Matching and Market design

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School Choice

- School choice: basic theory and recent developments.
  - The Boston Case
    - The Boston Mechanism
    - The DA Mechanism
    - The TTC
  - The New York City Case
  - Equity vs. Efficiency
  - Tie-Breaking

Overview

- School choice is a novel process: In many countries, children were automatically sent to a school in their neighborhoods.
- Recently, more and more cities employ school choice programs: school authorities take into account preferences of children and their parents.
- Typical goals of school authorities are:
  - (1) efficient placement,
  - (2) fairness of outcomes,
  - (3) easy for participants to understand and use, etc.

The Boston Experience

- Abdulkadiroglu and Sonmez (2003) showed that placement mechanisms used in many cities such as Boston are flawed, and proposed new mechanisms to improve upon existing placement mechanisms.
- Based on this and other studies, Boston and New York City changed their student placement mechanisms.
- Many studies are currently conducted to evaluate the current school choice mechanisms, and several mechanisms are proposed to improve the outcome.

The Model

- There is a finite sets S of students and C of schools.
- Each student can be matched to at most one school, and each school can admit at most q_c students.
- Each student s has strict preferences s over schools and being unmatched (denoted by s).
- For each school, there is a (for now, strict) priority order over students. $s \succ_c s'$ means "student s has higher priority for c than s' ."
The outcome is a matching, which specifies which student attends which school.

Stability

The model is isomorphic to the many-to-one matching, so a matching is stable if it is individually rational and it is not blocked by a school-student pair. Formally:

Definition

A matching μ is stable if:

- No blocking by an individual. $\mu(s)$ is acceptable to each student s, each $s \in \mu(c)$ is acceptable to c for each school c, and $\mid \mu(c) \mid \leq q_c$.
- No blocking by a pair. There is no pair s and c such that $c \succ_s \mu(s)$ and $\mid \mu(c) \mid < q_c$ and $,s \succ_c \emptyset$, or $s \succ_c s'$ for some $s' \in \mu(c)$.

Stability as fairness

- In school choice, stability can be understood as a fairness criterion.
- No blocking individual simply means no one is forced to attend an unacceptable school, and only qualified students can be admitted to a school (in some districts, all students are acceptable. Such cases are special cases.)
- No blocking pair means no justified envy. That is, there is no situation in which student s is matched to a worse school than school c , and c admits another student who has lower priority at c than s does.
- So stability may be a reasonable property we want for school choice mechanisms.

The Boston Mechanism

The Boston mechanism:

Step 0: Each student submits a preference ranking of the schools.

Step 1: In Step 1 only the top choices of the students are considered. For each school, consider the students who have listed it as their top choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her top choice.

Step k: Consider the remaining students. In Step k only the k-th choices of these students are considered. For each school still with available seats, consider the students who have listed it as their k-th choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her k-th choice.

The Boston Mechanism

Let us consider the following instance:
$\mathcal{S} = \{\alpha, \beta, \gamma\}$; $\mathcal{C} = \{A, B, C\}$; $q_A = q_B = q_C = 1$; and

Agents' Preferences	Boston Algorithm Step	A	B	C
$U_\alpha(A) > U_\alpha(B) > U_\alpha(C)$	1.a	α, β	γ	–
$U_\beta(A) > U_\beta(B) > U_\beta(C)$	1.b	α	γ	–
$U_\gamma(B) > U_\gamma(A) > U_\gamma(C)$	2.a			β
$P_A(\gamma) > P_A(\alpha) > P_A(\beta)$	2.b			β
$P_B(\alpha) > P_B(\beta) > P_B(\gamma)$				
$P_C(\beta) > P_C(\gamma) > P_C(\alpha)$				
	Outcome	α	γ	β

The Boston Mechanism

- Boston mechanism has lots of problematic features:
    1. It is not strategy-proof.
    2. Moreover, it is easy to manipulate it. Even if a student has a very high priority at a school, unless she lists it as her top choice she loses her priority to students who have top ranked that school.

The Boston Mechanism

- St. Petersburg Times (09/14/2003):

Make a realistic, informed selection on the school you list as your first choice. It's the cleanest shot you will get at a school, but if you aim too high you might miss. Here's why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices.

- The 2004-2005 BPS School Guide:

"For a better choice of your $<<$first choice$>>$ school . . . consider choosing less popular schools."

