Selenite Towers Move Faster Than Hanoï Towers, But Still Require Exponential Time

Jérémy Barbay*

Departamento de Ciencias de la Computación (DCC), Universidad de Chile, Santiago, Chile
`jeremy@barbay.cl`

Abstract

The Hanoï Tower problem is a classic exercise in recursive programming: the solution has a simple recursive definition, and its complexity and the matching lower bound correspond to the solution of a simple recursive function (the solution is so simple that most students memorize it and regurgitate it at exams without truly understanding it). We describe how some minor change in the rules of the Hanoï Tower yields various increases of difficulty in the solution, so that to require a deeper mastery of recursion than the classical Hanoï Tower problem. In particular, we analyze the Selenite Tower problem, where just changing the insertion and extraction positions from the top to the middle of the tower results in a surprising increase in the intricacy of the solution: such a tower of n disks can be optimally moved in a $\sqrt{3}^n$ moves for n even (i.e. less than a Hanoï Tower of same height), via five recursive functions following three distinct patterns.

1998 ACM Subject Classification F.2.2 [Analysis of Algorithms and Problem Complexity] Non-numerical Algorithms and Problems, F.2.m [Analysis of Algorithms and Problem Complexity] Miscellaneous

Keywords and phrases Brähma Tower, Disk Pile, HanoïTower, Levitating Tower, Recursivity

Digital Object Identifier 10.4230/LIPIcs.FUN.2016.5

1 Introduction

The Hanoï Tower problem is a classical problem often used to teach recursivity, originally proposed in 1883 by Édouard Lucas [5, 6], where one must move n disks, all of distinct size, one by one, from a peg A to a peg C using only an intermediary peg B, while ensuring no disk ever stands on a smaller one. As early as 1892, Ball [3] described an optimal recursive algorithm which moves the n disks of a HANOÏ TOWER in $2^n - 1$ steps. Many generalizations have been studied, allowing more than three pegs [4], coloring disks [7], and cyclic HANOÏ TOWERS [2]. Some problems are still open, as the optimality of the algorithm for 4-peg HANOÏ TOWER problem, and the analysis of the original problem is still a source of inspiration many years after its definition: for instance, Allouche and Dress [1] proved in 1990 that the movements of the Hanoï Tower problem can be generated by a finite automaton, making this problem an element of $SPACE(1)$.

The solution to the Hanoï Tower problem is simple enough that it can be memorized and regurgitated at will by students from all over the world: asking about it in an assignment or exam does not truly test a student's mastery of the concept of recursivity, pushing instructors to consider variants with slightly more sophisticated solutions. Some variants do not make the problem more difficult (e.g. changing the `insertion` and `removal` point to the bottom:

* Work supported by the Millennium Nucleus RC130003 "Information and Coordination in Networks".

© Jérémy Barbay;
licensed under Creative Commons License CC-BY
8th International Conference on Fun with Algorithms (FUN 2016).
Editors: Erik D. Demaine and Fabrizio Grandoni; Article No. 5; pp. 5:1–5:20

Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

the solution is exactly the same), some make it only slightly more difficult (e.g. considering the case where the disks are not necessarily of distinct sizes, described and analyzed in Appendix A), but some small changes can make it surprisingly more difficult.

We consider the SELENITE TOWER problem, which only differs from the HANOÏ TOWER problem in that the `insertion` and `removal` point in each tower is at the middle instead of the top (see Figure 1 for an illustration with SELENITE TOWERS of sizes $n = 3$ and $n = 4$, and Section 2.1 for the formal definition). One can poetically imagine that the Selenites[1] living on the moon can take advantage of the low gravity in order to remove and insert disks in the middle of the tower rather than merely at the top.

As for the classical HANOÏ TOWER, such `insertion` and `removal` rules guarantee that any move is *reversible* (i.e. any disk d removed from a peg X can always be immediately reinserted in the same peg X), that the `insertion` and `removal` positions are uniquely defined, that each peg can always receive a disk, and that each tower with one disk or more can always yield one disk. The problem is very similar to the HANOÏ TOWER problem: one would expect answering the following questions to be relatively easy, possibly by extending the answers to the corresponding questions on HANOÏ TOWERS[2]:

> Consider the problem of moving a SELENITE TOWER of n disks, all of distinct size, one by one, from a peg A to a peg C using only an intermediary peg B, while ensuring that at no time does a disk stand on a smaller one:
> **1.** Which sequences of steps permit moving such a tower?
> **2.** What is the minimal length of such a sequence?
> **3.** How many such shortest sequences are there?

We show that there is a unique shortest sequence of steps which moves a SELENITE TOWER of n disks of distinct sizes, and that it is of length at most $\sqrt{3}^n$ (i.e., exactly $\sqrt{3}^n = 3^{\frac{n}{2}}$ if n is even, and $\frac{3}{5}\sqrt{3}^{n-1} - \frac{2}{3} = 3^{\frac{n-1}{2}} + 2(3^{\frac{n-3}{2}} - 1) < \sqrt{3}^n$ if n is odd). As $\sqrt{3} \approx 1.733 < 2$, this sequence is exponentially shorter than the corresponding one for the HANOÏ TOWER problem (of length $2^n - 1$). We define formally the problem and its basic properties in Section 2: its formal definition in Section 2.1, some examples where such towers can be moved faster in Section 2.2, and some useful concepts on the `insertion` and `removal` order of a tower in Section 2.3. We describe a recursive solution in Section 3, via its algorithm in Section 3.1, the proof of its correctness in Section 3.2 and the analysis of its complexity in Section 3.3. The optimality of the solution is proved in Section 4, via an analysis of the graph of all possible states and transition (defined and illustrated in Section 4.1) and a proof of optimality for each function composing the solution (Section 4.2). We conclude with a discussion (Section 5) of other variants of similar or increased complexity, and share in Appendix A the text and the solution of a simpler variant successfully used in undergraduate assignments and exams.

2 Formal Definition and Basic Facts

In this section we define more formally the SELENITE TOWER (Section 2.1), how small examples already show that moving such towers requires less steps than moving a HANOÏ

---

[1] The word "selenite" is derived from the Greek lunar deity "$\Sigma\varepsilon\lambda\nu\eta\nu$", "Selene". The author H. G. Wells, in "The First Men In The Moon", referred to the inhabitants of the moon as selenites. The author Jules Verne also used this term in "From Earth to the Moon" and "Around the Moon".

[2] For a HANOÏ TOWER, the answer to those question is that there is a single such shortest sequence, of length $2^n - 1$, obtained by the recursion $h(n, A, B, C) = h(n-1, A, C, B).$"$A \to B;$"$.h(n-1, B, A, C)$ if $n > 0$ and $\emptyset$ otherwise.

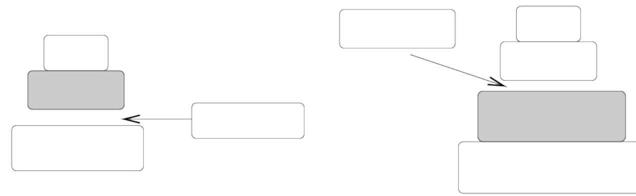

Figure 1 An illustration of the rules for the `insertion` and `removal` in a Selenite Tower, depending on the parity of its size (sizes $n = 3$ and $n = 4$ here). In each case, the shaded disk indicates the `removal` point and the arrow indicates the `insertion` point.

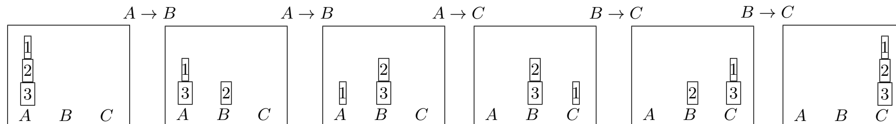

Figure 2 A Selenite Tower of three disks can be moved in just five steps.

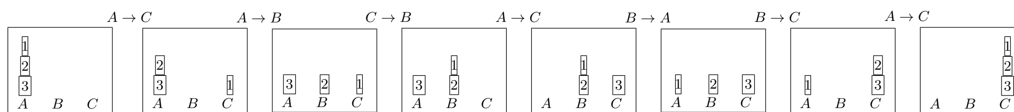

Figure 3 A Hanoï Tower of three disks requires seven steps to be moved between two pegs.

