Matching and Market design

Matteo Triossi

CEA, DII

August 17, 2017

Part II

- Second Session: Design of labor markets: National Resident Matching Program (NRMP)
  - Many-to-one Matching Markets
  - The NMRP

Many-to-one Matching Markets

- In this session we want to analyze markets where one sided of the market is matched with more than one element on the other side.
- Our model will reflect the features of University admissions but it can be use for Labor markets.
- We will present the case of the reform of the market for medical resident as an example o Market Design.

Many-to-one Matching Markets: The Model

- Let us consider a bilateral market with two finite disjoint sets $S = \{s_1, s_2, ..., s_m\}$ and $C = \{c_1, c_2, ..., c_n\}$,
  - Each student $s \in S$ can be matched with at most one college.
  - Each college $c \in C$ has a capacity of q_c : it can be matched to at most q_c students.

Preferences

- The preferences of each student $s \in S$ are represented by a linear order on $C \cup \{s\}$.
- The preferences of each College $c \in C$ are represented by a linear order on $S \cup \{c\}$.
- The college admission problem is fully described by a triplet (C, S, P) where P is a preference profile containing a full description of the agent's preferences.

The Basic Many-to-one Matching Model

Definition

A **matching** on (C, S) is a function $\mu : C \cup S \to 2^S \cup C$.

1. $\mid \mu(s) \mid = 1$ for every student s and $\mu(s) = s$ if $\mu(s) \notin C$;
2. $\mid \mu(c) \mid \leq q_c$ for every college c, and $\mu(c) \in 2^S \cup c$;
3. For all $s \in S$ and $c \in C$, we have that $\mu(s) = c \Leftrightarrow s \in \mu(c)$.

Preferences

Definition

The preference relation $\succ_c$ over set of students is responsive (to the preferences $P(c)$ over individual students) if, whenever $\mu'(c) = \mu(c) \cup \{s\} \setminus \sigma$ for $\sigma \in \mu(c)$, and $s \notin \mu(c)$, then $\mu'(c) \succ_c \mu(c)$ if an only if $sP(c)\sigma$.

Preferences

Example

(G. Katsenos) We assume that we have three students and two colleges. The colleges quotas are defined in their preferences:

- $P(s_1) = c_2, c_1, s_1$
- $P(s_2) = c_2, c_1, s_2$
- $P(s_3) = c_1, c_2, s_3$
- $\succ_{c_1} = \{s_1, s_3\}, \{s_1, s_2\}, \{s_2, s_3\}, \{s_1\}, \{s_2\}, \{c_1\},$
- $\succ_{c_2} = \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2\}, \{s_3\}, \{s_2\}, \{s_1\}, \{c_2\}$

Preferences

- $\mu_1 = \begin{matrix} c_1 & c_2 \\ \{s_1, s_3\} & \{s_2\} \end{matrix}$ - blocked by $(c_2\{s_1, s_2\})$
- $\mu_2 = \begin{matrix} c_1 & c_2 \\ \{s_1, s_2\} & \{s_3\} \end{matrix}$ - blocked by $(c_2\{s_1, s_3\})$
- $\mu_3 = \begin{matrix} c_1 & c_2 \\ \{s_2, s_3\} & \{s_1\} \end{matrix}$ - blocked by $(c_2\{s_1, s_2\})$
- $\mu_4 = \begin{matrix} c_1 & c_2 \\ \{s_2\} & \{s_1, s_3\} \end{matrix}$ - blocked by $(c_1\{s_2, s_3\})$
- $\mu_5 = \begin{matrix} c_1 & c_2 \\ \{s_1\} & \{s_2, s_3\} \end{matrix}$ - blocked by $(c_2\{s_1,s_3\})$

Preferences

- Notice that $\{s_1, s_3\} \succ_{c_1} \{s_1, s_2\}$ but $\{s_2\} \succ_{c_1} \{s_3\}$;
- Also, $\{s_1, s_3\} \succ_{c_2} \{s_2, s_3\}$ but $\{s_2\} \succ_{c_2} \{s_1\}$.
- The colleges' preferences are not responsive.
- Also responsiveness allows us to represent colleges' preferences as rankings of individual students.

