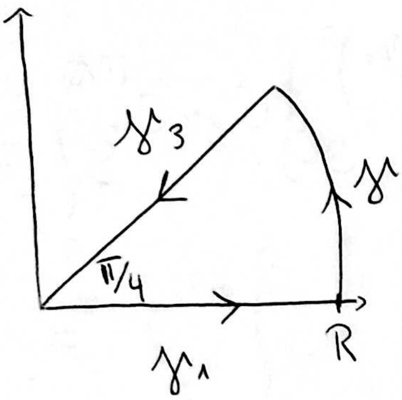
Pauta Auxiliar 10

P1) Siguiendo la indicación, consideramos la curva: $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, orientada positivamente



$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f \quad , f(z) = e^{iz^2}.$$

Calculemos cada integral del lado derecho.

$\underline{\gamma_1}$: $r_1(t) = t$, $t \in [0,R] \Rightarrow r_1'(t) = 1$.

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_0^R e^{it^2}\,dt = \int_0^R \cos(t^2)\,dt + i\int_0^R \operatorname{sen}(t^2)\,dt$$

$\underline{\gamma_2}$: $r_2(t) = Re^{it}$, $t \in [0, \pi/4] \Rightarrow r_2'(t) = Rie^{it}$.

Luego:

$$\left|\int_{\gamma_2} e^{iz^2}\,dz\right| = \left|\int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it}\,dt\right| = \left|\int_0^{\pi/4} e^{iR^2(\cos(2t)+i\operatorname{sen}(2t))} Rie^{it}\,dt\right|$$

$$\underset{\substack{\downarrow \\ |e^z| = e^{Re(z)}}}{\leq} R\int_0^{\pi/4} e^{-R^2\operatorname{sen}(2t)}\,dt \underset{\substack{\downarrow \\ \text{indicación, exp creciente} \\ (\operatorname{sen}(\alpha) \geq \frac{2\alpha}{\pi} \text{ en } [0,\pi/2])}}{\leq} R\int_0^{\pi/4} e^{-R^2\cdot\frac{4t}{\pi}}\,dt = -\frac{\pi}{4R} e^{\frac{-R^2 4t}{\pi}}\Big|_0^{\pi/4}$$

$$= \frac{-\pi}{4R}\left(e^{-R^2} - 1\right) \xrightarrow[R\to\infty]{} 0$$

$\underline{\gamma_3}$: $r_3(t) = (R-t)e^{i\pi/4}$, $t \in [0,R] \Rightarrow r_3'(t) = -e^{i\pi/4}$.

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$$\Rightarrow \int_{\gamma_3} e^{iz^2}\,dz = \int_0^R e^{i(R-t)^2 e^{i\pi/2}}(-e^{i\pi/4})\,dt \; ; \; ie^{i\pi/2} = i\cos(\pi/2) + i^2\operatorname{sen}(\pi/2) = i\cdot 0 - 1 = -1.$$

$$= \int_0^R e^{-(R-t)^2}\cdot(-e^{i\pi/4})\,dt \overset{u=R-t}{=} +e^{i\pi/4}\int_R^0 e^{-u^2}\,du$$

$$= -e^{i\pi/4}\int_0^R e^{-u^2}\,du \xrightarrow{R\to\infty} -e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

$$= -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\frac{\sqrt{\pi}}{2} = -\left(\frac{1}{\sqrt{2}}(1+i)\right)\frac{\sqrt{\pi}}{2}$$

$$= -\frac{1}{2}\left(\sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}\right)$$

Además, como f es holomorfa en γ $(\forall R)$, por Cauchy-Goursat,

$$\int_\gamma f = 0 \Rightarrow \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f = 0 \quad / R\to\infty$$

$$\Rightarrow \int_0^\infty \cos(t^2)\,dt + i\int_0^\infty \operatorname{sen}(t^2)\,dt + 0 + \frac{-1}{2}\left(\sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}\right) = 0$$