Tower (Section 2.2), and some properties of the order in which disks are inserted or removed on a peg to build or destroy a tower (Section 2.3).

2.1 Formal Definition

The "middle" disk of a tower of even size is not well defined, nor is the "middle" `insertion` point in a tower of odd size. We define both more formally in such a way that if n is odd, the `removal` position is the center one, and the `insertion` point is below it; while if n is even, the `insertion` point is in the middle of the tower, while the `removal` position is below the middle of the tower. See Figure 1 for an illustration with sizes $n = 3$ and $n = 4$.

More formally, on a peg containing n disks ranked by increasing sizes, the `removal` point is the disk of rank $\lfloor \frac{n}{2} \rfloor + 1$; and the `insertion` point is position $\lfloor \frac{n+1}{2} \rfloor$.

The `insertion` of disk d on peg X is *legal* if inserting d in the `insertion` point of X yields a legal configuration, where no disk is above a smaller one. A move from peg X to peg Y is *legal* if there is a disk d to remove from X, and if the `insertion` of d on the Y is legal.

2.2 Moving small towers – differences with Hanoï

For size one or two, there is no difference in the moving cost between a Hanoï Tower and a Selenite Tower. The first difference appears for size three, when only five steps are necessary to move a Selenite Tower (see the sequence of five steps to move a Selenite Tower of size $n = 3$ in Figure 2) as opposed to the seven steps required for moving a classical Hanoï Tower (see the sequence of seven steps to move a Hanoï Tower of size $n = 3$ in Figure 3).

When an odd number of disks is present on the peg A, and an even number is present on pegs B and C, a sub-tower of height 2 can be moved from A in 2 steps, when in a Hanoï Tower we need 3 steps to move any subtower of same height. In the Selenite Tower

problem, having a third disk "fixed" on A yields a reduced number of steps. We formalize this notion of "fixed" disk in the next section.

2.3 Structural facts on a single Peg

Before considering the complete problem over three pegs, we describe some concepts about single pegs, and on the order in which the disks are inserted and removed on a specific peg.

▶ **Definition 1.** We define the *removal order* as the order in which disks (identified by their rank in the final tower) can be removed from a Selenite Tower. Symmetrically, we define the *insertion order* as the order in which the disks are inserted in the tower.

The symmetry of the rules concerning the `insertion` and `removal` location of Selenite Towers yields that the *insertion* order is the exact reverse of the *removal* order (the `insertion` point of a tower is the `removal` point of a tower with one more disk), and each disk removed from a peg can be immediately put back exactly where it was.

In particular, a key argument to both the description of the solution in Section 3 and to the proof of its optimality in Section 4 is the fact that, when some (more extreme) disks are considered as "fixed" (i.e. the call to the current function has to terminate before such disks are moved), the order in which a subset of the disks is removed from a peg depends on the number of those "fixed" disks.

▶ **Definition 2.** When moving recursively n disks from a peg X with $x > n$ disks, the $x - n$ last disks in the `removal` order of X are said to be *fixed*. The *parity* of peg X is the parity of the number x of disks *fixed* on this peg.

Selenite Towers cannot be moved much faster than Hanoï Towers:

▶ **Lemma 3.** *Without a third peg, it is impossible to move more than one disk between two pegs with the same parity.*

Proof. Between two pegs of same parity, the `removal` order is the same. So the first disk needed on the final peg will be the last one removed from the starting peg. With more than one disk, we need the third peg to dispose temporally of the other disks. ◀

▶ **Lemma 4.** *It is impossible to move more than two disks between two pegs of opposite parities without a third peg.*

Proof. Between two pegs of opposite parities, the `removal` orders are different: But the definition of the middle is constant when the number of disks changes by 2. So after moving two disks the third cannot be inserted in the right place. ◀

The `removal` and `insertion` orders are changing with the parity of the Selenite Tower. Consider a peg with n disks on it:

- if $n = 2m + 1$ is odd, then the disks are removed in the following order:

$$(m+1, m+2, \quad m, m+3, \quad m-1, m+4, \quad \ldots, \quad 3, 2m, \quad 2, 2m+1, \quad 1)$$

- if $n = 2m$ is even, then the removal order is:

$$(m+1, m, \quad m+2, m-1, \quad m+3, m-2, \quad \ldots, \quad 2m-1, 2, \quad 2m, 1)$$

The relative order of m and $m+2$, of $m-1$ and $m+3$, and more generally of any pair of disks i and $m-i$ for $i \in [1..\lfloor n/2 \rfloor]$, are distinct. More specifically, disks are alternately extracted below and above the `insertion` point. This implies the two following connexity lemmas:

▶ **Lemma 5.** *The k first disks removed from the tower are contiguous in the original tower, and they are either all smaller or all larger than the $(k+1)$-th disk removed.*

▶ **Lemma 6.** *If k disks are all smaller than the disk below the `insertion` point, and all larger than the disk above the `insertion` point, then there exists an order for adding those k disks to the tower.*

Proof. By induction: for one disk it is true; for k disks, if the `insertion` point after the `insertion` of disc d is above d then add the larger and then the $k-1$ remaining disks, else add the smaller and then the $k-1$ disks left. ◀

In the next section, we present a solution to the Selenite Tower problem which takes advantage of the cases where two disks can be moved between the same two pegs in two consecutive steps.

3 Solution

One important difference between Hanoï Towers and Selenite Towers is that we do not always need to remove $n-1$ disks of a tower of n disks to place the n-th disk on another peg (e.g. in the sequence of steps shown in Figure 2, disk 3 was removed from A when there was still a disk sitting on top of it). But we need always to remove at least $n-2$ disks in order to release the n-th disk, as it is the last or the last-but-one disk removed. This yields a slightly more complex recursion than in the traditional case. We describe an algorithmic solution in Section 3.1, prove its correctness in Section 3.2, and analyze the length of its output in Section 3.3. We prove the optimality of the solution produced separately, in Section 4.

3.1 Algorithm

Note $|A|$ the number of disks on peg A, $|B|$ on B and $|C|$ on C. For each triplet $(x, y, z) \in \{0,1\}^3$, we define the function `movexyz`(n, A, B, C) moving n disks from peg A to peg C using peg B when $|A| \geq n$, $|A| - n \equiv x \mod 2$, $|B| \equiv y \mod 2$, $|C| \equiv z \mod 2$, and the n first disks extracted from A can be legally inserted on B and C. Less formally, there are x fixed disks on the peg A, y on B and z on C.

We need only to study three of those $2^3 = 8$ functions. First, as the functions are symmetric two by two: for instance, `move000`(n, A, B, C) behaves as `move111`(n, A, B, C) would if the `insertion` point in a tower of odd size was above the middle disk, and the `removal` point in a tower of even size was above the middle of the tower: in particular, they have exactly the same complexity. Second, the reversibility and symmetry of the functions yields a similar reduction: `move001`(n, A, B, C) has the same structure as the function `move100`(n, A, B, C) and the two have the same complexity.

We describe the python code implementing those functions in Figures 4 to 7, so that the initial call is made through the call `move000(n,"a","b","c")`, while recursive calls refer only to function `move000`(n, A, B, C) (Figure 4), function `move100`(n, A, B, C) (Figure 5), function `move001`(n, A, B, C) (similar to function `move100`(n, A, B, C) and described in Figure 6) and function `move010`(n, A, B, C) (Figure 7).

