mathbb{C}\setminus\mathbb{R}_-) = \sqrt{2}$.

c) $f''(z) = 2f(z)+1 \Rightarrow f'''(x) = 2f'(z) \Rightarrow f^{(k+2)}(z) = 2f^{(k)}(z) \qquad \forall k \geq 1$.

Dado que $f'(0) = 0$ se sigue que $f^{(k)}(0) = 0$ para todo k impar.

Asimismo, $f''(0) = 2f(0)+1 = 3 \Rightarrow f^{(2k)}(0) = 3\cdot 2^{k-1}\ \forall k > 1$. (0.5pts).

Con ello la serie resulta ser: $S(z) = 1 + \sum\limits_{k=1}^{\infty} 3\cdot\frac{2^{k-1}z^{2k}}{(2k)!} = \sum\limits_{n\geq 0} a_n z^n$ (0.5 pts)

$$a_n = \begin{cases} 0 & \text{si} & n \text{ impar} \\ 1 & \text{si} & n = 0 \\ 3\cdot\frac{2^{k-1}}{(2k)!} & \text{si} & n = 2k \end{cases}$$

El radio de convergencia es $\mathbb{R} = 1/\limsup\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1/\limsup\sqrt[2k]{\frac{3\cdot 2^{k-1}}{(2k)!}} = \frac{1}{0} = \infty$. (0.5 pts)

d) Sea $f(x) = u(x,y) + iv(x,y)$.

- Si $u(x,y) =$ constante $\Rightarrow \left.\begin{array}{r} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0 \\ \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} = 0 \end{array}\right\}$ Cauchy-Riemann
  $\Rightarrow \nabla v = 0 \Rightarrow v(x,y) =$ constante $\Rightarrow f(z)$ constante (0.5 pts)
- Si $v(x,y) =$ constante $\Rightarrow \left.\begin{array}{r} \frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0 \end{array}\right\} \Rightarrow \nabla u = 0 \Rightarrow$ constante
  $\Rightarrow f$ constante (0.5 pts)
- Si $|f(z)|$ es constante $\Rightarrow f(x)$ acotada y holomorfa en $\mathbb{C}$.
  Luego, por Teo. de Liouville $f(z)$ es constante. (0.5 pts)

P2. **a)** $f(z) = \frac{1}{z^2\,\text{senh}(z)}$ tiene un polo de orden 3 en $p = 0$ y polos simples en las restantes raíces de $\text{sen}\,h(x)$, vale decir, $p_k = k\pi i \quad k \in \mathbb{Z},\ k \neq 0$ (0.5 pts)

La curva $|z| = 4$ encierra los polos 0 ,πi, $-\pi i$.

Los residuos correspondientes son:

$Res(f,\pi i) = \lim\limits_{z\to\pi i}(z-\pi i)\frac{1}{z^2\,\mathrm{senh}(z)} = \frac{1}{(\pi i)^2\cosh(\pi i)} = \frac{1}{\pi^2}$

$Res(f,-\pi i) = \lim\limits_{z\to-\pi i}(z-\pi i)\frac{1}{z^2\,\mathrm{senh}(z)} = \frac{1}{(-\pi i)^2\cosh(-\pi i)} = \frac{1}{\pi^2}$ (0.5 pts)

$$\begin{aligned}
Res(f,0) &= \lim_{z\to0}\tfrac{1}{2!}\tfrac{d^2}{dz^2}\left[z^3\cdot\tfrac{1}{z^2\,\mathrm{senh}(z)}\right]\\
&= \lim_{z\to0}\tfrac{1}{2}\tfrac{d^2}{dz^2}\left[\tfrac{z}{\mathrm{senh}\,z}\right]\\
&= \lim_{z\to0}\tfrac{1}{2}\tfrac{d}{dz}\left[\tfrac{\mathrm{senh}\,z-z\cosh z}{\mathrm{sen}\,h^2 z}\right]\\
&= \lim_{z\to0}\tfrac{1}{2}\tfrac{d}{dz}\tfrac{[\cosh z-\cosh z-z\,\mathrm{senh}\,z]\,\mathrm{senh}^2 z-[\mathrm{senh}\,z-z\cosh z]2\,\mathrm{senh}\,z\cosh z}{\mathrm{senh}^4 z}\\
&= \lim_{z\to0}\tfrac{1}{2}\left[\tfrac{-z}{\mathrm{senh}\,z}-\tfrac{[\mathrm{senh}\,z-z\cosh z]2\cosh(z)}{\mathrm{senh}^3 z}\right]
\end{aligned}$$