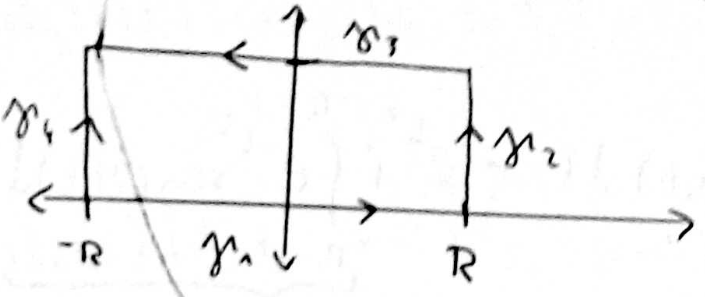
$$\Rightarrow \int_0^\infty \cos(t^2)\,dt + i\int_0^\infty \operatorname{sen}(t^2)\,dt = \frac{1}{2}\left(\sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}\right)$$

Paridad

$$\Rightarrow \int_{-\infty}^\infty \cos(t^2)\,dt + i\int_{-\infty}^\infty \operatorname{sen}(t^2)\,dt = \sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}},$$

igualando partes real e imaginaria, se concluye.

P2] Sea $\Gamma_R = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ el rectángulo descrito:



$$\Rightarrow \oint_{\Gamma_R} f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f \ , \quad f(z) = e^{-z^2}$$

$\underline{\gamma_1}$: $r_1(t) = t, \ t \in [-R, R] \Rightarrow r_1'(t) = 1$

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} e^{-t^2}\,dt \xrightarrow[R \to \infty]{} \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}.$$

$\underline{\gamma_2}$: $r_2(t) = R + it, \ t \in [0, b] \Rightarrow r_2'(t) = i$

Veamos que $\int_{\gamma_2} f \xrightarrow[R \to \infty]{} 0$

$$\left| \int_{\gamma_2} f(z)\,dz \right| = \left| \int_0^b e^{-(R+it)^2}\, i\,dt \right| = \left| \int_0^b e^{-R^2} \cdot e^{-2itR}\, e^{t^2}\, i\,dt \right|$$

$$\leq e^{-R^2} \int_0^b |e^{-2itR + t^2}|\,dt \underset{\substack{\downarrow \\ |e^z| = e^{Re(z)}}}{=} e^{-R^2} \cdot \int_0^b e^{t^2}\,dt \xrightarrow[R \to \infty]{} 0$$

(nula · acotada).

$\underline{\gamma_3}$: $r_3(t) = ib - t, \ t \in [-R, R] \Rightarrow r_3'(t) = -1.$

$$\Rightarrow \int_{\gamma_3} f(z)\,dz = -\int_{-R}^{R} e^{-(ib-t)^2}\,dt = -\int_{-R}^{R} e^{b^2 + 2ibt - t^2}\,dt =$$

$\longrightarrow$

$$= -e^{b^2}\int_{-R}^{R} e^{-t^2}(\cos(2bt) + i\,\mathrm{sen}(2bt))\,dt$$

$$= -e^{b^2}\int_{-R}^{R} e^{-t^2}\cos(2bt)\,dt + \underbrace{-e^{b^2} i\int_{-R}^{R} e^{-t^2}\mathrm{sen}(2bt)\,dt}_{=\,0 \text{ (impar en dom. simétrico)}}$$

$$\xrightarrow[R\to\infty]{} -e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt$$

γ_4: $r_4(t) = (b-t)i - R,\ t\in[0,b] \Rightarrow r_4'(t) = -i$

Luego:

$$\left|\int_{\gamma_4} f(z)\,dz\right| = \left|\int_0^b e^{-((b-t)i-R)^2}\,dt\right| \leq e^{-R^2}\cdot\int_0^b e^{(b-t)^2}\,dt$$

$$\xrightarrow[R\to\infty]{} 0 \quad (\text{mult. acotada}).$$

Por otro lado, como f es holomorfa en Γ_R, $\forall R$, por Cauchy Goursat, tenemos que:

$$\oint_{\Gamma_R} f(z)\,dz = 0$$

Así, $0 = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f \quad \Big| R\to\infty$

$$\Rightarrow 0 = \sqrt{\pi} + 0 - e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt + 0 \Rightarrow \int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\,dt = \frac{\sqrt{\pi}}{e^{b^2}}$$

Paridad
$$\Longrightarrow \int_0^\infty e^{-t^2}\cos(2bt)\,dt = \frac{\sqrt{\pi}}{2e^{b^2}} \quad //$$

