# Pauta Auxiliar 8



## P1]

Resumen:

- Si $r:[a,b] \longrightarrow \mathbb{C}$ es una parametrización de una curva $\Gamma$, con $r$ $\mathcal{C}^1$ por trozos, entonces, se define la integral de $f$ sobre $\Gamma$ como:

$$\int_{\Gamma} f(z)dz = \int_a^b f(r(t))\, r'(t)\, dt$$

Obs: Es el mismo procedimiento que se utilizaba al calcular integrales de línea.

a) Una parametrización es:

$$r(t) = 2e^{it}, \quad t \in [0, 2\pi] \Rightarrow r'(t) = 2ie^{it}$$

$$\Rightarrow \int_{\gamma} f(z)dz = \int_0^{2\pi} f(2e^{it})\cdot 2ie^{it}\, dt = \int_0^{2\pi} 2\cancel{e^{-it}} \cdot 2i\cancel{e^{it}}\, dt$$

$$= \int_0^{2\pi} 4i\, dt = 8\pi i$$

b) Parametrización: $r(t) = e^{it}$, $t \in [-\pi/2, \pi/2]$ (por la orientación de la curva). $\Rightarrow r'(t) = ie^{it}$

suele ser útil!!

Además, $Re(e^{it}) = Re(\cos(t) + i\operatorname{sen}(t)) = \cos(t) = \dfrac{e^{it} + e^{-it}}{2}$

$$\Rightarrow \int_{\gamma} f(z)dz = \int_{-\pi/2}^{\pi/2} \frac{(e^{it} + e^{-it})}{2} \cdot ie^{it}\, dt = \frac{i}{2}\int_{-\pi/2}^{\pi/2} (e^{2it} + 1)\, dt$$

$$= \frac{i}{2}\left(\frac{e^{2it}}{2i} + t\right)\Big|_{-\pi/2}^{\pi/2} = \frac{i}{2}\left(\frac{e^{i\pi}-e^{-i\pi}}{2i} + \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{i}{2}\left(\underbrace{\operatorname{sen}(\pi)}_{0} + \pi\right) = \frac{i\pi}{2}$$ //

c) Parametrización: $r(t) = e^{it},\ t\in[0,2\pi] \Rightarrow r'(t) = ie^{it}$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_0^{2\pi} e^{i\alpha t}\cdot ie^{it}\,dt = i\int_0^{2\pi} e^{it(\alpha+1)}\,dt \quad (*)$$

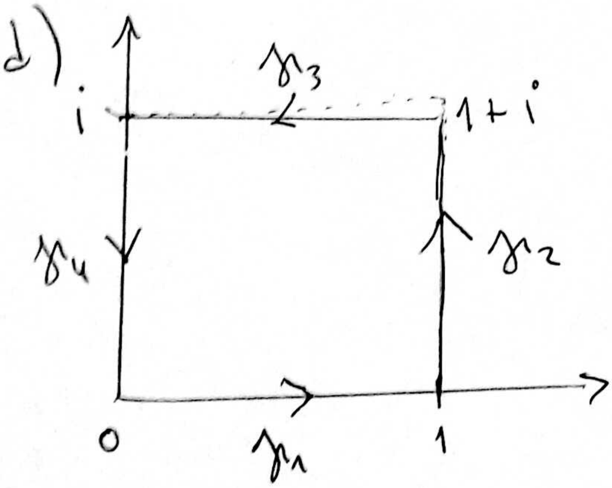
caso 1: $\alpha = -1$

$$\Rightarrow (*) = i\int_0^{2\pi} e^{it(-1+1)}\,dt = i\int_0^{2\pi} 1\,dt = 2\pi i$$ //

caso 2: $\alpha \neq 1$

$$\Rightarrow (*) = \cancel{i}\cdot \frac{e^{it(\alpha+1)}}{\cancel{i}(\alpha+1)}\Big|_0^{2\pi} = \frac{e^{2\pi(\alpha+1)i} - 1}{\alpha+1}$$ //

d)




$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$.

$$\Rightarrow \int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \dots + \int_{\gamma_4} f(z)\,dz.$$

Calculemos por separado:

$\gamma_1$: $r_1(t) = t,\ t\in[0,1] \Rightarrow r'(t) = 1$.