The Boston Mechanism

- The Boston mechanism is unfair, i.e., does not eliminate justified envy, either. Priorities are lost unless the school is ranked as the top choice.

Theorem

(Ergin and Sönmez 2006) The set of Nash equilibrium outcomes under the Boston mechanism is equal to the set of stable matchings.

The Boston Mechanism

- But the preference revelation game induced by the Boston mechanism is a "coordination game" among large numbers of parents in which there is incomplete information. So it may be unrealistic to expect to reach a Nash equilibrium in practice.
- Moreover, the Boston mechanism may produce an inefficient matching given students may behave strategically.
- In Boston many students end up unassigned, suggesting inefficiency.
- Given the problems of the popular Boston mechanism, what mechanism should we use instead?

Student-Proposing DA in School Choice

We can use the student-proposing DA (Gale and Shapley 1962; Abdulkadiroglu and Sonmez 2003).

Step 1: (a) Each student "applies" to her first choice school.

(b) Each school tentatively holds the applicants with highest priority up to its quota (if she is acceptable) and rejects all other students.

Step $t \geq 2$: (a) Each student rejected in Step $(t-1)$ applies to her next highest choice. (b) Each school considers both new applicants and the student (if any) held at Step $(t-1)$, tentatively holds the applicants with highest priority up to its quota from the combined set of students, and rejects all other students.

Terminate when no more applications are made. Termination happens in finite time.

Example

Gale and Shapley (1962): The Deferred Acceptance Algorithm

Agents' Preferences	Gale-Shapley Algorithm Step	A	B	C
$U_\alpha(A) > U_\alpha(B) > U_\alpha(C)$	1.a	α, β	γ	–
$U_\beta(A) > U_\beta(B) > U_\beta(C)$	1.b	α	γ	–
$U_\gamma(B) > U_\gamma(A) > U_\gamma(C)$	2.a	α	β, γ	–
$P_A(\gamma) > P_A(\alpha) > P_A(\beta)$	2.b	α	β	
$P_B(\alpha) > P_B(\beta) > P_B(\gamma)$	3.a	α, γ	β	–
$P_C(\beta) > P_C(\gamma) > P_C(\alpha)$	3.b	γ	β	
	4.a	γ	α, β	–
	4.b	γ	α	
	5.a	γ	α	β
	5.b	γ	α	β
	Outcome	γ	α	β

Difference of School Choice From Two-side Matching

- Schools are merely goods to be consumed, rather than players.
- If this is true, then we should take into account welfare of students only.
- Students' priorities at schools are often decided by law. In such a case, schools do not behave strategically.
- Given these differences, we can see some old results in new lights.

Stability

Theorem

(Gale and Shapley 1962; RS Theorem 2.12) There exists a student-optimal stable matching, that is, a stable matching that every student weakly prefers to any stable matching. The result of the student-proposing DA algorithm is the student-optimal stable matching.

- Because we consider welfare of students only, this theorem means that welfare is maximized by the student-proposing DA, subject to stability.
- We also learned the student-optimal stable matching is the unanimously worst stable matching for schools, but it is not costly any more (because we do not care about school's "welfare").

Strategy Proofness

Theorem

(Dubins and Freedman 1981, Roth 1982) The student-proposing DA is (group) strategy-proof. That is, telling the truth is a dominant strategy for every student (and even a joint deviation by a group of students cannot make everyone better o).

- In school choice problems, there exists a strategy-proof and stable mechanism. (Roth's impossibility theorem said there is no such thing when both students and schools can behave strategically.)
- DA is the only strategy-proof and stable mechanism.

More Properties of DA

- DA is the only stable mechanism that respects improvements, that is, a higher priority is always good for a student (Balinski and Sönmez 1999).
- DA is weakly Pareto optimal, that is, there is no other individually rational matching that every student strictly prefers to DA (Roth 1982).
- The student-proposing DA is the big winner among all stable mechanisms. Balinski and Sönmez (1999) shows that the mechanism used for college admission in Turkey is equivalent to the school-proposing DA, and advocated the change of the mechanism to the student-proposing DA.