P3 a) Queremos ver que

$$\left(\frac{x^m}{m!}\right)^2 = \frac{1}{2\pi i}\oint_{|z|=1} \frac{x^m e^{xz}}{\cancel{m!}\, z^{m+1}}\,dz \iff \frac{x^m}{m!} = \frac{1}{2\pi i}\oint_{|z|=1}\frac{e^{xz}}{z^{m+1}}\,dz$$

$$\iff x^m = \frac{m!}{2\pi i}\oint_{|z|=1}\frac{e^{xz}}{(z-0)^{m+1}}\,dz \quad (*)$$

El lado derecho motiva a definirnos $f(z)=e^{xz}$, para usar la fórmula de Cauchy sobre f, para $z_0=0$.

Notemos que $f^{(m)}(z) = x^m e^{xz} \Rightarrow f^{(m)}(0) = x^m$.

Luego, por fórmula de Cauchy:

$$x^m = \frac{m!}{2\pi i}\oint_{|z|=1}\frac{e^{xz}}{(z-0)^{m+1}}\,dz$$, que por $(*)$, es lo que queríamos.

b) Notemos que, para $N\in\mathbb{N}$,

$$\sum_{m=0}^{N}\left(\frac{x^m}{m!}\right)^2 \overset{(a)}{=} \sum_{m=0}^{N}\frac{1}{2\pi i}\oint_{|z|=1}\frac{x^m e^{xz}}{m!\, z^{m+1}}\,dz = \frac{1}{2\pi i}\oint_{|z|=1}\frac{e^{xz}}{z}\sum_{m=0}^{N}\frac{\left(\frac{x}{z}\right)^m}{m!}\,dz$$

$$\overset{N\to\infty}{\Longrightarrow} \sum_{m=0}^{\infty}\left(\frac{x^m}{m!}\right)^2 \overset{(\Delta)}{=} \frac{1}{2\pi i}\oint_{|z|=1}\frac{e^{xz}}{z}\cdot e^{x/z}\,dz = \frac{1}{2\pi i}\oint_{|z|=1}\frac{e^{x(z+1/z)}}{z}\,dz$$

$\longrightarrow$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{e^{x(e^{i\theta} + e^{-i\theta})}}{\cancel{e^{i\theta}}} \cancel{i}\cancel{e^{i\theta}} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{x(2\cos(\theta))} \, d\theta.$$ (6)

$z = e^{i\theta}, \theta \in [0, 2\pi]$.

$$\therefore \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \right)^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{2x\cos(\theta)} \, d\theta.$$

Obs: En (*) se pudo intercambiar límite con integral por convergencia uniforme, ya que la función
$\sum_{n=0}^{N} \frac{\left(\frac{x}{z}\right)^n}{n!}$ es uniformemente continua en $\{|z| = 1\}$,
por ser continua en un compacto.

P4 Sea $z_0 \in \mathbb{C}$ arbitrario. Luego, por fórmula de Cauchy:

$$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\,dz \,, \quad \forall R>0.$$

Además,

$$\left|\frac{f(z)}{(z-z_0)^2}\right| \leq \frac{\ln(|z|+1)}{R^2} \overset{(*)}{\leq} \frac{\ln(R+|z_0|+1)}{R^2} \quad \forall |z-z_0| = R.$$

$(*)$: $|z| = |z - z_0 + z_0| \leq |z - z_0| + |z_0| = R + |z_0|$.

Así,

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\oint_{|z-z_0|=R} \frac{f(z)dz}{(z-z_0)^2}\right| \leq \frac{1}{2\pi}\oint_{|z-z_0|=R} \frac{|f(z)\,dz|}{|(z-z_0)^2|}$$

$$\leq \frac{\ln(R+|z_0|+1)}{2\pi R^2}\int_{|z-z_0|=R} |dz| = \frac{\ln(R+|z_0|+1)}{R}$$

Luego, $$\lim_{R\to\infty} \frac{\ln(R+|z_0|+1)}{R} \overset{L'H}{=} \lim_{R\to\infty}\frac{1}{R+|z_0|+1} = 0.$$

$\Rightarrow f'(z_0) = 0$. Como $z_0 \in \mathbb{C}$ es arbitrario

$\Rightarrow f' \equiv 0$, y como $\mathbb{C}$ es conexo, se concluye que f es constante.

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