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_0^1 |t|^2\cdot 1\,dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

$\underline{\gamma_2}$: $r_2(t) = 1 + it$ , $t \in [0,1] \Rightarrow r_2'(t) = i$

$$\Rightarrow \int_{\gamma_2} f(z)\,dz = \int_0^1 |1+it|^2 \cdot i\,dt = i\int_0^1 (1+t^2)\,dt \rightsquigarrow \left(\begin{array}{l}\text{recuerdo:}\\ |z| = |a+ib| \\ \quad = \sqrt{a^2+b^2}\end{array}\right)$$

$$= i\left(t + \frac{t^3}{3}\right)\Big|_0^1 = \frac{4}{3}i$$

$\underline{\gamma_3}$: $r_3(t) = (1-t) + i \Rightarrow r_3'(t) = -1$
$t \in [0,1]$

$$\Rightarrow \int_{\gamma_3} f(z)\,dz = -\int_0^1 |(1-t)+i|^2\,dt = -\int_0^1 ((1-t)^2+1)\,dt$$

$$= -\int_0^1 (1-2t+t^2+1)\,dt = -\left(2t - t^2 + \frac{t^3}{3}\right)\Big|_0^1 = -\left(2-1+\frac{1}{3}\right)$$

$$= -\frac{4}{3}$$

$\underline{\gamma_4}$: $r_4(t) = (1-t)i$ , $t \in [0,1] \Rightarrow r_4'(t) = -i$

$$\Rightarrow \int_{\gamma_4} f(z)\,dz = -i\int_{\gamma_4} |(1-t)i|^2 dt = -i\int_0^1 (1-2t+t^2)\,dt$$

$$= -i\left(t - t^2 + \frac{t^3}{3}\right)\Big|_0^1 = \frac{-i}{3}$$

$$\therefore \int_\gamma f(z)\,dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = i - 1.$$

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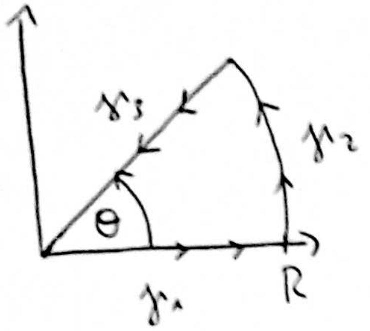
P2) [Cauchy-Goursat: $f$ holomorfa en $\Omega$, abierto convexo $\Rightarrow \int_\Gamma f(z)dz = 0$, $\Gamma \subset \Omega$ regular por trozos, simple] (4)

a). Caso $\theta = 0$ es directo, pues $e^{i\theta} = e^{i0} = 1$.

· Caso $\theta \in (0, \pi/2]$: Tomando la curva indicada:



$$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$\Rightarrow \oint_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

Calculemos cada integral $\gamma_i$

$\underline{\gamma_1}$: $r_1(t) = t$, $t \in [0, R] \Rightarrow r_1'(t) = 1$.

$$\Rightarrow \int_{\gamma_1} f(z)dz = \int_0^R f(r_1(t)) \cdot r_1'(t)dt = \int_0^R f(t)dt$$

$\underline{\gamma_2}$: $r_2(t) = Re^{it}$, $t \in [0, \theta] \Rightarrow r_2'(t) = iRe^{it}$

$$\Rightarrow \int_{\gamma_2} f(z)dz = \int_0^\theta f(Re^{it})iRe^{it}dt$$

$\underline{\gamma_3}$: $r_3(t) =$ (Recta que une al punto inicial, $Re^{it}$, con $0$)