Stability

- Like in the one-to-one matching problem a matching is stable if it is individually rational and it is not blocked by a college-student pair. Formally:

Definition

A matching μ is stable if:

- $\nexists s \in S$ such that $s \succ_s \mu(s)$.
- $\nexists c \in C$ such that $\mu(c) \setminus \{s\} \succ_c \mu(c)$ for some $s \in \mu(c)$.
- $\nexists (s, c) \in S \times C$ such that $c \succ_s \mu(s)$ and $\mu(c) \setminus \sigma \cup \{s\} \succ_c \mu(c)$, for some some $\sigma \in \mu(c)$.

Group Stability

- We have seen that a matching can be block by deviations of more than two agents. If it is not we can say that the matching is group stable Formally,

Definition

A matching μ is blocked by a coalition if there is a group $A \subseteq S \cup C$ and a matching $\mu' \neq \mu$ such that

- For each $s \in A$, $\mu'(s) \in A$ and $\mu'(s) \succ_s \mu(s)$;
- For all colleges $c \in A$, we have $\sigma \in \mu'(c) \setminus \mu(c)$ implies that $\sigma \in A$ and $\mu'(c) \succ_c \mu(c)$.

Group Stability

Definition

A matching μ is group stable if it is not blocked by a coalition $A \subseteq S \cup C$.

- **Lemma:** Under responsive and transitive preferences a matching μ is group stable if and only if it is stable.
- **Proof:** Clearly group stability implies stability. The other direction: Suppose that a matching is not group stable and the blocking coalition A includes an agent that blocks the matching μ individually. Then the allocation μ is not stable. Under responsive and transitive preferences the preferences over a group will not decrease if we change a student for a better one so a couple college-student can block μ.

Existence of Stable Matchings

Theorem

(GS 1962) When preferences are responsive, there is a stable matching in the college admissions problem.

Many-to-One DA Algorithm

- Step 1:
  - a) Each student proposes to his first choice. If this is preferred to remaining unmatched. Otherwise he does not propose.
  - b) Each college **tentatively** accepts the q_c most preferred applications it receives, among those which are acceptable; some positions might be empty - copies of itself -.
- Step 2:
  - a) Each student that has been rejected in Step 1 proposes **to his next choice** if it is preferred to remaining unmatched. Otherwise he does not propose.
  - b) Each college **tentatively** accepts the q_c most preferred applications it receives, possibly rejecting previous applications; some positions might be empty - copies of itself -.

The Deferred Acceptance Algorithm

- Step k:
  - a)Each student that has been rejected in Step k-1 proposes **to his next choice** if it is preferred to remaining unmatched. Otherwise he does not propose.
  - b) Each college **tentatively** accepts the q_c most preferred applications it receives, possibly rejecting previous applications; some positions might be empty - copies of itself -.
- Last Step: The algorithm ends after a step in which no proposals are made. At this point all tentative allocation become final and constitute the outcome of the algorithm.

The Equivalent Marriage Market

- Using the college admission model we can transform it into a related marriages market:
  - Men: The students $S = \{s_1, s_2, ..., s_m\}$.
  - Women: Each of the colleges' open positions: $C = \left\{c_1^1, c_1^2, ..., c_1^{q_1}, c_2^1, c_2^2, ..., c_2^{q_2}, c_n^1, c_n^2, ..., c_n^{q_n}\right\}$,

The Equivalent Marriage Market

- Men's preferences: $c_k^l \succ_s c_{k'}^{l'} \Leftrightarrow c_k \succ_s c_{k'}$ to make the preferences strict we use a tie-braking rule: $c_k^l \succ_s c_k^{l'} \Leftrightarrow l < l'$.
- Women's preferences: $s \succ_{c_k^l} s' \Leftrightarrow s \succ_{c_k} s'$.

The Equivalent Marriage Market

- **Lemma:** Under responsive preferences a matching μ in the college admission problem is stable if and only if the corresponding matchings in the related marriage market are stable.
- **Remark:** If the preferences in the college admission problem are strict, a matching in that problem correspond to a unique matching in the associated marriage problem. Otherwise they might be several corresponding matchings.