Efficiency Cost of Stability

Agents' Preferences	Comparing μ^{GS} and μ^{B}	A		B	C
$U_\alpha(A) > U_\alpha(B) > U_\alpha(C)$					
$U_\beta(A) > U_\beta(B) > U_\beta(C)$	μ^{GS}	$\boxed{\gamma}$	$\leftrightarrows$	$\boxed{\alpha}$	β
$U_\gamma(B) > U_\gamma(A) > U_\gamma(C)$					
$P_A(\gamma) > P_A(\alpha) > P_A(\beta)$	μ^{B}	α		γ	$\boxed{\beta}$
$P_B(\alpha) > P_B(\beta) > P_B(\gamma)$					$\updownarrow$
$P_C(\beta) > P_C(\gamma) > P_C(\alpha)$					$\boxed{\text{B}}$

Efficiency Cost of Stability

- For school choice, the student-proposing DA may not produce a Pareto efficient matching.
- Since the student-proposing DA is Pareto dominant among stable matchings, no stable matching is Pareto efficient in the last example.
- Also, in school choice stability may be desirable but may not be indispensable: It depends on school districts, presumably (compare it to labor markets, in which stability seems necessary just to sustain an orderly assignment).
- If the school districts can tolerate somewhat unfair matchings, how can we design a more efficient mechanism?

The TTC

The TTC algorithm (Abdulkadiroglu and Sonmez 2003):

1. Assign a counter for each school that keeps track of how many seats are still available at the school. Initially set the counters equal to the capacities of the schools.
2. Each student "points to" her favorite school. Each school points to the student who has the top priority.
3. There is at least one cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. Counters of all other schools are unchanged.
4. Repeat above steps for the remaining school seats and students.

- TTC allows students to trade priorities, starting with the students with highest priorities.

The TTC

Theorem

(Abdulkadiroglu and Sonmez 2003) The TTC mechanism is Pareto-efficient and strategy-proof.

- The TTC for school choice is a common generalization of:
  1. Serial Dictatorship in the house allocation problem,
  2. Gale's TTC in the housing market,
  3. YRMH-IGYT (TTC) in house allocation with existing tenants, and the school choice TTC inherits good properties from these, as shown by the Theorem.

Boston School Match

Based on (Abdulkadiroglu, Pathak, Roth and Sonmez 2005, 2008)

- Students entering grades K, 6, and 9 submit preferences over schools.
- Students have priorities at schools set by the school system:

  1. Students who already attend the school,
  2. Students who live in a walk zone and have their siblings already attending the school,
  3. Students whose siblings are already attending the school,
  4. Students who live in a walk zone,
  5. All other students.

Priorities are weak, i.e., there are many students in each priority class: This is going to be important (later topic) but for now let's ignore the issue.

Boston School Match

- Remember the problem with the old Boston mechanism: It is very easy to manipulate!
- Especially, reported first choices are important: if you don't receive your first choice, you might drop far down list.
- Indeed, there seem to be gaming of preferences: the vast majority are assigned to their stated "first choices."
- Chen and Sönmez (2005): experimental evidence on preference manipulation under Boston mechanism.

Boston School Match

- Advice from the West Zone Parent's Group meeting, 10/27/03

One school choice strategy is to find a school you like that is under subscribed and put it as a top choice, OR, and a school that you like that is popular and put it as a first choice and find a school that is less popular for a "safe" second choice.

Boston School Match

- Of the 15,135 students on whom Abdulkadiroglu et al. analyzed, 19% (2910) listed two over-demanded schools as their top two choices, and about 27% (782) of these ended up unassigned.
- Such behavior is clearly a bad choice, and people suffer from not being sophisticated enough to game the system (Abdulkadiroglu et al. advocate the idea that strategy-proofness is a certain fairness criterion).

Boston School Match

- Since priorities are set by law for Boston schools, Abdulkadiroglu et al. recommended not only DA but also TTC
  - remember TTC is more efficient than DA.
- However the school system finally chose DA: the story says the idea of "trading priorities" in TTC did not appeal to policy makers. DA was implemented in Boston in 2006 and is in use.

New York City

Abdulkadiroglu, Pathak and Roth (2005, 2008). In New York over 90,000 students enter high schools each year.

The old NYC system was decentralized:

1. Each student can submit a list of at most 5 schools.
2. Each school obtains the list of students who listed it, and independently make offers.
3. There were waiting lists (run by mail), and 3 rounds of move waiting lists.

Problems with the old system:

1. The system left 30,000 children unassigned to any of their choices and they are administratively assigned.
2. Strategic behavior by schools: school principals were concealing capacities (Sönmez 1997; further studies by Konishi and Ünver 2006; Kojima 2008).