$\lim\limits_{z\to0}\frac{\mathrm{senh}\,z-z\cosh z}{\mathrm{senh}^3 z} = \lim\frac{\cosh z-\cosh x-z\,\mathrm{senh}\,z}{3\,\mathrm{sen}\,h^2 z\cosh z} = \lim\limits_{z\to0}-\frac{z}{3\,\mathrm{senh}\,z\cosh z} = -\frac{1}{3}$

$\Rightarrow Res(f,0) = \frac{1}{2}[-1-2(-\frac{1}{3})] = \frac{1}{2}[\frac{2}{3}-1] = \frac{1}{2}\cdot\frac{(-1)}{3} = -\frac{1}{6}$ (1.0 pts)

$\Rightarrow \int\limits_{|z|=4}\frac{1}{z^2\,\mathrm{sen}\,hz}dz = 2\pi i\cdot[\frac{1}{\pi^2}+\frac{1}{\pi^2}-\frac{1}{6}] = 2\pi i[\frac{2}{\pi^2}-\frac{1}{6}]$ (0.5pts)

b)

$$\begin{aligned}
\int\limits_{|z|=1}\tfrac{\exp(\alpha z)}{z^{n+1}}dz &= 2\pi i\ Res(\tfrac{e^{\alpha z}}{z^{n+1}},0) = 2\pi i\cdot\lim_{z\to0}\tfrac{1}{n!}\tfrac{d^n}{dz^n}[z^{n+1}\cdot\tfrac{e^{\alpha z}}{z^{n+1}}]\\
&= \tfrac{2\pi i}{n!}\tfrac{d^n}{dz^n}[e^{\alpha z}]|_{z=0}\\
&= \tfrac{2\pi i}{n!}\alpha^n e^{\alpha z}|_{z=0}\\
&= 2\pi i\tfrac{\alpha^n}{n!}
\end{aligned}$$

$\Rightarrow (\frac{\alpha^n}{n!})^2 = \frac{1}{2\pi i}\int\limits_{|z|=1}\frac{\alpha^n}{n!}\frac{\exp(\alpha z)}{z^{n+1}}dz$ (1.0 pts)

Alternativa: usar directamente las fórmulas de Cauchy $\frac{1}{2\pi}\int\limits_{|z|=1}\frac{f(z)}{z^{n+1}}dz = \frac{f^{(n)}(0)}{n!}$)

$$\begin{aligned}
\sum_{n=0}^{\infty}(\tfrac{\alpha^n}{n!})^2 &= \tfrac{1}{2\pi i}\int\limits_{|z|=1}\tfrac{\exp(\alpha z)}{z}\underbrace{(\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{\alpha}{z})^n)}_{\exp(\frac{\alpha}{z})}dz\\
&= \tfrac{1}{2\pi i}\int\limits_{|z|=1}\tfrac{\exp(\alpha z)}{z}\cdot\exp(\tfrac{\alpha}{z})dz\\
&= \tfrac{1}{2\pi i}\int\limits_{|z|=1}\tfrac{\exp(\alpha(z+\frac{1}{z}))}{z}dz \qquad \text{(1.0 pts)}\\
&= \tfrac{1}{2\pi i}\int_0^{2\pi}\tfrac{\exp(\alpha(e^{i\theta}+\frac{1}{e^{i\theta}}))}{e^{i\theta}}\cdot ie^{i\theta}d\theta\\
&= \tfrac{1}{2\pi}\int_0^{2\pi}\exp(\alpha\underbrace{(e^{i\theta}+e^{-i\theta})}_{2\cos(\theta)})d\theta\\
&= \tfrac{1}{2\pi i}\int_0^{2\pi}\exp(2\alpha\cos\theta)d\theta. \qquad \text{(1.0 pts)}
\end{aligned}$$

