Aux #8

$\boxed{P1}$ $I_n = \int_0^{\pi/4} \tan^n(x)\,dx$

pdq $\quad I_n + I_{n-2} = \dfrac{1}{n-1} \qquad \forall n \geq 2$

Dem: notar que no es un problema de conectar recurrencias, nos piden probar una igualdad

Partamos por la izq

$$I_n + I_{n-2} = \int_0^{\pi/4} \operatorname{Tan}^n(x)\,dx + \int_0^{\pi/4} \operatorname{Tan}^{n-2}(x)\,dx$$

$$= \int_0^{\pi/4} \operatorname{Tan}^n(x) + \operatorname{Tan}^{n-2}(x)\,dx$$

$$= \int_0^{\pi/4} \operatorname{Tan}^{n-2}(x)\underbrace{(\operatorname{Tan}^2(x) + 1)}_{\sec^2(x)}\,dx$$

$$= \int_0^{\pi/4} \underbrace{\operatorname{Tan}^{n-2}(x)}_{u^{n-2}}\,\underbrace{\sec^2(x)\,dx}_{du}$$

Cambio de Variable

$u = \tan(x)$; $du = \sec^2(x)dx$

Si $x = 0 \Rightarrow u = \operatorname{Tan}(0) = 0$

Si $x = \pi/4 \Rightarrow u = \operatorname{Tan}(\frac{\pi}{4}) = 1$

$$= \int_0^1 u^{n-2}\,du = \left.\frac{u^{n-1}}{n-1}\right|_0^1$$

$$= \frac{1^{n-1}}{n-1} - \frac{0^{n-1}}{n-1} = \frac{1}{n-1}$$

$\boxed{P2}$ $u, v: \mathbb{R} \to \mathbb{R}$ derivables y $G(x) = \int_{v(x)}^{u(x)} f(t)\,dt$.

a) pdq $G'(x) = f(u(x))\,u'(x) - f(v(x))\,v'(x)$

En efecto por TFC, si: $F' = f$ pues f es continua

$\Rightarrow G(x) = \int_{V(x)}^{u(x)} f(t)\,dt \underset{TFC}{=} F(u(x)) - F(V(x))$

$\Rightarrow G'(x) = \left(F(u(x)) - F(v(x))\right)'$

$= f(u(x))\,u'(x) - f(v(x))\,v'(x)$

algebra de derivadas y regla de la cadena

b) $F(x) = \int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt$

pdq: $F(x)$ no depende de x y calcule su valor

que no dependa de x, es lo mismo que $F(x) = C$, $C \in \mathbb{R}$
luego si $F'(x) = 0 \Rightarrow F(x) = C$

$\Rightarrow$ Aplicando lo de (a) sobre cada integral

$F'(x) = \frac{1}{1+x^2} \cdot (x)' - \cancel{\frac{1}{1+0^2} \cdot 0'} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(\frac{1}{x}\right)' - \cancel{\frac{1}{1+0^2} \cdot 0'}$

$= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2}$

$= \frac{1}{1+x^2} + \frac{-1}{1+x^2} = 0 \Rightarrow F(x) = C \quad \forall x > 0$

Como $F(x) = c \quad \forall x > 0$, podemos tomar $x = 1$ y seguirá cumpliendose que $F(x) = c$

$$\Rightarrow F(1) = \int_0^1 \frac{1}{1+t^2}\,dt + \int_0^1 \frac{1}{1+t^2}\,dt$$

$$= 2 \int_0^1 \frac{1}{1+t^2}\,dt = 2\left(\arctan(t)\Big|_0^1\right)$$

$$= 2(\arctan(1) - \arctan(0))$$

Recuerda
- $\tan\left(\frac{\pi}{4}\right) = 1 \quad / \arctan()$

$\Rightarrow \frac{\pi}{4} = \arctan(1)$

- $\tan(0) = 0 \quad / \arctan()$

$\Rightarrow 0 = \arctan(0)$

$$= 2\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{2}$$

$$\therefore F(x) = \frac{\pi}{2}$$

$\boxed{P3}$

$$\lim_{x\to 0} \frac{\int_0^{x^2} \operatorname{sen}(t^2)\,dt}{1 - e^{x^6}} \xrightarrow{x\to 0} \frac{\int_0^0 \operatorname{sen}(t^2)\,dt}{1 - e^0} = \frac{0}{0} \quad \text{L'hôpital}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\operatorname{sen}((x^2)^2)\cdot(x^2)' - \operatorname{sen}(0^2)\cdot 0'}{-e^{x^6}\cdot 6x^5}$$

$$= \lim_{x\to 0} \frac{\operatorname{sen}(x^4)\cdot \cancel{2}x}{-e^{x^6}\cdot \cancel{6}x^5} = -\frac{1}{3} \lim_{x\to 0} \underbrace{\frac{\operatorname{sen}(x^4)}{x^4}}_{\to 1} \cdot \underbrace{\frac{1}{e^{x^6}}}_{\to 1}$$

$$= -\frac{1}{3}$$

b) $G(x) = \int_0^x \frac{\operatorname{sen}(t)}{t}\,dt$, pdq $\int_0^{\pi/2} G(x)\,dx = \frac{\pi}{2}\int_0^{\pi/2} \frac{\operatorname{sen}(t)}{t}\,dt - 1$