New York City

- In New York City, schools behave strategically.
- Deputy Chancellor of Schools (NYT 11/19/04):
    - Before you might have had a situation where a school was going to take 100 new children for 9th grade, they might have declared only 40 seats and then placed the other 60 children outside the process.
- So, unlike Boston, the market seems to be really two-sided, i.e., we should treat both students and schools are strategic players.

New York City

Since NYC is a two-sided matching market, the student-proposing DA is the big winner! Remember DA has a lot of good properties:

1. The student-proposing DA implements a stable matching (probably more important for NYC than for Boston.)
2. The student-proposing DA is strategy-proof for students: it is a dominant strategy for every student to report true preferences (Dubins and Freedman 1981; Roth 1982).
3. There is no stable mechanism that is strategy-proof for schools (Roth 1982)
4. When the market is large, it is almost strategy-proof for schools to report true preferences (Roth and Peranson 1999; Kojima and Pathak 2008). Recall there are 90,000 students and over 500 public high schools in New York City.

New York City

Abdulkadiroglu et al. and NYC Department of Education changed the mechanism to the student-proposing DA, except for some details:

1. Students can rank only 12 schools.
2. Seats in a few schools, called specialized high schools (such as Stuyvesant and Bronx High School of Science), is assigned in an earlier round, separately from the rest.
3. Some top students are granted to get into a school when they rank the school as their first choices.
4. All unmatched students in the main round will be assigned in the supplementary round, where the random serial dictatorship is used.

These features come from historical constraints and could not be changed. This make it technically incorrect to use standard results in two-sided matching, but they seem to be small enough a problem (it may be interesting to study if this is true and why or why not.)

Effect of changes in the mechanism

Over 70,000 students were matched to one of their choice schools: an increase of more than 20,000 students compared to the previous year match.

An additional 7,600 students matched to a school of their choice in the third round.

3,000 students did not receive any school they chose, a decrease from 30,000 who did not receive a choice school in the previous year.

Priority Structure and Efficiency

Based on Ergin (2002). Another question to ask is, when is DA costly in school choice? What priority structure ensures efficiency of DA? To investigate this issue, it is useful to see another example where DA is inefficient.

DA Causes Inefficiency

Example

(Fuhito Kojima, Class notes) Let $S = i, j, k$; $C = a, b$, each school has one position and

$\succ_i$: b, a,
$\succ_j$: a,
$\succ_k$: a, b,
$\succ_a$: i, j, k,
$\succ_b$: k, i,
DA results $\mu = \{(i, a); (j, \emptyset), (k, b)\}$, while $\mu' = \{(i, b); (j, \emptyset), (k, a)\}$, is better for every one.

DA Causes Inefficiency

There is a chain of applications and rejections in the above example, as we have seen before.

Such a chain is caused by a "cycle" of priorities, that is, two schools' priorities are

$\succ_a$: $i, j, k,$

$\succ_b$: $k, i,$

DA Causes Inefficiency

Because of such a cycle, in DA,

1. k applies to her favorite a but j displaces k ,
2. k is forced to apply to her second choice b, displacing i from his favorite b,
3. i is forced to apply to his second choice a, displacing j . In the end, j is displaced by school a anyway, with the result being just causing more rejections and making i and k worse off.

Ergin says the priority structure of the schools is acyclic if there is no such cycle.

Priority Structure and Efficiency

Theorem

(Ergin 2002) DA is Pareto efficient for all possible student preferences if and only if the priority structure of the schools is acyclic.

This theorem is bad news for school systems, because most priority structures are cyclic. Kesten (2006) defined a stronger version of acyclicity and showed

Theorem

(Kesten 2006) DA and TTC coincide if and only if the priority structure of the schools is Kesten-acyclic. Taken together, it is rare for DA to have no efficiency cost, and most likely there is tension between stability and efficiency.

Fairness: Reconciling Equity and Efficiency

First Questions

- Is there a regular procedure always providing Pareto Efficient Allocations?
  - Yes: The TTC.
- Is there a regular procedure always providing Equitable Allocations?
  - Yes: The (new) Boston System or students optimal DA.
- Is there a regular procedure always providing Fair (Pareto Efficient and Equitable) Allocations?
  - No: It is impossible.