P3. **a)** (Primera solución)

$$\int_{-\pi/2}^{\pi/2} \frac{1}{a+\operatorname{sen}^2\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{a+\operatorname{sen}^2\theta} d\theta = \frac{1}{2}\int_{|z|=1} \frac{1}{a+(\frac{z-1/z}{2i})^2}\cdot\frac{1}{iz}\cdot dz$$

$$= \frac{1}{2i}\int_{|z|=1} \frac{4z}{\underbrace{4az^2-|z^2-1|^2}_{q(z)}} dz \qquad \text{(0.5 pts)}$$

Polos: $q(z) = 4az^2 - (z^2-1)^2 = 0 \Leftrightarrow 4az^2 = (z^2-1)^2 \Leftrightarrow z^2 - 1 = \pm 2\sqrt{a}z$ (1.0 pts)

$$\begin{aligned} z^2 - 2\sqrt{a}z - 1 = 0 \quad &\Leftrightarrow \quad z = \sqrt{a} \pm \sqrt{a+1} \\ z^2 + 2\sqrt{a}z - 1 = 0 \quad &\Leftrightarrow \quad z = -\sqrt{a} \pm \sqrt{a+1} \end{aligned}$$

Así tenemos 4 polos simples: $p_1 = \sqrt{a} + \sqrt{a+1}$, $p_2 = \sqrt{a} - \sqrt{a+1}$, $p_3 = -\sqrt{a} + \sqrt{a+1}$, $p_4 = -\sqrt{a} - \sqrt{a+1}$

Solamente p_2 y p_3 están dentro del círculo unitario.

Residuos: (1.0 pts)

$$\begin{aligned} Res(f,p_2) &= \lim_{z\to p_2}(z-p_2)\frac{4z}{q(z)} = \frac{4p_2}{q'(p_2)} = \frac{4p_2}{8ap_2 - 2(p_2^2-1)2p_2} = \frac{4}{8a-4(p_2^2-1)} \\ &= \frac{4}{8a-4(a+(a+1)-2\sqrt{a(a+1)}-1)} = \frac{4}{8\sqrt{a(a+1)}} = \frac{1}{2\sqrt{a(a+1)}} \\ Res(f,p_3) &= \lim_{z\to p_3}(z-p_3)\frac{4z}{q(z)} = \frac{4p_3}{q'(p_3)} = \frac{4}{8a-4(p_3^2-1)} = \frac{1}{2\sqrt{a(a+1)}} \end{aligned}$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{1}{a+\operatorname{sen}^2\theta} d\theta = \frac{1}{2i}\cdot 2\pi i\cdot[Res(f,p_2) + (Res(f,p_3)] = \frac{\pi}{\sqrt{a(a+1)}} \qquad \text{(0.5 pts)}$$

a) (Segunda solución) $\operatorname{sen}^2\theta = \frac{1-\cos 2\theta}{2}$

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \frac{1}{a+\operatorname{sen}^2\theta} d\theta &= \int_{-\pi/2}^{\pi/2} \frac{1}{a+\frac{1-\cos 2\theta}{2}} d\theta = \int_{-\pi}^{\pi} \frac{1}{a+\frac{1-\cos 2}{2}}\frac{1}{2}dx \\ &= \int_0^{2\pi} \frac{1}{2a+1-\cos\theta} d\theta \\ &= \int_{|z|=1} \frac{1}{2a+1-(\frac{z+1/z}{2})}\cdot\frac{1}{iz}dz \\ &= \frac{2}{i}\int_{|z|=1} \frac{1}{2(2a+1)z - z^2 - 1} dz \end{aligned} \qquad \text{(1.0 pts)}$$

Polos:

$$\begin{aligned} z^2 - 2(2a+1)z + 1 = 0 \quad &\Leftrightarrow \quad z = (2a+1) \pm \sqrt{(2a+1)^2 - 1} \\ &\Leftrightarrow \quad z = (2a+1) \pm \sqrt{4a^2+4a} \quad \text{polos simples} \end{aligned}$$