Some Results

- The transformation of the college admissions problem into a marriage market allows us to generalize some previous results:
  - Optimal Matching Theorem: In the college admissions problem the College-optimal (Student-Optimal) is given by the DA algorithm when colleges (students) proposing.
  - Lattice Theorem: In the college admissions problem the set of stable matching forms a lattice with respect to the partial other of students (or colleges) preferences.
  - Opposite preferences: In the college admissions problem the preferences of colleges and student over stable matching are opposed.
  - Rural Hospital Theorem: In the college admissions problem the set of students or colleges positions that remain unmatched are the same.

Strategic Behavior

- In the college admissions problem colleges can misrepresent their preferences and its quota. Therefore:

Theorem

In the college admissions problem, there is no strategy proof mechanism.

.

Theorem

In the college admissions problem, with one strategic side:

- When only colleges behave strategically, no matching mechanism is strategy proof.
- When only students behave strategically, the student-proposing DA algorithm is strategy proof.

The NRMP

- We will discuss the redesign of the NRMP (at http://www.nrmp.org/) algorithm in 1990s.
- 1900s: The matching began as a decentralized market.
- Unraveling:
  - 1930s: Hiring was completed 1/2 year before graduation.
  - 1940s: Hiring was done up to 2 years before graduation.
  - 1945: To halt unraveling medical schools agree not to release information about students until shortly before graduation.

The NRMP

- Congestion:
  - 1945: Offers remain open for 10 days.
  - 1949. Offers remain open for 12 hours.
- In 1952 the NRMP was introduced due to the dissatisfaction with the previous match procedure (Roth, Alvin; Elliott Peranson,1999) that suffers from unraveling and congestion.
- The NRMP was a hospital-proposing DA algorithm.

The NRMP

- 1950s: 95% appointments made though the NIMP.
- 1970s: It suffers for a decline in the participation rates to 85%.
- 1980s: the algorithm was specially unpopular with couples. It changes to accommodate them and participation increases.
- 1990s: Participation declines again.
- The NRMP algorithm saw only minor and incremental changes after its institution in 1952 until 1997.

The NRMP

- Groups such as the American Medical Student Association and others advocated to reconsider the current algorithm.
- In 1995 the Board of Directors of the NRMP commissioned a preliminary research program for the evaluation of the current algorithm and of changes to be considered in its operation and description, and a study comparing a new algorithm with the existing one.
- The new algorithm by Roth and Peranson was adopted in May 1997 and has been in use since its first application in March 1998.
- It was based on the student-proposing DA algorithm but accommodating couples.
- The study showed that the net effect of the change on actual matches has been minimal.

What are the issues?

- The original NRMP favors the hospitals.
- The original NRMP was manipulable.
- The NRMP has special features called "match variations":
  - Couples,
  - Hospital programs that fill even number of positions,
  - Hospital programs with positions that, if vacant, revert to other programs.

What are the issues?

- The NRMP is / can be made to be stable.
- Can some be math or not depending on the algorithm.
- Theory points to potential problem with variations: We can do numerical analysis and that is what Roth and Peranson did.

Descriptive Statistic of NRMP

	1987	1993	1994	1995	1996
APPLICANTS					
Applicants with ROLs	20071	20916	22353	22937	24749
Applicants who are coupled	694	854	892	998	1008
PROGRAMS					
Active Programs with ROL	3170	3622	3662	3745	3758
Programs with Even Match	4	2	6	7	8
Total Quota Before Match	19973	22737	22801	22806	22578

The New Roth and Peranson Algorithm

- The RP algorithm is based on the student-proposing DA algorithm but accommodating couples.
- The algorithm allows couples to express preferences for pairs of hospital programs.
- It first run the DA without couples and then add the couples one by one.
- If some one is displaced, then he or she is allowed to apply later in the algorithm.

An example at:
http://www-personal.umich.edu/~jeffshuo/nrmpcouples.html

The New Roth and Peranson Algorithm

- Roth and Vande Vate (1990) proof that starting for a any matching, there is a sequence of blocking pairs that leads to a stable matching in one to one matching without couples.
- Ma, Jinpeng (1996) not all stable matchings can be reached by a Roth-Vande Vate mechanism, and some are more likely to appear than others.