Fairness: Reconciling Equity and Efficiency

Non Existence of Equitable and Efficient Procedures

Theorem

There is no matching mechanism always selecting an allocation satisfying Equity and Pareto Efficiency for each $\mathcal{SAP}$.

Agents' Preferences	Comparing μ^{GS} and μ^{B}				
		A		B	C
$U_\alpha(A) > U_\alpha(B) > U_\alpha(C)$					
$U_\beta(A) > U_\beta(B) > U_\beta(C)$	μ^{GS}	$\boxed{\gamma}$	$\leftrightarrows$	$\boxed{\alpha}$	β
$U_\gamma(B) > U_\gamma(A) > U_\gamma(C)$					
$P_A(\gamma) > P_A(\alpha) > P_A(\beta)$	μ^{B}	α		γ	$\boxed{\beta}$
$P_B(\alpha) > P_B(\beta) > P_B(\gamma)$					$\updownarrow$
$P_C(\beta) > P_C(\gamma) > P_C(\alpha)$					$\boxed{\text{B}}$

The Idea of Fairness

A Novel Thought

If we want to reconcile both ideas we need something different.

An alternative solution concept that must satisfy two requirements:

- It must capture an idea of how to understand Equity, and
- It has to be compatible with Pareto Efficiency.

The Idea of Fairness

Our proposal follows the idea of the Bargaining Set, introduced by Aumann and Mashler (1964) for Transferable Utility Cooperative Games. The logic is the following:
"μ will be implemented. If you objects it (based on equity arguments), you must propose a new matching. If no other student would claim that your proposal fails to be equitable, it will be implemented. Otherwise , your objection will be dismissed ."

ρ-Equity A Novel Notion of Equity

- Definition . $[Equitable\ Objection]$.
  Let $\mathcal{SAP}$ be a School Allocation Problem, and let μ be a matching for such a problem. An Equitable Objection from student $s_i \in \mathcal{S}$ against μ is a pair (s_i, μ') such that
  1. $U_i(\mu'(s_i)) > U_i(\mu(s_i))$, and
  2. $|\mu(\mu'(s_i))| < q_{\mu(\mu'(s_i))}$, or $P_{\mu'(s_i)}(s_i) > P_{\mu'(s_i)}(s_h)$ for some $s_h \in \mu(\mu'(s_i))$.
- Definition . $[Counter - Objection]$.
  Let (s_i, μ') be an Equitable objection against matching μ. A counter-objection from student s_h against (s_i, μ') is a pair (s_h, μ'') that constitutes an Equitable objection against matching μ'.

From ρ-Equity to τ-Fairness Conciliating Equity and Pareto Efficiency

- We say that (s_i, μ') is a justified equitable objection against μ if it cannot be counter-objected.
- Definition. $[\rho - Equity]$.
  Let $\mathcal{SAP}$ be a School Allocation Problem. We say that matching μ is ρ-equitable if any equitable objection against it can be counter-objected.
- Definition. $[\tau - Fairness]$.
  Let $\mathcal{SAP}$ be a School Allocation Problem. We say that matching μ is τ-fair if it is Pareto Efficient and ρ-equitable .

From ρ-Equity to τ-Fairness Conciliating Equity and Pareto Efficiency

Natural Questions Arising from the Above Ideas:

- Can we guarantee that any School Allocation Problem has ρ-equitable allocations? Yes. The reason is that any equitable (stable) allocation is also ρ-equitable.
- Can we ensure that any School Allocation Problem has τ-fair allocations? Yes.

The Exchanging Places Mechanism A Procedure

Working Idea : Given a School Allocation Problem,

- First : Select a ρ-equitable matching, for instance, the students-optimal stable matching, μ^{SO} .
- Second : Let's allow students to exchange their places, as if they had property rights on them:
  "Since this is my place, I can freely exchange it."

Consequences:

- Due to the second step : the outcome will be Pareto Efficient .
- Due to the first step : the outcome will be ρ-equitable .
- Therefore , the outcome will be τ-fair .

The Exchanging Places Mechanism An Informal Definition

As an input for the algorithm we need the following elements:

- A School Allocation Problem, and
- A matrix $\Sigma \in \mathcal{M}_{n\times n}$ reflecting students' interest on whom to exchange with.

Then, the algorithm can be defined as a two-step procedure:

- First Step: For the given School Admissions Problem, let compute its students optimal stable matching , μ^{SO}.