The algorithm for `move000`(n, A, B, C) (in Figure 4) has the same structure as the corresponding one for moving Hanoï Towers, the only difference being in the parity of the pegs

```
def move(a,b):
 print "("+a+",",
 print b+")",

def move000(n,a,b,c):
 if n>0 :
  move100(n-1,a,c,b)
  move(a,c)
  move001(n-1,b,a,c)
```

Figure 4

```
def move100(n,a,b,c):
 if n == 1 :
  move(a,c)
 elif n>1 :
  move100(n-2,a,c,b)
  move(a,c)
  move(a,c)
  move010(n-2,b,a,c)
```

Figure 5

```
def move001(n,a,b,c):
 if n == 1 :
  move(a,c)
 elif n>1 :
  move010(n-2,a,c,b)
  move(a,c)
  move(a,c)
  move001(n-2,b,a,c)
```

Figure 6

```
def move010(n,a,b,c):
 if n == 1 :
  move(a,c)
 elif n == 2 :
  move(a,b)
  move(a,c)
  move(b,c)
 elif n>2 :
  move010(n-2,a,b,c)
  move(a,b)
  move(a,b)
  move010(n-2,c,b,a)
  move(b,c)
  move(b,c)
  move010(n-2,a,b,c)
```

Figure 7

in the recursive calls, which implies calling other functions than `move000`(n, A, B, C), in this case `move001`(n, A, B, C) and `move100`(n, A, B, C). The algorithms for `move100`(n, A, B, C) (in Figure 5) and `move001`(n, A, B, C) (in Figure 6) and are taking advantage of the difference of parity between the two extreme pegs to move two consecutive disks in two moves, but still has a similar structure to the algorithm for `move000`(n, A, B, C) and the corresponding one for moving Hanoï Towers (just moving two disks instead of one).

The algorithm for `move010`(n, A, B, C) is less intuitive. Given that the `removal` and `insertion` orders on the origin peg A and on the destination peg C are the same (because the parity of those pegs is the same), $n-1$ disks must be removed from A before the last disk of the `removal` order, which yields a naive algorithm such as described in Figure 8. Such a strategy would yield a correct solution but not an optimal one, as it reduces the size only by one disk at the cost of two recursive calls and one step (i.e. reducing the size by two disks at the cost of four recursive calls and three steps). Another strategy (described in the algorithm in Figure 7) reduces the size by two at the cost of three recursive calls and four steps: moving $n-2$ disks to C, the two last disks of the `removal` order on B, then $n-2$ disks to A, the two last disks of the `removal` order on C, then finally the $n-2$ disks to C. The first strategy ($f(n) = 2f(n-1) + 2 = 4f(n-2) + 3$) yields a complexity within $\Theta(2^n)$ while the second strategy ($f(n) = 3f(n-2)+4$) yields a complexity within $\Theta(3^{\frac{n}{2}})$. We show in Section 3.2 that moving two disks at a time is correct in this context and in Section 4 that the latter yields the optimal solution.

```
def nonOptimalMove010(n,a,b,c):
 if n==1:
  move(a,c)
 else:
  move101(n-1,a,c,b)
  move(a,c)
  move101(n-1,b,a,c)
```

Figure 8 Alternative (non optimal) take on $move010(n, A, B, C)$.

```
def nonOptimalMove101(n,a,b,c):
 if n==1:
  move(a,c)
 else:
  move010(n-1,a,c,b)
  move(a,c)
  move010(n-1,b,a,c)
```

Figure 9 Alternative (non optimal) take on $move101(n, A, B, C)$.

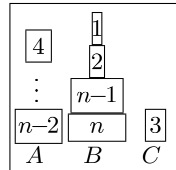

Figure 10 *Requirement for insertion* (i)*:* disks 4 to $n-2$ can be inserted on B as the `insertion` point of B is between 2 and $n-1$; and on C as the `insertion` point of C is under 3.

3.2 Correctness of the algorithm

We prove the correctness of our solution by induction on the number n of disks.

▶ **Theorem 7.** *For any positive integer value n, and any triplet $(x, y, z) \in \{0, 1\}^3$ of booleans, the function* `movexyz`(n, A, B, C) *produces a sequence of legal steps which moves a* Selenite Tower *from A to C via B.*

The proof is based on the following invariant, satisfied by all recursive functions on entering and exiting:

▶ **Definition 8.** *Requirement for insertion* (i)*:* The disks above the `insertion` point of B or C are all smaller than the first n disks removed from A; and the disks below the `insertion` point of B or C are all larger than the first n disks removed from A (see an illustration in Figure 10).

Proof. Consider the property $H(n)$ = "$\forall (x, y, z) \in \{0, 1\}^3$, $\forall i \leq n$, `movexyz`(i, A, B, C) is correct". $H(0)$ is trivially true, and $H(1)$ can be checked for all functions at once. For all values x, y, z, the function `movexyz`$(1, A, B, C)$ is merely performing the step `move`(A, C). The hypothesis $H(1)$ follows. Now, for a fixed $n > 1$, assume that $H(n-1)$ holds: we prove the hypothesis $H(n)$ separately for each function.

- Analysis of `move000`(n, A, B, C):
  1. According to $H(n-1)$ the call to `move100`$(n-1, A, B, C)$ is correct if (i) and $(p)_{100}$ are respected. (i) is implied by (i) on `move000`$(n-1, A, B, C)$; $(p)_{100}$ is implied by $(p)_{000}$ and the remaining disk on A $(a - n \mod 2 \equiv 0 \Rightarrow a - (n-1) \mod 2 \equiv 1 \mod 2)$.
  2. The step `move`(A, C) is both possible and legal because of the precondition (i) for `move000`(n, A, B, C): the disk moved was in the n first removed from A, and so can be introduced on C.
  3. The call to `move001`(n, A, B, C) is symmetrical to 1, and therefore correct.
  4. We can check the final state by verifying that the number of disks removed from A and added to C is $(n-1) + 1 = n$.

(a) (i): n odd: a is removed first, y is removed second.

(b) (ii): n even: y is removed first, x is removed second.

Figure 11 Removal order of the last two disks.

So `move000`(n, A, B, C) is correct.

- Analysis of `move100`(n, A, B, C):
  1. `move100`$(n-2, A, B, C)$ is correct according to $H(n-1)$, as the requirements are also: The requirement (i) is given by (i) for the initial call, and the parity $(p)_{100}$ is respected because we move two disks less than in the current call to `move100`(n, A, B, C).
  2. The two disks left (let us call them α and β) are in position (given Fig. 11, (i)) such that the removal order on A is (α, β) and the `insertion` order on C is (β, α) (see Fig. 11, (ii)). They can be inserted on C because of requirement (i). So the two disks are correctly moved in two steps.
  3. The requirements for `move010`$(n-2, A, B, C)$ are satisfied:
     - (i) stand as a consequence of the precondition (i) for the current call, as the $n-2$ disks to be moved on C were on A before the original call, in the middle of α and β.
     - $(p)_{010}$: The number of disks on C is still even as we added two disks. The number of disks on A is still odd as we removed two disks.

     Therefore, because of $H(n-2)$, `move010`$(n-2, A, B, C)$ is correct.

  This finishes the proof that `move100`(n, A, B, C) is correct.
- Analysis of `move001`(n, A, B, C): This function, being similar to `move100`(n, A, B, C), for a task symmetric, has the same proof of correctness.
- Analysis of `move010`$(2, A, B, C)$: The two disks (let us call them α and β) are in position (given Fig. 11, (ii)) such that the `removal` order on A is (β, α) and the `insertion` order on C is $(\alpha, \beta,)$, as A and C have the same parity. β can be inserted on B and they can both be inserted on C because of requirement (i). So the two disks are correctly moved in three steps, using peg B to temporally dispose of the disk β. So `move010`$(2, A, B, C)$ is correct.
- Analysis of `move010`(n, A, B, C) if $n > 2$: Note that fixing two disks on the same peg does not change the parity of this peg.
  1. `move010`$(n-2, A, B, C)$ is correct as: from (i) for the initial call results (i) for the first recursive call; $(p)_{010}$ is a natural consequence of $(p)_{010}$ for the initial call (because the parity of the peg is conserved when two disks are fixed on it). So $H(n-1)$ implies that `move010`$(n-2, A, B, C)$ is correct.
  2. As A and B have different parities, we can move two consecutive disks in two consecutive calls, as for `move100`(n, A, B, C).
  3. The second recursive call to `move010`$(n-2, A, B, C)$ verifies conditions (i) and $(p)_{010}$ as only two extreme disks (the smallest and the largest) have been removed from A.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f_{010}	0	1	3	7	13	25	43	79	133	241	403	727	1213	2185	3643
f_{100}	0	1	2	4	7	13	22	40	67	121	202	364	607	1093	1822
f_{000}	0	1	3	5	9	15	27	45	81	135	243	405	729	1215	2187
$3^{\lceil n/2 \rceil}$	1	3	3	9	9	27	27	81	81	243	243	729	729	2187	2187

Figure 12 The first values of f_{010},f_{100} and f_{000}, computed automatically from the recursion. Those corroborate the intuition that $f_{100}(n) < f_{000}(n)$ for values of n larger than 1.