Difference Between Hospital and College Proposing DA

http://www.stanford.edu/~alroth/phase1.html

	1987	1993	1994	1995	1996
APPLICANTS					
Number of Applicants Affected	20	16	20	14	21
Applicant Proposing Preferred	12	16	11	14	12
Program Proposing Preferred	8	0	9	0	9
New Matched	0	0	0	0	1
New Unmatched	1	0	0	0	0
PROGRAMS					
Number of Programs Affected	20	15	23	15	19
Applicant Proposing Preferred	8	0	12	1	10
Program Proposing Preferred	12	15	11	14	9
Prog. with New Position(s) Filled	0	0	2	1	1
Prog. with New Unfilled Position(s)	1	0	2	0	0

Possible Manipulation by Students

- Upper limit in the number of applicant who could benefit by truncating their list at one above their original match point. This is called truncation and for students the truncation correspondence is exhaustive (Roth and Vande Vate, 1991).

	1987	1993	1994	1995	1996
Program-Proposing Algorithm	12	22	13	16	11
Applicant-Proposing Algorithm	0	0	2	2	9

Possible Manipulation by Hospitals

- Upper limit in the number of hospitals who could benefit by truncating their list at one above their original match point. (For Hospitals the truncation correspondence is not exhaustive (Kojima and Pathak, 2009).

	1987	1993	1994	1995	1996
Program-Proposing Algorithm	15	12	15	23	14
Applicant-Proposing Algorithm	27	28	27	36	18

Capacity Manipulation

- Upper limit number of programs that could improve their remaining matches by reducing quota (Sonmez 1997).

	1987	1993	1994	1995	1996
Program-Proposing Algorithm	28	16	32	8	44
Applicant-Proposing Algorithm	8	24	16	16	32

- Hospitals can both manipulate capacities and preferences and do not need to use truncations but this was not considered by Roth and Peranson.

Conclusion

- The R-P Algorithm does reasonably well but we need to know more. Simulations on generated data (Kojima y Pathak 2009) and additional theoretical analysis are our tools of trade.

Capacity Manipulation

The ability of a mechanism to achieve stable allocations is decisive in its success and its endurance (see Roth and Sotomayor 1990 and Roth 2003). Unfortunately stable matching are prone to manipulations:

- preference manipulation (Dubins and Friedman 1981 ....)
- capacity manipulation (Sönmez 1997, Konishi and Unver 2006)
- capacity and preference manipulation in large markets (Kojima y Pathak 2009).

Capacity Manipulation

The possibility of strategically reducing capacity is a concern when designing school mechanisms.

There is no stable-revelation mechanism able to prevent capacity manipulation!

Sönmez, T., 1997. Manipulation via capacities in two-sided matching markets. Journal of Economic Theory 77 (1), 197 – 204.

Capacity Manipulation

Romero-Medina, Antonio and Triossi, Matteo, (2013). Games with Capacity Manipulation: Incentives and Nash Equilibria. Social Choice and Welfare

- games of capacity manipulation: necessary and sufficient condition for non-manipulability.
- games with capacity manipulation: some results.

Matching markets

Matching Market (H, I, q, P)

- $H = \{h_1, ..., h_m\}$ hospitals
- $I = \{i_1, ..., i_n\}$
- $P_H = (P_{h_1}, ..., P_{h_m})$ hospitals' preferences over subset of interns, responsive.
- $P_I = (P_{i_1}, ...P_{i_n})$ be a list of interns' preferences over hospitals.
- i is acceptable to h if $\{i\} P_h \emptyset$. $A(h) \subseteq I$ interns who are acceptable to h.
- h is acceptable to i if $hP_i i$. $A(i) \subseteq H$ hospitals that are acceptable to i.
- q_h: maximum numbers of interns hospital h can hire is h's **capacity**

Matching Markets

Hospitals' preferences over subset of interns P_H are responsive iff:
for all $I' \subset I$ and for all interns $i, i' \in I$:

- $I' \cup \{i\} P_h I' \cup \{i'\} \Leftrightarrow i P_h i'$
- $I' \cup \{i\} P_h I' \Leftrightarrow i \in A(h)$.

A hospital h has **strong monotonic preferences** if it strictly prefers group of acceptable interns of larger cardinality to sets of acceptable interns of smaller cardinality.

Capacity Reporting Games

Under responsiveness the set of stable matchings is not empty:

- φ^I student optimal stable matching
- φ^H hospital optimal stable matching
- The set of stable matchings is a Lattice

The **capacity reporting game** induced by φ^V $(V = H, I)$ is a normal form game of complete information.
The set of players is H and the strategy space of hospital h is $\{1, ..., q_h\}$.
The outcome function is φ^V.