The Exchanging Places Mechanism An Informal Definition

- Second Step : Given μ^{SO} , obtained in the previous step, and Σ, we can define, for each student s_i, a complete, linear preordering $\succ_i$ as follows. Given any two students, s_h and s_k, then
  - If $U_i\left(\mu^{SO}\left(s_h\right)\right) > U_i\left(\mu^{SO}\left(s_k\right)\right)$, then $s_h \succ_i s_k$;
  - If $U_i\left(\mu^{SO}\left(s_h\right)\right) = U_i\left(\mu^{SO}\left(s_k\right)\right)$, then $s_h \succ_i s_k$ if, and only if, $\Sigma_{ih} > \Sigma_{ik}$.

  Given the above students' preferences, defined over students' endowments, we can define the induced Housing Market, as modeled by Shapley and Shubik (1974), and apply to it the Gale's Top Trading Cycle . This process will lead a new matching, which is the outcome for the Exchange Places Mechanism.

The Exchanging Places Mechanism An Example

Let us consider the following Schools Allocation Problem.
$\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathcal{C} = \{a, b, c, d\}$, the capacity for each school is 2; and the Utilities and Priorities functions are

$$U = \begin{bmatrix} 3 & 4 & 2 & 1 \\ 3 & 1 & 4 & 2 \\ 2 & 3 & 4 & 1 \\ 1 & 3 & 2 & 4 \\ 4 & 1 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{bmatrix}; \text{ and } \qquad \Pi = \begin{bmatrix} 7 & 6 & 1 & 8 & 2 & 5 & 3 & 4 \\ 3 & 7 & 8 & 4 & 1 & 5 & 6 & 2 \\ 2 & 4 & 3 & 1 & 7 & 6 & 8 & 5 \\ 1 & 4 & 6 & 5 & 8 & 7 & 2 & 3 \end{bmatrix}$$

The Exchanging Places Mechanism An Example

We first compute the student-optimal stable matching as follows

Step	a		b		c		d
1	5, 6, 7		1, 8		2, 3		4
2	6, 7		1, 8		2, 3, 5		4
3	6, 7		1, 3, 8		2, 5		4
4	6, 7, 8		1, 3		2, 5		4
8	1, 2, 6		3, 7		5, 8		4
9	1, 2		3, 6, 7		5, 8		4
10	1, 2		3, 7		5, 6, 8		4
11	1, 2		3, 7		5, 6		4, 8
μ^{SO} :=	1, 2		3, 7		5, 6		4, 8

The Exchanging Places Mechanism An Example

Given μ^{SO} and an specific matrix Σ, we compute the Gale's Top Trading Cycle as follows

1. The successive incidence matrices are

$$I_{S}^{\Sigma^{SO}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, students 1, 3 and 5 will exchange the places that μ^{SO} assigns to them , and student 4 will keep her place under μ^{SO}.

The Exchanging Places Mechanism An Example

So $\mathcal{S}_1 = \{2, 6, 7, 8\}$; and thus

1. The next incidence matrices are

$$I_{\mathcal{S}_1}^{\Sigma^{SO}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } I_{\mathcal{S}_2}^{\Sigma^{SO}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus, students 2 and 6 will exchange the seats that μ^{SO} assigned to them. Then student 7 will keep her place under μ^{SO}. And, finally, student 8 will not be able to exchange her place.

The Exchanging Places Mechanism A Comparison with other Methods

Let us consider the following table. It associates each student two items: the school in which she gets a place (in blue) , and the position of such school in the student's ranking (in red)

	Comparing Systems															
	1		2		3		4		5		6		7		8	
μ^{BM}	b	1	c	1	c	1	d	1	d	3	a	1	a	1	b	1
μ^{TTC}	b	1	c	1	c	1	d	1	a	1	b	2	a	1	d	4
μ^{SO}	a	2	a	2	b	2	d	1	c	2	c	3	b	2	d	4
μ^{EP}	b	1	c	1	c	1	d	1	a	1	a	1	b	2	d	4

τ-Fairness

Theorem

For a given $\mathcal{SAP}$ *a matching* μ *is* τ*-fair if and only if it is Pareto efficient and* $U_i\left(\mu^{SO}(s_i)\right) \leq U_i(\mu(s_i))$ *for all* s_i*.*