4. The two next steps are feasible because of the difference of parity between B and C (same argument as point 2).
5. The last recursive call is symmetric to the first call, as we move the $n-2$ disks back between the two extreme disks, but this time on C.

Therefore `move010`(n, A, B, C) is correct. ◀

We analyze the complexity of this solution in the next section.

3.3 Complexity of the algorithm

Let $f_{xyz}(n)$ be the complexity of the function `movexyz`(n, A, B, C), when $|A| \geq n$, $|A|-n \equiv x \mod 2$, $|B| \equiv y \mod 2$ and $|C| \equiv z \mod 2$. The algorithms from Figures 4 to 7 yield a recursive system of four equations.

$$\begin{cases} \forall x,y,z & f_{xyz}(0) &= 0 \\ \forall x,y,z & f_{xyz}(1) &= 1 \\ & f_{010}(2) &= 3 \\ & & \\ \forall n>1, & f_{000}(n) &= f_{100}(n-1)+1+f_{001}(n-1) \\ \forall n>1, & f_{100}(n) &= f_{100}(n-2)+2+f_{010}(n-2) \\ \forall n>1, & f_{001}(n) &= f_{010}(n-2)+2+f_{001}(n-2) \\ \forall n>2, & f_{010}(n) &= 3f_{010}(n-2)+4 \end{cases}$$

As f_{001} is defined exactly as f_{100} (because of the symmetry between `move001`(n, A, B, C) and `move100`(n, A, B, C)), we can replace each occurrence of f_{001} by f_{100}, hence reducing the four equations to a system of three equations:

$$\begin{cases} \forall x,y,z & f_{xyz}(0) &= 0 \\ \forall x,y,z & f_{xyz}(1) &= 1 \\ & f_{010}(2) &= 3 \\ & & \\ \forall n>1, & f_{000}(n) &= 2f_{100}(n-1)+1 \\ \forall n>1, & f_{100}(n) &= f_{100}(n-2)+2+f_{010}(n-2) \\ \forall n>2, & f_{010}(n) &= 3f_{010}(n-2)+4 \end{cases}$$

Lemmas 9 to 11 resolve the system function by function. The function $f_{010}(n)$ can be solved independently from the others:

▶ **Lemma 9.**

$$f_{010}(n) = \begin{cases} 0 & \textit{if } n = 0; \\ 1 & \textit{if } n = 1; \\ 3 & \textit{if } n = 2; \\ 3^{\frac{n+1}{2}} - 2 & \textit{if } n \geq 3 \textit{ is odd; and} \\ 5 \times 3^{\frac{n}{2}-1} - 2 & \textit{if } n \geq 4 \textit{ is even.} \end{cases}$$

Proof. Consider the recurrence $X_{k+1} = 3X_k + 4$ at the core of the definition of f_{010}: a mere extension yields the simple expression $X_k = 3^k(X_0 + 2) - 2$.

- When $n \geq 3$ is odd, set $k = \frac{n-1}{2} \geq 1$, $U_0 = 1$ and $U_{k+1} = 3U_k + 4$ so that $f(2k+1) = U_k = 3^k(1+2) - 2$. Then $f_{010}(n) = 3 \times 3^k - 2 = 3^{k+1} - 2$ for $n \geq 3$ and odd.
- When $n \geq 4$ is even, set $k = \frac{n}{2} \geq 1$, $V_0 = 3$ and $V_{k+1} = 3V_k + 4$ so that $f(2k) = V_k = 3^k(3+2) - 2$, so that $f_{010}(n) = 5 \times 3^k - 2$ for $n \geq 4$ and even.

Gathering all the results yields the final expression. ◀

The expression for the function f_{010} yields the expression for the function f_{100}:

▶ **Lemma 10.**

$$f_{100}(n) = \begin{cases} 0 & \textit{if } n = 0; \\ 1 & \textit{if } n = 1; \\ 2 & \textit{if } n = 2; \\ 4 & \textit{if } n = 3; \\ \frac{5}{2} \times 3^{\frac{n}{2}-1} + 2 & \textit{where } n \geq 4 \textit{ is even; and} \\ \frac{3^{\frac{n+1}{2}} - 1}{2} & \textit{where } n \geq 5 \textit{ is odd.} \end{cases}$$

Proof. Consider the projection of the system to just f_{100}:

$$f_{100}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{100}(n-2) + 2 + f_{010}(n-2) & \text{if } n \geq 2 \end{cases}$$

For any integer value of $k \geq 0$, we combine some change of variables with the results from Lemma 9 to yield two linear systems, which we solve separately:

- $V_k = f_{100}(2k)$ and $V_0 = f_{100}(0) = 0$ so that $f_{100}(n) = V_k$ if n is even and $k = \frac{n}{2}$; and
- $U_k = f_{100}(2k+1)$ and $U_0 = f_{100}(1) = 1$ so that $f_{100}(n) = U_k$ if n is odd and $k = \frac{n-1}{2}$.

On one hand, $U_k = U_{k-1} + 2 + f_{010}(2k+1-2)$ for $k > 0$ and $U_0 = 1$. This yields a linear recurrence which we develop as follows:

$$\begin{aligned} U_k &= U_{k-1} + 2 + f_{010}(2k-1) \text{ by definition;} \\ &= U_{k-1} + 2 + 3 \times 3^{\frac{(2k-1)-1}{2}} - 2 \text{ via Lemma 9 because } 2k-1 \text{ is odd;} \\ &= U_{k-1} + 3^k \text{ by mere simplification;} \\ &= U_0 + \frac{3}{2}(3^k - 1) \text{ by resolution of a geometric series;} \\ &= \frac{3^{k+1} - 1}{2} \text{ because } U_0 = 1. \end{aligned}$$

Since $f_{100}(n) = U_{\frac{n-1}{2}}$ when n is odd, the solution above yields $f_{100}(n) = \frac{3^{\frac{n+1}{2}} - 1}{2}$ if n is odd.

On the other hand, $V_k = V_{k-1} + 2 + f_{010}(2k-2)$ for $k > 0$ and $V_0 = 0$. The initial conditions of f_{010} for $n = 0, 1$ and 2 yield the three first values of V_k: $V_0 = 0$; $V_1 =$

$V_0 + 2 + f_{010}(0) = 0 + 2 + 0 = 2$; and $V_2 = V_1 + 2 + f_{010}(2) = 2 + 2 + 3 = 7$. We then develop the recursion for $k \geq 3$ similarly to U_k:

$$\begin{aligned}
V_k &= V_{k-1} + 2 + f_{010}(2k-2) \text{ by definition;} \\
&= V_{k-1} + 2 + 5 \times 3^{\frac{(2k-2)}{2}-1} - 2 \text{ for } 2k-2 \geq 4 \text{ even, or any } k \geq 3 \text{ via Lemma 9;} \\
&= V_{k-1} + 5 \times 3^{k-2} \text{ by mere simplification (still only for } k \geq 3\text{);} \\
&= V_2 + 5\left(3^1 + \cdots + 3^{k-2}\right) \text{ by propagation;} \\
&= V_2 + 5\frac{3^{k-1} - 2}{2} \text{ by resolution of a geometric series;} \\
&= 7 + \frac{5}{2}(3^{k-1} - 2) \text{ because } V_2 = 7; \\
&= \frac{5}{2}3^{k-1} + 2 \text{ by simplification.}
\end{aligned}$$

Since $f_{100}(n) = V_{\frac{n}{2}}$ when n is even, the solution above yields $f_{100}(n) = \frac{5}{2}3^{\frac{n}{2}-1} + 2$ if n is even.