$= (R-t)e^{i\theta}$, $t \in [0, R] \Rightarrow r_3'(t) = -e^{i\theta}$

$$\Rightarrow \int_{\gamma_3} f(z)dz = -\int_0^R f((R-t)e^{i\theta})e^{i\theta}dt \overset{\substack{u=R-t,\ du=-dt \\ \uparrow}}{=} \int_R^0 f(ue^{i\theta})e^{i\theta}du$$

$$= -\int_0^R f(ue^{i\theta})e^{i\theta}du.$$

$$\Rightarrow \oint_\Gamma f(z)dz = \int_0^R f(t)dt + \int_0^\theta f(Re^{it})iRe^{it}dt - \int_0^R f(e^{i\theta}t)e^{i\theta}dt \quad (1)$$

Por otro lado, como $f$ es holomorfa en $D$, y notando que $\forall R>0$, $\Gamma \subset \overline{D}$, entonces, por Cauchy-Goursat:

$$\oint_\Gamma f(z)dz = 0 \quad (2)$$

$$\overset{(1)=(2)}{\Longrightarrow} \int_0^R f(t)dt + \int_0^\theta f(Re^{it})iRe^{it}dt - \int_0^R f(e^{i\theta}t)e^{i\theta}dt = 0$$

$$\Rightarrow \int_0^R f(t)dt + \underbrace{\int_0^\theta f(Re^{it})iRe^{it}dt}_{(*)} = \int_0^R f(e^{i\theta}t)e^{i\theta}dt \quad (3)$$

Notemos que, si tomamos $R\to\infty$ en (3), y $(*) \xrightarrow{R\to\infty} 0$, se obtiene el resultado buscado.

En efecto:

$$0 \leq \left| \int_0^\theta f(Re^{it})iRe^{it}dt \right| \leq \int_0^\theta |f(Re^{it})|\underbrace{|i|}_{1}|R|\underbrace{|e^{it}|}_{1}dt$$

$$\leq \int_0^\theta \frac{M}{|Re^{it}|^2}\cdot R\,dt = \int_0^\theta \frac{M}{R\underbrace{|e^{it}|}_{1}}dt = \frac{M\theta}{R} \xrightarrow[R\to\infty]{} 0.$$

(por sandwich se concluye).

Así, tomando $R\to\infty$ en (3), obtenemos:

$$\int_0^\infty f(t)dt + 0 = e^{i\theta}\int_0^\infty f(e^{i\theta}t)dt$$

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b) Notemos que, para $z \in D$:

$$|f(z)| = \left|\frac{e^{iz}}{(1+z)^2}\right| = \frac{|e^{ix-y}|}{|(1+z)|^2} = \frac{e^{-y}}{|(1+z)|^2}$$

$e^{-y}$ decrec.
$y>0$ en $D$

$$\leq \frac{e^0}{|1+z|^2} := \frac{M}{|1+z|^2} = \frac{M}{|(1+x)+iy|^2}$$

$$= \frac{M}{(1+x)^2+y^2} \overset{x>0 \text{ en } D}{\leq} \frac{M}{x^2+y^2} = \frac{M}{|z|^2}.$$

Por lo que $f$ cumple las hipótesis de (a)
(pues además, claramente es continua en $\bar{D}$ y holomorfa en $D$).

