Integral de flujo y teo. Gauss

Sea $S \subseteq \mathbb{R}^3$ superficie suave y simple con param.
$\vec{\varphi} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$. $\hookrightarrow \mathcal{C}^1$ φ inyectiva

Dados $(u_0, v_0) \in int(D)$, $u \longmapsto \vec{\varphi}(\cdot, v_0)$ definen
$v \longmapsto \vec{\varphi}(u_0, \cdot)$ curvas sobre S

Supongamos $\frac{\partial \vec{\varphi}}{\partial u} \neq 0$; $\frac{\partial \vec{\varphi}}{\partial v} \neq 0$ en (u_0, v_0)

Def (Vectores tangentes): $\hat{t}_u = \frac{\vec{\varphi}_u}{\|\vec{\varphi}_u\|}$; $\hat{t}_v = \frac{\vec{\varphi}_v}{\|\vec{\varphi}_v\|}$.

Def $\vec{\varphi}$ se dice regular si $\hat{t}_u, \hat{t}_v$ son l.i.

Def (Vector normal) $\hat{n} = \frac{\hat{t}_u \times \hat{t}_v}{\|\hat{t}_u \times \hat{t}_v\|} = \frac{\vec{\varphi}_u \times \vec{\varphi}_v}{\|\vec{\varphi}_u \times \vec{\varphi}_v\|}$
Es único salvo signo.

Def Una superficie S regular es orientable si los vectores normales están definidos como una fn. cont. sobre S.

Integral de Flujo

$\vec{F}$: Campo de Velocidades
$\vec{\varphi}$: parametrización de S

$\vec{F}(\vec{\varphi}(u,v)) \cdot \hat{n}(u,v)$ = rápidez en la dirección normal con que las part. que pasan por $\vec{\varphi}(u,v)$ atraviesan S.

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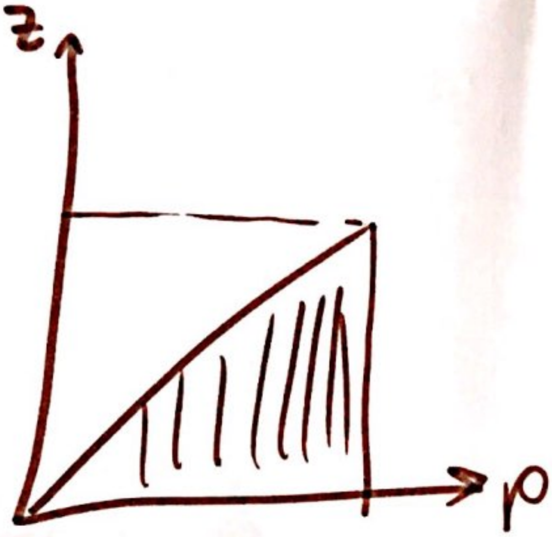
im: $\Delta V \approx \vec{F}(\vec{\varphi}(u,v)\cdot \hat{n})\Delta t \underbrace{\Delta A}_{\approx \|\vec{\varphi}_u \times \vec{\varphi}_v\| \Delta u \Delta v}$

$$\Rightarrow \Delta V \approx \vec{F}(\vec{\varphi}(u,v))(\vec{\varphi}_u \times \vec{\varphi}_v)\Delta u \Delta v \Delta t$$

$$Q = \iint_D \vec{F}(\vec{\varphi}(u,v))\,(\vec{\varphi}_u \times \vec{\varphi}_v)\,du\,dv$$

$$= \iint_S \vec{F}\cdot d\vec{A} = \iint_S \vec{F}\cdot \hat{n}\, dA.$$

P1



$\rho \in [0,1]$

$z \in [0,\rho]$

$\theta \in [0,2\pi)$

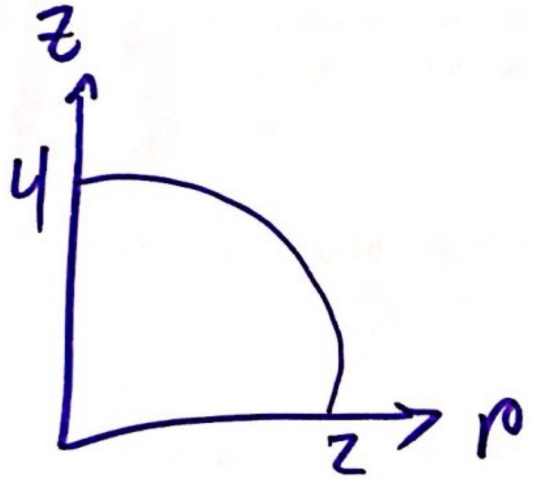
$$\int_0^1 \int_0^{\rho} \int_0^{2\pi} 2z\,\rho^2\,\rho\,d\theta\,dz\,d\rho$$