Reporting those results in the definition of f_{100} yields the final formula:

$$f_{100}(n) = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ 2 & \text{if } n = 2; \\ 4 & \text{if } n = 3; \\ \frac{5}{2} \times 3^{\frac{n}{2}-1} + 2 & \text{where } n \geq 4 \text{ is even; and} \\ \frac{3^{\frac{n+1}{2}}-1}{2} & \text{where } n \geq 5 \text{ is odd.} \end{cases}$$

◀

Finally, the expression for the function f_{100} directly yields the expression for the function f_{000}:

▶ **Lemma 11.**

$$f_{000}(n) = \begin{cases} 1 & \textit{if } n = 1 \\ 3 & \textit{if } n = 2 \\ 5 & \textit{if } n = 3 \\ 3^{\frac{n}{2}} & \textit{where } n \geq 4 \textit{ is even; and} \\ 5(3^{\frac{n-3}{2}} + 1) & \textit{where } n \geq 5 \textit{ is odd.} \end{cases}$$

Proof.

$$f_{100}(n) = \begin{cases} 1 & \text{if } n = 1; \\ 2 & \text{if } n = 2; \\ 4 & \text{if } n = 3; \\ \frac{5}{2}3^{\frac{n}{2}-1} + 2 & \text{where } n \geq 4 \text{ is even; and} \\ \frac{3^{\frac{n+1}{2}}-1}{2} & \text{where } n \geq 5 \text{ is odd.} \end{cases}$$

From these results, deduce the value of $f_{000}(n)$ using the fact that $f_{000}(n) = 2f_{100}(n-1) + 1$.

$$f_{000}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ 5 & \text{if } n = 3 \\ 5 \times 3^{\frac{n-1}{2}-1} + 5 & \text{where } n \geq 5 \text{ is odd; and} \\ 3^{\frac{n}{2}} & \text{where } n \geq 6 \text{ is even.} \end{cases}$$

As $\sqrt{3} \approx 1.73 < 2$, this value is smaller than the number $2^n - 1$ of steps required to move a Hanoï Tower. We prove that this is optimal in the next section.

4 Optimality

Each legal state of the Selenite Tower problem with three pegs and n disks can be uniquely described by a word of length n on the alphabet $\{A, B, C\}$, where the i-th letter indicates on which peg the i-th largest disk stands. Moreover, each word of $\{A, B, C\}^n$ corresponds to a legal state of the tower, so there are 3^n different legal states (even though not all of them are reachable from the initial state).

To prove the optimality of our algorithm, we prove that it moves the disks along the shortest path in the *configuration graph* (defined in Section 4.1) by a simple induction proof (in Section 4.2).

4.1 The configuration graph

The configuration graph of a Selenite Tower has 3^n vertices corresponding to the 3^n legal states, and two states s and t are connected by an edge if there is a legal move from state s to state t. The reversibility of moves (seen in Section 2.3) implies that the graph is undirected.

Consider the initial state $A \ldots A$ $(= A^n)$. The smallest disk 1 cannot be moved before the other disks are all moved to peg B or all moved to peg C: we cannot remove disk 1 from peg A if there is a disk under it, and we cannot put it on another peg if a larger disk is already there. This partitions G into three parts, each part being characterized by the position of disk 1; these parts are connected by edges representing a move of disk 1 (see the recursive decomposition of $G(n)$ in Figure 13).

Each part is an instance of the configuration graph $G'(n-1)$ defining all legal steps of $(n-1)$ disks $\{2, \ldots, n\}$ given that disk 1 is fixed on its peg.

Let us consider this subgraph $G'(n-1)$, when disk 1 (the smallest) is fixed on one peg (say on peg A). Note each state of this graph $aX \ldots Z$, where a stands for the disk 1 fixed on peg A, and $X \ldots Z \in \{A, B, C\}^{n-1}$ describes the positions of the other disks. The removal order changes from those observed in G each time $|A|$ is odd.

To remove the two extreme disks 2 and n (not moving disk 1, since it is fixed), it is necessary to move all other disks to a single other peg (same argument as for $G(n)$), so we can divide our configuration graph in subsets of states corresponding to different positions where disks 2 and n are fixed.

This defines 9 parts, as each of the two fixed disks can be on one of the three pegs. Of those 9 parts, we need to focus only on 5:

- two parts of the graph cannot be accessed from the initial state $aA \ldots A$, (see an illustration in Figure 14); and

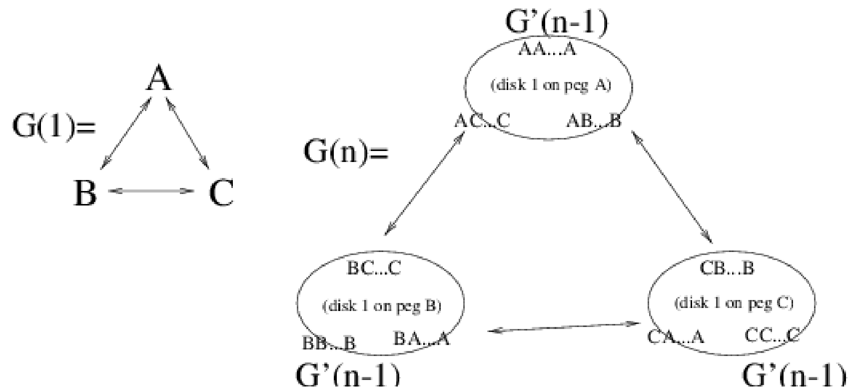

Figure 13 First decomposition of the configuration graph of the SELENITE TOWER problem.

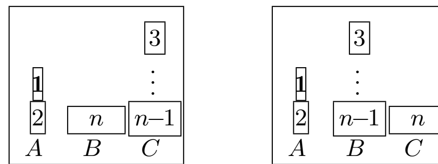

Figure 14 States where disk 2 is on A and disk n is on another peg (i.e. B or C) cannot be accessed from the initial state $A \ldots A$ for $n > 4$. No move is possible from these states as A cannot receive a larger disk than 2 (and all are), B cannot receive a smaller disk than n (and all disks are of size smaller or equal to n), and C cannot receive the disks 2 nor n if $n > 4$.

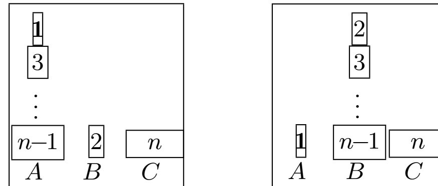

Figure 15 States $aBA \ldots AC$ and $aBB \ldots BC$ are not connected in the subgraph where disks 2 and n are fixed on B and C, and disk 1 is fixed on A: As no disk can be inserted under n, if $n > 4$ it is impossible to move the $n - 3 > 1$ unfixed disks from A to B (as to move more than one disk between two pegs of same parity require a third peg).

- the part of the graph where disk 2 is fixed on B and disk n is fixed on C contains two parts, which are not connected for $n > 4$ (see an illustration in Figure 15).

The five remaining parts are very similar. Three of them are of particular importance as each contains one key state, which are $aA \ldots A$, $aB \ldots B$ and $aC \ldots C$. Consider first the graphs $G'(n)$ for $n \in \{1, 2, 3\}$ ($n + 1$ disks in total if we count the fixed one): they are represented in Figure 16. When one disk is fixed on A, the task of moving disks from A to B is symmetric with moving them from A to C, but quite distinct from the task of moving disks from B to C.

Now, consider the part of the graph $G'(n-1)$ where the smallest and the largest disks (2 and n) are fixed on A. This part contains the initial state $A \ldots A$. The only way to free the smallest disk is to move the $n - 3$ other disks to another peg.