Results

Lemma: The equilibria of any capacity manipulation games are stable if and only if truth-telling is a dominant strategy.

Cycles and Stability

If an hospital has strong monotonic preference then any capacity reporting game has a Pure Strategy NE (Konishi and Unver).
But under the hospital optimal stable rule it does not necessary yields a stable outcome.

Cycles and Stability: Example

Hospitals: h_1, h_2.
Interns i_1, i_2.
Monotonic $P_{h_1} : \{i_1, i_2\}, \{i_1\}, \{i_2\}$ $P_{h_2} : \{i_1, i_2\}, \{i_2\}, \{i_1\}$.
Let$P_{i_1} : h_2, h_1$ and let$P_{i_2} = h_1, h_2$.
When the quotas are $(2,2), (1,2)$ and $(2,1)$, the unique stable matching is

$$\mu_1 \begin{array}{cc} h_1 & h_2 \\ \{i_2\} & \{i_1\} \end{array}$$

When the quota is $(1,1)$ the hospital-optimal stable matchings is

$$\mu_2 \begin{array}{cc} h_1 & h_2 \\ \{i_1\} & \{i_2\} \end{array}$$

$(1,1)$ is a NE when the quota is $(2,2)$ yielding μ_2 as outcome which is blocked by (h_1, i_1).

Cycles and Stability

Definition

A **cycle (of length $T+1$) in hospitals preferences** is given by $h_0, ..., h_T$ with $h_l \neq h_{l+1}$[a] for $i = 0, ...T$ and distinct $i_0, i_1, ..., i_T$ such that

1. $i_0 P_{h_0} i_T P_{h_T} i_{T-1} .... i_1 P_{h_1} i_0$,
2. for every l, $i_{l+1} \in A(h_l) \cap A(h_{l+1})$.

Hospitals preferences are *acyclical* if they have no cycles of length 2.

[a] From now on indices are considered modulo $T+1$.

Cycles and Stability

There is an alternating list of hospitals and interns "on a circle" such that every hospital in the cycle prefers the intern on its clockwise side to the intern on its counterclockwise side and find both acceptable.

Cycles and Stability

Acyclicity is weaker than common preference (Konishi and Unver (2006)).
The **definition of a cycle in interns' preference** is specular.

Cycles and Stability

Acyclicity is weaker than common preference (Konishi and Unver (2006)).
The **definition of a cycle in interns' preference** is specular.

A **simultaneous cycle** arises when there is an alternating list of hospitals and interns "on a circle" such that every hospital (resp. intern) prefers the intern (resp. hospital) on its clockwise side to the intern (resp hospital) on its counterclockwise side and find both acceptable.

General Results

Proposition: Assume that no simultaneous cycle exists and let $V = H, I$. Then:

1. Stating the true capacities is a dominant strategy under any stable rule φ.
2. The stable set of (H, I, q, P) is a singleton for every q.
3. The capacity revelation games induced by all stable rules φ have the same pure strategy Nash equilibrium outcome for every q: the unique stable matching of (H, I, q, P).

General Results

Assume that either the preferences of the hospitals or the preference of the interns are acyclical and let φ be a stable rule.

1. Stating the true capacities is a dominant strategy under φ.
2. The stable set of (H, I, q, P) is a singleton for every q.
3. The capacity revelation games induced by all stable rules φ have the same Nash equilibrium outcome for every q: the unique stable matching of (H, I, q, P).

Acyclicity of the preferences of one side of the market is the minimal conditional guaranteeing 1., 2. and 3 under φ^H.

The Intern Optimal Stable Matching.

A **generalized cycle (of length $T+1$) at** h is given by a cycle in hospital's preferences $h = h_0, ..., h_T$, $i_0, i_1, ..., i_T$ and by i_{-1} such that: $i_0 P_{h_0} i_{-1} P_{h_0} i_T$. Hospitals preferences are *weakly acyclical* if, there is no generalized cycle at any h. We will call intern i_{-1}, **intruder**.