El único polo encerrado por $|z| = 1$ es $p = (2a+1) - \sqrt{(2a+1)^2 - 1}$ (0.5 pts)

$$\begin{aligned} Res(f,p) = \lim_{z\to p}(z-p)f(z) &= \frac{1}{q'(p)} = \frac{1}{2(2a+1)-2p} = \frac{1}{2(2a+1)-2(2a+1)+2\sqrt{(2a+1^2)-1}} \\ &= \frac{1}{2\sqrt{4a^2+4a}} = \frac{1}{4\sqrt{a(a+1)}} \end{aligned}$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{1}{a+\operatorname{sen}^2\theta} d\theta = \frac{2}{i}2\pi i\cdot\frac{1}{4\sqrt{a(a+1)}} = \frac{\pi}{\sqrt{a(a+1)}} \qquad \text{(0.5 pts)}$$

b)

$$0 = \int_C \exp(-z^2)dz = \underbrace{\int_{-R}^{R} \exp(-x^2)dx}_{I_1} + \underbrace{\int_0^{\beta} \exp(-(R+iy)^2)idy}_{I_2} + \underbrace{\int_R^{-R} \exp(-(x+i\beta)^2)dx}_{I_3}$$
$$+ \underbrace{\int_{\beta}^{0} \exp(-(-R+iy)^2)idy}_{I_4} \quad (1)$$

(1.0 pts)

$$\lim_{R\to\infty} I_1 = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \qquad (0.5 \text{ pts})$$

$$\begin{aligned}
\lim_{R\to\infty} |I_2| &= \lim_{R\to\infty} \left| \int_0^{\beta} e^{-R^2+y^2-2Riy} idy \right| \\
&\le \lim_{R\to\infty} \int_0^{\beta} e^{-R^2} e^{\beta^2} dy = 0 \\
\lim_{R\to\infty} |I_4| &= \lim_{R\to\infty} \left| \int_0^{\beta} e^{-R^2+y^2+2Riy} idy \right| \\
&\le \lim_{R\to\infty} \int_0^{\beta} e^{-R^2+\beta^2} dy = 0
\end{aligned}$$

$$\begin{aligned}
\lim_{R\to\infty} I_3 = -\int_{-\infty}^{\infty} e^{-x^2} e^{\beta^2} e^{2ixB} dx &= -e^{\beta^2} \int_{-\infty}^{\infty} e^{-x^2} [\cos 2x\beta + i \operatorname{sen} 2x\beta] dx \\
&= -2e^{\beta^2} \int_0^{\infty} e^{-x^2} \cos 2x\beta dx
\end{aligned}$$

Reemplazando en **(1)**

$\Rightarrow \sqrt{\pi} = 2e^{\beta^2} \int_0^{\infty} e^{-x^2} \cos 2x\beta dx \Rightarrow \int_0^{\infty} e^{-x^2} \cos 2x\beta dx = \frac{\sqrt{\pi}}{2} e^{-\beta^2}$ (0.5 pts)

b) (Segunda forma) (0.5 pts)

$$0 = \int_C e^{-z^2}dz = \underbrace{\int_0^R e^{-x^2}dx}_{\to \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}} + \underbrace{\int_0^{\beta} e^{-(R+iy)^2} idy}_{\to 0} + \underbrace{\int_R^0 e^{-x+i\beta)^2}dx}_{\to -\int_0^{\infty} e^{-x^2+\beta^2-2xi\beta}dx} + \int_{\beta}^{0} e^{-(iy)^2} idy \qquad (1.0 \text{ pts})$$

Nota: la segunda integral se resuelve igual que en la primera forma.

$\Rightarrow \int_0^{\infty} e^{-x^2} e^{\beta^2} [\cos 2x\beta + i \operatorname{sen} 2x\beta] dx = \frac{\sqrt{\pi}}{2} - \int_0^{\beta} e^{y^2} idy$ (1.0 pts)

Igualando las partes reales obtenemos

$$e^{\beta^2} \int_0^{\infty} e^{-x^2} \cos 2x\beta dx = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \int_0^{\infty} e^{-x^2} \cos 2x\beta dx = \frac{\sqrt{\pi}}{2} e^{-\beta^2}$$

(0.5 pts)