Once disks 2 and n are fixed on the same peg (in addition to disk 1), the situation is similar to the entire graph, with two fewer disks. It is the case each time two extreme disks

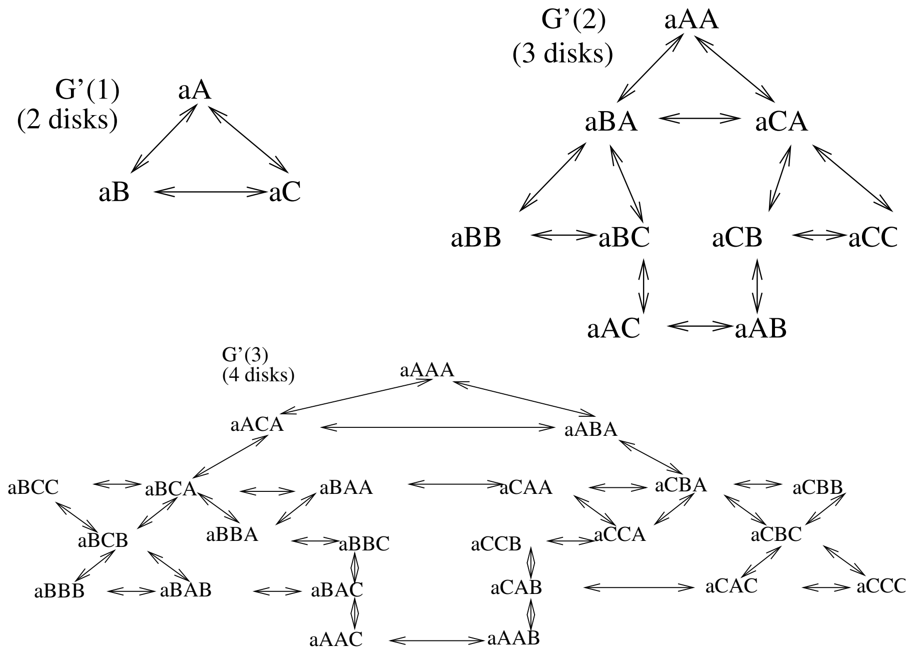

Figure 16 Subgraphs $G'(n)$ with one disk fixed on the peg A for $n \in \{1, 2, 3\}$.

are fixed on the same peg: when 2 and n are fixed on peg C or B, or when 1 and n are fixed on peg A; the process can then ignore the two fixed disks to move the $n - 3$ remaining disks, as the parity of the peg is unchanged. See the definitions of the graph $G'(n)$ in Figure 16 for $n \in \{1, 2, 3\}$ and in Figure 17 for $n > 3$.

4.2 Proof of optimality

To prove the optimality of the solution described in Section 3, we prove that the algorithm is taking the shortest path in the configuration graph defined in the last section. A side result is that this is the unique shortest solution.

▶ **Theorem 12.** $\forall(x, y, z) \in \{0, 1\}^3$, $\forall n \geq 0$, `movexyz`(n, A, B, C) *moves optimally* n *disks from* A *to* C.

Proof. Define the induction hypothesis $H(n)$ as "$\forall(x, y, z) \in \{0, 1\}^3$ `movexyz`(n, A, B, C) moves optimally n disks from A to C". Trivially $H(0)$ and $H(1)$ are true. Suppose that there exists an integer $N > 1$ such that $\forall n < N$, the induction hypothesis $H(n)$ is true. We prove that $H(N)$ is then also true.

- `move000`(N, A, B, C) is optimal: `move000`(N, A, B, C) for $N > 0$ consists of one call to `move100`(N, A, C, B), one unitary step, and one call to `move001`(N, B, A, C).
  Therefore it moves optimally (by $H(N - 1)$) from $aA \dots A$ to $aB \dots B$, and then to $cB \dots B$, and after that to $cC \dots C$. (In Figure 13 the right edge of the triangle.)
  A path not going through states $aB \dots B$ or $cB \dots B$ would take more steps:
  - if we do not go through the state $aB \dots B$, then the state $aC \dots C$ is necessary, with a cost of $f_{100}(N - 1)$, and also the state $bC \dots C$ (with a cost of 1), and at the end of the path we have to go through the state $cA \dots A$, which optimal path to go to the final $cC \dots C$ state is of length $f_{100}(N-1)$: this path is of length $f_{100}(N-1)+1+f_{100}(N-1)$ and is already as long as the one given by `move000`(N, A, B, C).

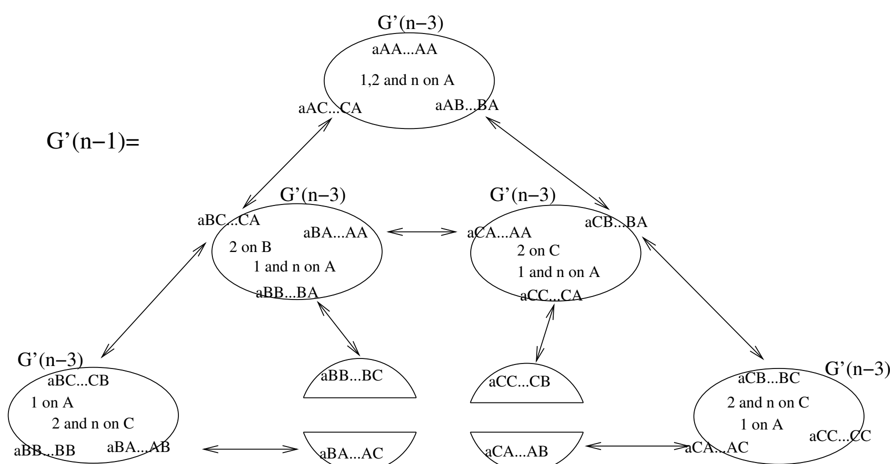

Figure 17 Recursive definition of $G'(n)$, the graph of all legal steps when one disk is fixed on the first peg, for $n > 3$. There is no way to connect the states $aBB....BC$, $aBA...AC$, $aCC...CB$ and $aCA...AB$ without moving some of the disks from $\{1, 2, n\}$.

- if we go through $aB \dots B$, but not through $cB \dots B$, then the path is not optimal as it must go through $aC \dots C$ and the optimal path from $aA \dots A$ to $aC \dots C$ does not go through $aB \dots B$.

So `move000`(N, A, B, C) is optimal.

- `move100`(N, A, B, C) is optimal:

`move100`(N, A, B, C) for $N > 1$ consists of one call to `move100`$(N-2, A, C, B)$), two steps, and one call to `move010`$(N-2, B, A, C)$).
As before, we shall consider these recursive calls of order smaller than N as optimal because of $H(N-2)$. So we know how to move optimally from $aAA \dots AA$ to $aAB \dots BA$, to $aCB \dots BA$, then to $aCB \dots BC$ and to $aCC \dots CC$ (in Figure 17), this corresponds to the left edge of the triangle).
We must now prove that other paths take more steps:
  - We cannot avoid the state $aCB \dots BC$, neither $aCB \dots BA$, as there is no other way out of $aCC \dots CC$.
  - if we avoid the state $aAB \dots BA$ then the optimal path to $aCB \dots BA$ necessarily passes by $aBA \dots AA$ and $aCA \dots AA$, and is of length $f_{100}(N-2) + 1 + f_{010}(N-2) + 1 + f_{010}(N-2)$, which is longer than the whole solution given by the algorithm, of length $f_{100}(N) = f_{100}(N-2) + 2 + f_{010}(N-2)$.

So `move100`(N, A, B, C) is optimal.