(a), $\theta = \pi/2$

$$\Longrightarrow \int_0^\infty \frac{e^{ix}}{(1+x)^2}dx = e^{i\pi/2}\int_0^\infty \frac{e^{i(e^{i\pi/2}x)}}{(1+e^{i\pi/2}x)^2}dx$$

donde $e^{i\pi/2} = \cos(\pi/2) + i\,\text{sen}(\pi/2) = 0 + i\cdot 1 = i$

$$\Rightarrow \underbrace{\int_0^\infty \frac{e^{ix}}{(1+x)^2}dx}_{(1)} = \underbrace{i\int_0^\infty \frac{e^{-x}}{(1+ix)^2}dx}_{(2)}$$

Para que $(1) = (2)$, en particular, $Re(1) = Re(2)$,

donde: $\cdot\, Re(1) = Re\left(\int_0^\infty \frac{e^{ix}}{(1+x)^2}dx\right) = \int_0^\infty \frac{\cos(x)}{(1+x)^2}dx$

$\cdot\, Re(2) = \left(i\int_0^\infty \frac{e^{-x}\,dx}{(1+ix)^2}\right) = Re\left(i\int_0^\infty \frac{e^{-x}(1-ix)^2\,dx}{[(1+ix)(1-ix)]^2}\right)$

$$= \mathrm{Re}\left( \int_0^\infty \frac{e^{-x}(i+2x-ix^2)\,dx}{(1+x^2)^2} \right) = \int_0^\infty \frac{2xe^{-x}}{(1+x^2)^2}$$

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Así, $\mathrm{Re}(1) = \mathrm{R}(2) \iff \displaystyle\int_0^\infty \frac{\cos(x)\,dx}{(1+x)^2} = \underbrace{\int_0^\infty \frac{2e^{-x}}{(1+x^2)^2}\,dx}_{(3)}$ $(*)$

Usando vaca para resolver (3):

$$u = e^{-x} \Rightarrow du = -e^{-x}dx; \quad dv = \frac{2x\,dx}{(1+x^2)^2} \Rightarrow v = \frac{-1}{1+x^2}$$

$$\overset{\text{vaca}}{\Longrightarrow} \int_0^\infty \frac{2e^{-x}x\,dx}{(1+x^2)^2} = \frac{-e^{-x}}{1+x^2}\Big|_0^\infty - \int_0^\infty \frac{e^{-x}}{1+x^2}\,dx$$

$$= 1 - \int_0^\infty \frac{e^{-x}}{1+x^2}\,dx$$

Reemplazando en $(*)$:

$$\int_0^\infty \frac{\cos(x)\,dx}{(1+x)^2} + \int_0^\infty \frac{e^{-x}}{1+x^2}\,dx = 1.$$

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P3

[Fórmula integral de Cauchy:

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{m+1}}\, dz$$
]

a) Sea $f(z) = e^{az}$. Aplicando la fórmula integral de Cauchy para $f$, $z_0 = 0$, $m = 0$, $\Gamma = \mathcal{C}$:

$$\underbrace{f^{(0)}(0)}_{} = \frac{0!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z^{0+1}}\, dz$$

$$\underbrace{e^0}_{1} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{e^{az}}{z}\, dz \Rightarrow 2\pi i = \oint_{\mathcal{C}} \frac{e^{az}}{z}\, dz$$

b) Sea $f(z) = \tan(z/2)$. Luego, las singularidades de $f$ son los ceros de $\cos(z/2)$, lo cual ocurre si $\frac{z}{2} = (m + \frac{1}{2})\pi$, $\forall m \in \mathbb{Z}$.

Ahora bien, notemos que todas estas singularidades están fuera del cuadrado, pues $2\left(-m + \frac{1}{2}\right)\pi < -2 < 2 < 2\left(m + \frac{1}{2}\right)\pi$, $\forall m \in \mathbb{N}$.

Luego, $f$ es holomorfa en el interior de $\Gamma$.

Como $x_0$ está encerrado por $\Gamma$ (pues $-2 < x_0 < 2$), podemos aplicar fórmula de Cauchy, con $z_0 = x_0$, $m = 1$:

$$f'(x_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{\tan(z/2)}{(z-x_0)^2}\, dz \;;\; f'(x_0) = \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right)$$

$$\therefore \oint_{\Gamma} \frac{\tan(z/2)}{(z-x_0)^2}\,dz = i\pi \sec^2\left(\frac{x_0}{2}\right).$$