Strategy-Proofness

Example 20 *Benefiting from a loss and the SOCE manipulation*

Let us consider the three-students-three-schools problem, $\mathcal{P}$, where $\mathcal{S} = \{1, 2, 3\}$, $\mathcal{C} = \{a, b, c\}$, and each school quota is $q_j = 1$. The students' preferences and colleges' priorities are

$$\begin{array}{lll} a \succ_1 b \succ_1 c & \qquad & 3 \, P_a \, 2 \, P_a \, 1 \\ b \succ_2 a \succ_2 c & \qquad & 1 \, P_b \, 3 \, P_b \, 2 \\ b \succ_3 c \succ_3 a & \qquad & 2 \, P_c \, 1 \, P_c \, 3 \end{array}$$

In such a case the application of the SOCE mechanism yields matching $\mu^{ce}(\mathcal{P})$ described as

$$\mu^{ce}(\mathcal{P}) := \begin{array}{ccc} 1 & 2 & 3 \\ a & b & c \end{array} .$$

Since students 1 and 2 obtain a place at their preferred schools, they cannot manipulate the SOCE at this problem. Nevertheless, when student 3 reports preferences $\succ_3'$, with $b \succ_3' a \succ_3' c$, the application of the SOCE yields matching $\mu^{ce}(\mathcal{P}')$ described as

$$\mu^{ce}(\mathcal{P}') := \begin{array}{ccc} 1 & 2 & 3 \\ a & c & b \end{array} .$$

Note that $\mu^{ce}(3; \mathcal{P}') = b \succ_3 c = \mu^{ce}(3; \mathcal{P})$. This implies that 3 can manipulate the SOCE at $\mathcal{P}$ via $\succ_3'$.

Tie-breaking

- A new issue in school choice is how to do tie breaking. We find tradeoffs between efficiency, stability, and strategy-proofness.

Matching with indifference

- When we were using matching theory for labor markets, strict preferences seemed like a good assumption.
- But in school choice, indifferences (weak priorities) are important:
- In Boston, for example, students have priorities at schools set by the school system:

    1. Students who already attend the school,
    2. Students who live in a walk zone and have their siblings already attending the school,
    3. Students whose siblings are already attending the school,
    4. Students who live in a walk zone,
    5. All other students.

Student-Proposing DA in School Choice

Step 0: Arbitrarily break all ties in preferences.

Step 1:

(a) Each student "applies" to her first choice school.

(b) Each school tentatively holds the applicants with highest priority up to its quota (if she is acceptable) and rejects all other students.

Step $t \geq 2$:

(a) Each student rejected in Step (t-1) applies to her next highest choice.

(b) Each school considers both new applicants and the student (if any) held at Step (t-1), tentatively holds the applicants with highest priority up to its quota from the combined set of students, and rejects all other students. Terminate when no more applications are made.

Termination happens in finite time.

Tie Breaking

- There are at least two ways to break the ties:
  1. Single tie breaking Use one lottery to decide order on all students and, whenever two students are in the same priority class, break the tie using the ordering.
  2. Multiple tie breaking Draw one lottery for each school, and whenever two students are in the same priority class for a school, break the tie using the ordering for that particular school.

Tie-Breaking DA

- The outcome of a DA may not be a student-optimal stable matching (that is, there may be a stable matching that is better for everyone).

Example

Example

(Fuhito Kojima, Class notes) Let $S = \{s_1; s_2; s_3\}$; $C = \{c_1; c_2; c_3\}$, each college has one seat,

$\succ_{c_1}: s_1, \{s_2, s_3\},$ $\qquad$ $\succ_{s_1}: c_2, c_1, c_3,$

$\succ_{c_2}: s_2, \{s_1, s_3\},$ $\qquad$ $\succ_{s_2}: c_3, c_2, c_1,$

$\succ_{c_3}: s_3, \{s_1, s_2\},$ $\qquad$ $\succ_{s_3}: c_2, c_3, c_1,$

Assume ties are broken in the order s_1, s_2, s_3 for each college, that is, we pretend

$\succ_{c_1}: s_1, s_2, s_3,$

$\succ_{c_2}: s_2, s_1, s_3,$

$\succ_{c_3}: s_3, s_1, s_2,$

DA with this tie-breaking $\mu = \{(s_1, c_1), (s_2, c_2), (s_3, c_3)\}$, but everyone prefers $\mu' = \{(s_1, c_1), (s_2, c_3), (s_3, c_2)\}$, and μ' is stable with respect to the original priority.