- `move010`(N, A, B, C) is optimal:

`move010`$(1, A, B, C)$ and `move010`$(2, A, B, C)$ are special cases, we can see in graphs $G'(1)$ and $G'(2)$ on figure 16 page 14 that the optimal paths between $aB \dots B$ and $aC \dots C$ are of length 1 and 3, as the solutions produced by the algorithm. So `move010`$(1, A, B, C)$ and `move010`$(2, A, B, C)$ are proven optimal.
`move010`(N, A, B, C) for $N > 2$ corresponds to a path going through the states (the first disk being fixed on b): (please report to Fig. 17 from $aCC \dots CC$ to $aBB \dots BB$ down left to down right.)

$$aCC \dots CC \xrightarrow{f_{010}(N-2)} aCB \dots BC \xrightarrow{2} aAB \dots BC$$
$$\xrightarrow{f_{010}(N-2)} aAC \dots CC \xrightarrow{2} aBC \dots CB \xrightarrow{f_{010}(N-2)} aBB \dots BB$$

We shall demonstrate that all other paths take more steps:

- The states $bAC\dots CA$ and $bCA\dots AC$ are mandatory, for connectivity, and so are $bAC\dots CB$ and $bCA\dots AB$.
- going through the state $bBC\dots CB$ makes the state $bBA\dots AB$ mandatory.
- going around the state $bBC\dots CB$ makes the states $bAB\dots BB$, $bCB\dots BB$ and $bCA\dots AB$ mandatory: the total path would be of length $3+4f_{010}(N-2)$, to be compared with $4+3f_{010}(N-2)$ (We trade one step with one recursive call). As $f_{010}(N-2)\geq 1$ for $N-2\geq q$ (i.e. $N\geq 3>2$), `move010`(N,A,B,C) is optimal for $N>2$.

So `move010`(N,A,B,C) is optimal. ◀

We discuss further extensions of those results in the next section.

5 Discussion

All the usual research questions and extensions about the Hanoï Tower problem are still valid about the Selenite Tower problem. We discuss only a selection of them, such as the space complexity in Section 5.1, and the extension to other proportional `insertion` and `removal` points in Section 5.2.

5.1 Space Complexity

Allouche and Dress [1] showed that the optimal sequence of steps required to move a Hanoï Tower of n disks can be obtained by a simple function from the prefix of an infinite unique sequence, which itself can be produced by a finite automaton. This proves that the space complexity of the Hanoï Tower problem is constant.

The same technique does not seem to yield constant space for Selenite Towers: whereas the sequences of steps generated by each of the functions `move100`(n,A,B,C), `move010`(n,A,B,C) and `move001`(n,A,B,C) are prefixes of infinite sequences, extracting those suffixes and combining them in a sequence corresponding to `move000`(n,A,B,C) would require a counter using logarithmic space in the length of the sequences to be extracted, i.e. $\log_2(\sqrt{3}^n)\in\Theta(n)$, which would still be linear in the number of disks.

5.2 Levitating Towers

An extension of the Selenite Tower problem is to parametrize the `insertion` point, so that the `removal` point is at position $\lfloor\alpha n\rfloor+1$ and the `insertion` point is under the disk at position $\lfloor\alpha(n+1)\rfloor$ in a tower of n disks, for $\alpha\in[0,\frac{1}{2}]$ fixed (the problem is symmetrical for $\alpha\in[\frac{1}{2},1]$). By analogy with Selenite Towers, we call this variant a α-Levitating Tower. This parametrization creates a continuous range of variants, of which the Hanoï Tower problem and the Selenite Tower problem are the two extremes:

- for $\alpha=0$, the `removal`/`insertion` point is always at the top, which corresponds to a Hanoï Tower, while
- for $\alpha=\frac{1}{2}$ the problem corresponds to a Selenite Tower.

The complexity of moving a α-Levitating Tower cannot be smaller than the one of a Selenite Tower, as the key configuration permitting to move 2 disks in 2 steps between the same pegs is less often obtainable in a α-Levitating Tower.

Acknowledgements. We would like to thank Claire Mathieu, Jean-Paul Allouche and Srinivasa Rao for corrections and encouragements, and Javiel Rojas-Ledesma and Carlos Ochoa-Méndez for their comments on preliminary drafts.

Funding. Jérémy Barbay is partially funded by the Millennium Nucleus RC130003 "Information and Coordination in Networks".

References

1 J.-P. Allouche and F. Dress. Tours de Hanoï et automates. *RAIRO, Informatique Théorique et applications*, 24(1):1–15, 1990.
2 M.D. Atkinson. The cyclic towers of Hanoï. *Information Processing Letters (IPL)*, 13(118-119), 1981.
3 W. R. Ball. *Mathematical Recreations and Essays.* McMillan, London, 1892.
4 J. S. Frame and B. M. Stewart. Solution of problem no 3918. *American Mathematics Monthly (AMM)*, 48:216–219, 1941.
5 Édouard Lucas. La tour d'Hanoï, véritable casse-tête annamite. In a puzzle game., Amiens, 1883. Jeu rapporté du Tonkin par le professeur N.Claus (De Siam).
6 Édouard Lucas. *Récréations Mathématiques*, volume II. Gauthers-Villars, Paris, quai des Augustins, 55, 1883.
7 D. Wood. The towers of Brahma and Hanoï revisited. *Journal of Recreational Mathematics (JRM)*, 14(1):17–24, 1981.

A Disk Pile Problem

The Hanoï Tower problem is a classic example on recursivity, originally proposed by Édouard Lucas [5] in 1883. A recursive algorithm is known since 1892, moving the n disks of a Hanoï Tower in $2^n - 1$ unit moves, this value being proven optimal by a simple lower bound [3].

Consider the Disk Pile problem, a very simple variant where we allow some disks to be of the same size. This obviously introduces some much easier instances, including an extreme one where the disks are all the same size and the resulting tower can be moved in linear time (see Figure 18 for the sequence of steps moving such a tower of size 3 with a single size of disks).

1. Give a recursive algorithm to move a Disk Pile from one peg to the other, using only one extra peg, knowing that $\forall i \in \{1, \ldots, s\}$, n_i is the number of disks of size i. Your algorithm must be efficient for the cases where all the disks are the same size, and where all the disks are of distinct sizes.
   Solution: We present an algorithm in Figure 19. It is very similar to the algorithm for moving a Hanoï Tower, the only difference being that it moves the n_i disks of size i at the same time, in n_i consecutive moves. ◀

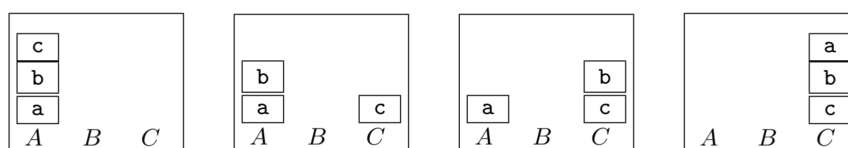

Figure 18 Moving a Disk Pile of size 3.

```
def diskPileMove(n,sizes,a,b,c):
  if n>0 :
    move(n-sizes[-1],sizes[0:-1],a,c,b)
    for i in range(0,sizes[-1]):
      move(a,c)
    move(n-sizes[-1],sizes[0:-1],b,a,c)
```

Figure 19 Python code to move a Disk Pile.

2. Give and prove the worst case performance of your algorithm over all instances of fixed s and vector $(n_1, \dots, n_s)$.
Solution: By solving the recursive formula directly given by the recursion of the algorithm, one gets that the n_s largest disks are moved once, the n_{s-1} second largest disks are moved twice, the n_{s-2} third largest disks are moved four times, and so on to the n_1 smallest disks, which are each moved 2^{s-1} times. Summing all those moves gives the number of moves performed by the algorithm:
$$\sum_{i\in\{1,\dots,s\}} n_i 2^{s-i}.$$
Note that for $s = n$ and $n_1 = \dots = n_s = 1$, this yields $\sum_{i=1}^{s-1} 2^i = 2^n - 1$, the solution to the traditional HANOÏ TOWER problem. ◀
3. Prove that a performance of $\sum_{i\in\{1,\dots,s\}} n_i 2^{s-i}$ is optimal.
Solution: We prove a lower bound of $\sum_{i\in\{1,\dots,s\}} n_i 2^{s-i}$, for n disks of s distinct sizes, with n_i disks of size i by induction on the number of types of disks. We prove by induction on the number of types of disk s that any pile of disks of sizes $(n_1, \dots, n_s)$ requires $\sum_{i\in\{1,\dots,s\}} n_i 2^{s-i}$ disk moves to be moved to another peg.
   - *Initial Case*: for $s = 1$ the bound is n_1 and is obviously true, since each disk must be individually moved from one peg to the other.
   - *Inductive Hypothesis*: suppose there is some $\sigma \geq 1$ so that any pile of disks sizes $(n_1, \dots, n_\sigma)$ requires $\sum_{i\in\{1,\dots,\sigma\}} n_i 2^{\sigma-i}$ disk moves to be moved to another peg.
   - *Inductive Step*: consider a pile of disks of sizes $(n_1, \dots, n_{\sigma+1})$: clearly all the disks of sizes smaller than $\sigma + 1$ need to be gathered on a unique peg before the largest disks can be moved, to allow those last ones to be moved in $n_{\sigma+1}$ disk moves, after which all the disks of sizes smaller than $\sigma + 1$ need to be stacked above the largest ones. By the inductive hypothesis, moving the smaller disks will require $\mathbf{2}\sum_{i\in\{1,\dots,\sigma\}} n_i 2^{\sigma-i}$ disk moves, to be added to the $n_{\sigma+1}$ disk moves. Hence, any pile of disks of sizes $(n_1, \dots, n_{\sigma+1})$ requires $\sum_{i\in\{1,\dots,\sigma+1\}} n_i 2^{\sigma+1-i}$ disk moves to be moved to another peg.
   - *Conclusion*: The inductive hypothesis is verified for the initial case where $s = 1$, and propagates to any value of $s \geq 1$ through the inductive step. We conclude that any pile of disks of sizes $(n_1, \dots, n_s)$ for $s \geq 1$ requires $\sum_{i\in\{1,\dots,s\}} n_i 2^{s-i}$ disk moves to be moved to another peg.