En efecto, de izq a derecha

$\int_0^{\pi/2} G(x)\,dx$ por partes $u = G(x) \Rightarrow du = G'(x)\,dx$

$dv = dx \Rightarrow v = x$

$= x\,G(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} x\,G'(x)\,dx$ con $G'(x) = F'(x) - \cancel{F'(0)}^{0}$

donde $F' = \frac{\operatorname{sen}(t)}{t}$

$= \frac{\pi}{2} G\left(\frac{\pi}{2}\right) - \cancel{0\,G(0)}^{0} - \int_0^{\pi/2} x \left(\frac{\operatorname{sen}(x)}{x}\right) dx$

$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\operatorname{sen}(t)}{t}\,dt - \int_0^{\pi/2} \operatorname{sen}(x)\,dx$

$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\operatorname{sen}(t)}{t}\,dt - \left(-\cos(x)\Big|_0^{\pi/2}\right)$

$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\operatorname{sen}(t)}{t}\,dt + \cancel{\cos\left(\frac{\pi}{2}\right)}^{0} - \cancel{\cos(0)}^{1}$

$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\operatorname{sen}(t)}{t}\,dt - 1$ ■

$\boxed{P4}$ a) pdq: $\left| \int_1^{\sqrt{3}} \frac{e^{-x}\operatorname{sen}(x)\,dx}{1+x^2} \right| \leq \frac{\pi}{12e}$

Recuerdo: $\left|\int f\right| \leq \int |f|$ y $\int f < \int g$ si $f < g$

$$\Rightarrow \left| \int_1^{\sqrt{3}} \frac{e^{-x}\operatorname{sen}(x)\,dx}{1+x^2} \right| \leq \int_1^{\sqrt{3}} \left| \frac{e^{-x}\operatorname{sen}(x)}{1+x^2} \right| dx$$

$|\operatorname{sen}(x)| \leq 1$

$$\leq \int_1^{\sqrt{3}} \frac{e^{-x}}{1+x^2}\,dx$$

$$\leq \int_1^{\sqrt{3}} \frac{e^{-1}}{1+x^2}\,dx$$

Recordemos que e^{-x} es decreciente $\Rightarrow e^{-1} < e^{x}, \forall x \in [1, \sqrt{3}]$

$$= \frac{1}{e} \int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{e} \left(\arctan(x) \Big|_1^{\sqrt{3}} \right)$$

$$= \frac{1}{e} \left(\arctan(\sqrt{3}) - \arctan(1) \right)$$

$$= \frac{1}{e} \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$= \frac{1}{e} \frac{\pi}{12} = \frac{\pi}{12e} \quad \blacksquare$$

Recordemos

$\tan\left(\frac{\pi}{3}\right) = \frac{\operatorname{sen}(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\tan\left(\frac{\pi}{4}\right) = 1$

$\boxed{P4}$ b) $\int_{-a}^{a} f(t)\,dt = 0$ Con f impar

$$\int_{-a}^{a} f(t)\,dt = \underbrace{\int_{-a}^{0} f(t)\,dt}_{\text{Cambio de variable}} + \int_{0}^{a} f(t)\,dt$$

Cambio de variable

$u = -t \quad du = -dt$

$\hookrightarrow t = 0 \Rightarrow u = 0$

$t = -a \Rightarrow u = a$

$$= \int_{a}^{0} f(-u) - du + \int_{0}^{a} f(t)\,dt$$

Recuerdo

$\int_a^b f = -\int_b^a f$

$$= -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(t)\,dt$$

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(t)\,dt$$

impar

$$= \cancel{-\int_{0}^{a} f(u)\,du} + \cancel{\int_{0}^{a} f(t)\,dt} = 0$$

$\boxed{P5}$ $f: \mathbb{R} \to \mathbb{R}$, $f \in C^2(\mathbb{R})$, $f''(x) > 0 \quad \forall x \in \mathbb{R}$

$G(x) = \int_x^{x+1} (x-t) f(t)\,dt$. PDQ: $G''(x) < 0 \quad \forall x \in \mathbb{R}$

En efecto, calculemos usando TFC

notar que $G(x) = x\int_x^{x+1} f(t)\,dt - \int_x^{x+1} t f(t)\,dt$

$$\Rightarrow G'(x) = \left(x \int_x^{x+1} f(t)\,dt\right)' - \left(\int_x^{x+1} t f(t)\,dt\right)'$$

$$= 1\int_x^{x+1} f(t)\,dt + x\left[f(x+1) - f(x)\right] - \left[(x+1)f(x+1) - x f(x)\right]$$

$$= \int_x^{x+1} f(t)\,dt + \cancel{x f(x+1)} - \cancel{x f(x)} - \cancel{(x+1)} f(x+1) + \cancel{x f(x)}$$

$$= \int_x^{x+1} f(t)\,dt - f(x+1)$$

$$\Rightarrow G''(x) = f(x+1) - f(x) - f'(x+1) \cdot \cancel{(x+1)'}^{\,1}$$

$$= f(x+1) - f(x) - f'(x+1)$$

$$= \underbrace{\frac{f(x+1) - f(x)}{(x+1) - x}} - f'(x+1)$$

$x + 1 - x = 1$

Como f es derivable y continua $\overset{TVM}{\Rightarrow}$ $\exists\, \xi \in (x, x+1)$ $\dfrac{f(x+1) - f(x)}{(x+1) - x} = f'(\xi)$

$$\Rightarrow G''(x) = f(x+1) - f(x) - f'(x+1)$$

$$= f'(\xi) - f'(x+1)$$

TVM

Como $f'' > 0 \Rightarrow f'$ es creciente y como $\xi < x+1$

$$\Rightarrow f'(\xi) < f'(x+1) \Rightarrow f'(\xi) - f'(x+1) < 0$$

$$\Rightarrow G''(x) = f'(\xi) - f'(x+1) < 0 \quad \boxtimes$$