Tie-Breaking DA

- Tie-breaking may hurt efficiency but DA with tie-breaking is still strategy-proof.
- If we need to use DA, what tie-breaking should be used? Recall there are at least two ways to break the ties:
  1. Single tie breaking Use one lottery to decide order on all students and, whenever two students are in the same priority class, break the tie using the ordering.
  2. Multiple tie breaking Draw one lottery for each school, and whenever two students are in the same priority class for a school, break the tie using the ordering for that particular school.

Tie-Breaking DA

- Policymakers from the NYC Department of Education believed that DA with single tie-breaking is less equitable than multiple tie-breaking:

<<If we want to give each child a shot at each program, the only way to accomplish this is to run a new random. [...] I cannot see how the children at the end of the line are not disenfranchised totally if only one run takes place. I believe that one line will not be acceptable to parents. When I answered questions about this at training sessions, (it did come up!) people reacted that the only fair approach was to do multiple runs.>>

Choice	Deferred Acceptance Single Tie-Breaking DA-STB (1)	Deferred Acceptance Multiple Tie-Breaking DA-MTB (2)
1	32,105.3 (62.2)	29,849.9 (67.7)
2	14,296.0 (53.2)	14,562.3 (59.0)
3	9,279.4 (47.4)	9,859.7 (52.5)
4	6,112.8 (43.5)	6,653.3 (47.5)
5	3,988.2 (34.4)	4,386.8 (39.4)
6	2,628.8 (29.6)	2,910.1 (33.5)
7	1,732.7 (26.0)	1,919.1 (28.0)
8	1,099.1 (23.3)	1,212.2 (26.8)
9	761.9 (17.8)	817.1 (21.7)
10	526.4 (15.4)	548.4 (19.4)
11	348.0 (13.2)	353.2 (12.8)
12	236.0 (10.9)	229.3 (10.5)
unassigned	5,613.4 (26.5)	5,426.7 (21.4)

Figure: New York City Grade 8 students in 2006-2007.

Tie-Breaking DA

- Simulation suggests that single tie breaking is better in efficiency, although it is not too clear-cut.
- Abdulkadiroglu, Che and Yasuda (2008), Che and Kojima (2008) show that, when there is no intrinsic priority and the market is large, DA-STB is more efficient than DA-MTB.
- Intuition: DA's inefficiency comes from students displacing each other. That is less likely in STB than in MTB. NYC decided to use single tie breaking. But even single tie-breaking can cause inefficient matching. Any way to improve efficiency while keeping stability?

Stable Improvement Cycles

Based on Erdil and Erging 2008.

- Given a stable matching , say student s desires c if $c \succ_s \mu(s)$.
- Let B_c be the set of highest $\succeq c$-priority students among those who desire c .
- A stable improvement cycle consists of distinct students $s_1, ..., s_{n-1}, s_n = s_0$ such that, for any $k = 1, ..., n$.

  1 $\mu(s_k) \in C$,
  2 s_k desires $\mu(s_{k+1})$, and
  3 $s_k \in B_{\mu(s_{k+1})}$.

- Given a stable improvement cycle, define a new matching μ' by

  1 $\mu'(s_k) = \mu(s_{k+1})$ for all k , and
  2 $\mu'(s) = \mu(s)$ for all $s \notin \{s_1, ..., s_n\}$.

Stable Improvement Cycles

Theorem

(1) μ' is stable and it Pareto dominates μ .
(2) Whenever a stable matching is not student-optimal, there is a stable improvement cycle.

The theorem implies that we can find a student-optimal stable matching by

1. running DA with some tie-breaking, and
2. applying SIC repeatedly until it reaches student-optimal stable matching.

Stable Improvement Cycles

- Is SIC without a problem?
- Is there any cost of improving efficiency of the matching?
- The SIC procedure turned out not to be strategy-proof. Intuition: A student can rank a popular school higher, and use the school for "trading" purpose in SIC.

Stable Improvement Cycles

More generally,

Theorem

For any tie-breaking rule, there is no strategy-proof mechanism that always results in a better matching than the DA with the tie-breaking.

In other words, whatever improvement over DA with tie breaking may become non-strategy-proof. But note that the requirement is extremely strong: "always results in a better matching." TTC does not meet the criterion, but usually TTC produce better matchings than DA.

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