   ◀
4. What is the worst case complexity of the DISK PILE problem over all instances of fixed value s and fixed total number of disks n? *Solution:* The worst case (of both the algorithms and the most precise lower bound with the number of disks of each size fixed) occurs when $n_1 = n - s + 1$ and $n_2 = \dots = n_s = 1$. Using the previous results, it yields a complexity of $2^{s-1}(n - s + 1) + \sum_{i=1}^{s-1} 2^i = 2^{s-1}(n - s + 2) - 1$ steps in the worst case over all instances of fixed value s and fixed total number of disks n. This correctly yields $2^n - 1$ when $s = n$, in the worst case over all instances of fixed total number of disks n. ◀

B Code to generate the sequences of moves

See Figure 20 for the `python` code used to print the sequence of moves for a given Selenite Tower. For $n = 10$ it generates the following sequences:

```
move000(1,'a','b','c') =  (a,c)

move000(2,'a','b','c') = (a,b) (a,c) (b,c)

move000(3,'a','b','c') = (a,b) (a,b) (a,c) (b,c) (b,c)

move000(4,'a','b','c') = (a,c) (a,b) (a,b) (c,b) (a,c) (b,a) (b,c)
(b,c) (a,c)

move000(5,'a','b','c') = (a,c) (a,c) (a,b) (a,b) (c,a) (c,b) (a,b)
(a,c) (b,c) (b,a) (c,a) (b,c) (b,c) (a,c) (a,c)

move000(6,'a','b','c') = (a,b) (a,c) (a,c) (b,c) (a,b) (a,b) (c,b)
(c,a) (c,a) (b,c) (a,b) (a,b) (c,b) (a,c) (b,a) (b,c) (b,c) (a,b)
(c,a) (c,a) (b,a) (b,c) (b,c) (a,b) (a,c) (a,c) (b,c)

move000(7,'a','b','c') = (a,b) (a,b) (a,c) (a,c) (b,a) (b,c) (a,c)
(a,b) (a,b) (c,a) (c,b) (a,b) (c,a) (c,a) (b,a) (b,c) (a,c) (a,b)
(a,b) (c,a) (c,b) (a,b) (a,c) (b,c) (b,a) (c,a) (b,c) (b,c) (a,c)
(a,b) (c,b) (c,a) (c,a) (b,c) (b,a) (c,a) (b,c) (b,c) (a,c) (a,b)
(c,b) (a,c) (a,c) (b,c) (b,c)

move000(8,'a','b','c') = (a,c) (a,b) (a,b) (c,b) (a,c) (a,c) (b,c)
(b,a) (b,a) (c,b) (a,c) (a,c) (b,c) (a,b) (a,b) (c,b) (c,a) (c,a)
(b,c) (a,b) (a,b) (c,b) (c,a) (c,a) (b,c) (b,a) (b,a) (c,b) (a,c)
(a,c) (b,c) (a,b) (a,b) (c,b) (c,a) (c,a) (b,c) (a,b) (a,b) (c,b)
(a,c) (b,a) (b,c) (b,c) (a,b) (c,a) (c,a) (b,a) (b,c) (b,c) (a,b)
(a,c) (a,c) (b,a) (c,b) (c,b) (a,b) (c,a) (c,a) (b,a) (b,c) (b,c)
(a,b) (c,a) (c,a) (b,a) (b,c) (b,c) (a,b) (a,c) (a,c) (b,a) (c,b)
(c,b) (a,b) (a,c) (a,c) (b,a) (b,c) (b,c) (a,c)
```

C Code used to generate the values of the complexity

See Figure 21 for the `python` code used to experimentally generate the values of the complexity. For $n = 16$, it generates the following array:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_{010}	0	1	3	7	13	25	43	79	133	241	403	727	1213	2185	3643	6559
f_{100}	0	1	2	4	7	13	22	40	67	121	202	364	607	1093	1822	3280
f_{000}	0	1	3	5	9	15	27	45	81	135	243	405	729	1215	2187	3645
$3^{\lceil n/2 \rceil}$	1	3	3	9	9	27	27	81	81	243	243	729	729	2187	2187	6561

```
def move(a,b):
  print "("+a+","+b+")",

def move010(n,a,b,c):
  if n == 1 :
    move(a,c)
  elif n == 2:
    move(a,b)
    move(a,c)
    move(b,c)
  elif n>2 :
    move010(n-2,a,b,c)
    move(a,b)
    move(a,b)
    move010(n-2,c,b,a)
    move(b,c)
    move(b,c)
    move010(n-2,a,b,c)

def move100(n,a,b,c):
  if n == 1 :
    move(a,c)
  elif n>1 :
    move100(n-2,a,c,b)
    move(a,c)
    move(a,c)
    move010(n-2,b,a,c)

def move001(n,a,b,c):
  if n == 1 :
    move(a,c)
  elif n>1:
    move010(n-2,a,c,b)
    move(a,c)
    move(a,c)
    move001(n-2,b,a,c)

def move000(n,a,b,c):
  if n == 1 :
    move(a,c)
  elif n>1 :
    move100(n-1,a,c,b)
    move(a,c)
    move001(n-1,b,a,c)

for i in range(1,9):
  print("move000("
   +str(i)+",'a'"
   +",'b'"+",'c') = "),
  move000(i,'a','b','c')
  print
  print
```

Figure 20 Python code to move a Selenite Tower.

```
def f010(n):
  if n < 2 :
    return n
  elif n == 2:
    return 3
  else:
    return 3*f010(n-2)+4

def f100(n):
  if n < 2 :
    return n
  else:
    return f100(n-2)+f010(n-2)+2

def f000(n):
  if n < 2 :
    return n
  else:
    return 2*f100(n-1)+1

def power(n):
  if n % 2 == 0 :
    return 3**(n/2)
  else:
    return 3**((n+1)/2)

def printArray(n):
  print("n\t "),
  for i in range (0,n):
    print(" &\t "+str(i) ),
  print("\\\\ \\hline")
  print("f_{010}\t "),
  for i in range (0,n):
    print(" &\t" + str(f010(i))),
  print("\\\\ \\hline")
  print("f_{100}\t "),
  for i in range (0,n):
    print(" &\t" + str(f100(i))),
  print("\\\\ \\hline")
  print("f_{000}\t "),
  for i in range (0,n):
    print(" &\t" + str(f000(i))),
  print("\\\\ \\hline")
  print("3^{\\lceil n/2\\rceil }\t "),
  for i in range (0,n):
    print(" &\t "+str(power(i)) ),
  print("\\\\")

printArray(16)
```

Figure 21 Python code to generate experimentally the values of the complexity.