A **non-monotonic cycle at** h is given by $M, M' \subseteq I$, with $|M| < |M'|$ such that:

1. $M P_h M'$
2. Let $M' \setminus M = \{i^1, ..., i^s\}$. For $k = 1, ..., s$ there is a generalized cycle at h, $h_0^k, ..., h_{T^k}^k, i_{-1}^k, i_0^k, i_1^k, ..., i_{T^k}^k$, $T^k \geq 1$ such that $i^k = i_0^k$ and $i_{-1}^k, i_{T^s}^k \in M \setminus M'$.
3. For $k \neq k'$, $i_l^k \neq i_{l'}^{k'}$ for all $l = 0, ...T^k$, $l' = 0, ..., T^{k'}$.

The Intern Optimal Stable Matching is more difficult to manipulate!

Proposition: Assume that no non-monotonic cycle exists. Then:

1. Stating the true capacities is a dominant strategy under φ^I.
2. The capacity revelation game induced by φ^I yields the intern-optimal stable matching at equilibrium.

The condition is the minimal one that guarantees 1. and 2.

The Intern optimal stable matching: Corollary

Corollary: Under the following conditions stating the true capacity is a dominant strategy in capacity revelation game induced by φ^I. The game yields the intern optimal stable matching at every Nash equilibrium.

1. The preferences of the hospitals are strongly monotonic in population.
2. The maximum length of every preference cycle is two.

Games with capacity manipulation

Games where:

1. Hospitals state their capacities.
2. Agents play an extensive form game given the capacity reported in 1.

Games with capacity manipulation: Findings

1. Hospitals state their capacities.
2. Agents reports their preferences given the capacity reported in 1 and φ^I is implemented.

Proposition

If preferences are acyclical, the games implements the unique stable matching under iterated elimination of weakly dominated strategies.

Notice

The result does not hold if φ^H is used, nor under no non-monotonic cycles.

Pre-arranged matching

What if hospitals and doctors can get to an agreement before the mechanism is played?

Sönmez, T., 1999. Can pre-arranged matches be avoided in two-sided matching markets? Journal of Economic Theory 86 (1), 148 – 156.

Conclusion

- intern-optimal stable matching is harder to manipulate than hospitals'.
- However, in games with capacity manipulation non manipulability seems to require stronger conditions.

References

- Dubins LE, Freedman DA (1981) Machiavelli and the Gale-Shapley algorithm. Am Math Mon 88:485–494
- Gale, D. and Ll. Shapley, (1962) "College Admissions and the Stability of Marriage", American Mathematical Monthly 69, 9-15.
- Kojima, F. and Parag A. Pathak, (2009) "Incentives and Stability in Large Two-Sided Matching Markets" American Economic Review 99, pp 608-627.
- Konishi I, Unver MU (2006) Games of capacity manipulation in hospital-intern markets. Soc Choice Welfare 27:3-24

References

- Ma, Jinpeng (1996), “On randomized matching mechanisms,” Economic Theory, Volume 8 Issue 2 (1996) pp 377-381 .
- Roth, Alvin; Elliott Peranson (1999). “The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design”. The American Economic Review 89 (4): 748–780.
- Roth, Alvin E. and M. Sotomayor, (1990) “Two-Sided Matchings Markets: A Study in Game Theoretic Modeling and Analysis”, Econometrics Society Monograph. Cambridge, MA; Cambridge University Press.
- Roth, A.E. and Vande Vate, J.H., “Random Paths to Stability in Two-Sided Matching,” Econometrica, 58, 1990, 1475-1480.
- Roth, A.E. and Vande Vate, J.H., “Incentives in Two-Sided Matching with Random Stable Mechanisms,” Economic Theory, 1991, Vol. 1, No. 1, 31-44.

References

- Romero-Medina, Antonio and Triossi, Matteo, 2013. Games with Capacity Manipulation: Incentives and Nash Equilibria. Social Choice and Welfare: Volume 41, Issue 3, Page 701-720, DOI: 10.1007/s00355-012-0703-1.
- Sonmez, Tayfun (1997), "Manipulation via Capacities in Two-Sided Matching Markets," Journal of Economic Theory, 77, 1, November, 197-204.
- Sönmez, T., 1999. Can pre-arranged matches be avoided in two-sided matching markets? Journal of Economic Theory 86 (1), 148 – 156.