$$= 4\pi \int_0^1 \int_0^{\rho} z\rho^3\,dz\,d\rho = 4\pi \int_0^1 \frac{\rho^2}{2}\rho^3\,d\rho$$

$$= \frac{2\pi}{6} = \frac{\pi}{3}$$

4) $\vec{F}(x,y,z) = \begin{pmatrix} xz \\ 0 \\ 1 \end{pmatrix}$

$$S = \left\{ \begin{array}{ll} (x,y,z)\in\mathbb{R}^3 & z = 4-\rho^2 \\ 0 \leq \rho \leq 2 & \rho = \sqrt{x^2+y^2} \\ \theta \in [0,2\pi] & \theta = \operatorname{arctg}\left(\frac{y}{x}\right) \end{array} \right\}$$



Parametrizar S:

$$\sigma(\rho,\theta) = \rho\hat{\rho} + (4-\rho^2)\hat{k}$$
$$= (\rho\cos\theta, \rho\operatorname{sen}\theta, 4-\rho^2)$$

$$\Rightarrow \frac{\partial\sigma}{\partial\rho} = (\cos\theta, \operatorname{sen}\theta, -2\rho) = \hat{\rho} - 2\rho\hat{k}$$

$$\frac{\partial\sigma}{\partial\theta} = (-\rho\operatorname{sen}\theta, \rho\cos\theta, 0) = \rho\hat{\theta}$$

$$\Rightarrow \frac{\partial\sigma}{\partial\rho} \times \frac{\partial\sigma}{\partial\theta} = \begin{vmatrix} \hat{\rho} & \hat{\theta} & \hat{k} \\ 1 & 0 & -2\rho \\ 0 & \rho & 0 \end{vmatrix} = 2\rho^2\hat{\rho} + \rho\hat{k}.$$

Además, $\vec{F}(\sigma(\rho,\theta)) = \vec{F}(\rho\cos\theta, \rho\sin\theta, 4-\rho^2)$

$$= (\rho\cos\theta(4-\rho^2), 0, 1)$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\, dA = \int_0^{2\pi}\int_0^2 (4\rho\cos\theta - \rho^3\cos\theta, 0, 1)\cdot(2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho)\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (8\rho^3\cos^2\theta - 2\rho^5\cos^2\theta + \rho)\, d\rho\, d\theta$$

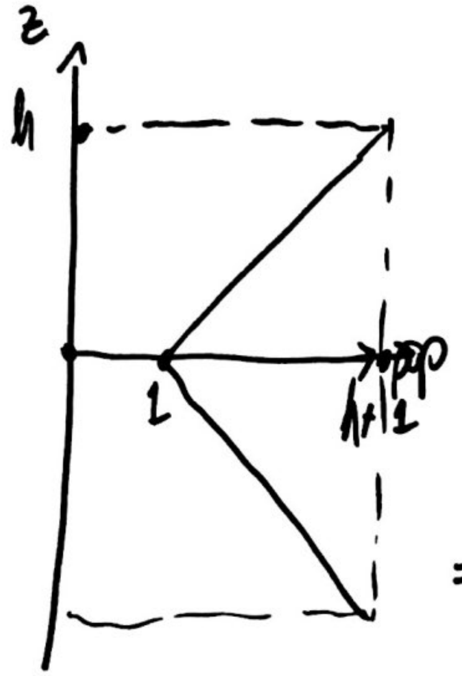
$$= \int_0^{2\pi} \left(32\cos^2\theta - \frac{64}{3}\cos^2\theta + 2\right) d\theta$$

$$= \int_0^{2\pi} \frac{32}{3}\cos^2\theta\, d\theta + 4\pi$$

$$= \frac{32}{3}\pi + 4\pi \quad //$$

$\rho = 1 + z \quad , \quad 0 \le z \le h$

$\rho = 1 - z \quad , \quad -h \le z < 0$



$$\sigma(\theta, z) = (1 + |z|)\,\hat{\rho} + z\,\hat{k}$$

$$\frac{\partial \sigma}{\partial \theta} = (1 + |z|)\,\hat{\theta}$$

$$\frac{\partial \sigma}{\partial z} = \frac{z}{|z|}\,\hat{\rho} + \hat{k}$$

$$\Rightarrow \frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial z} = \begin{vmatrix} \hat{\rho} & \hat{\theta} & \hat{k} \\ 0 & 1+|z| & 0 \\ \frac{z}{|z|} & 0 & 1 \end{vmatrix}$$

$$= (1 + |z|)\,\hat{\rho} - \left(\frac{z}{|z|} + z\right)\hat{k}$$

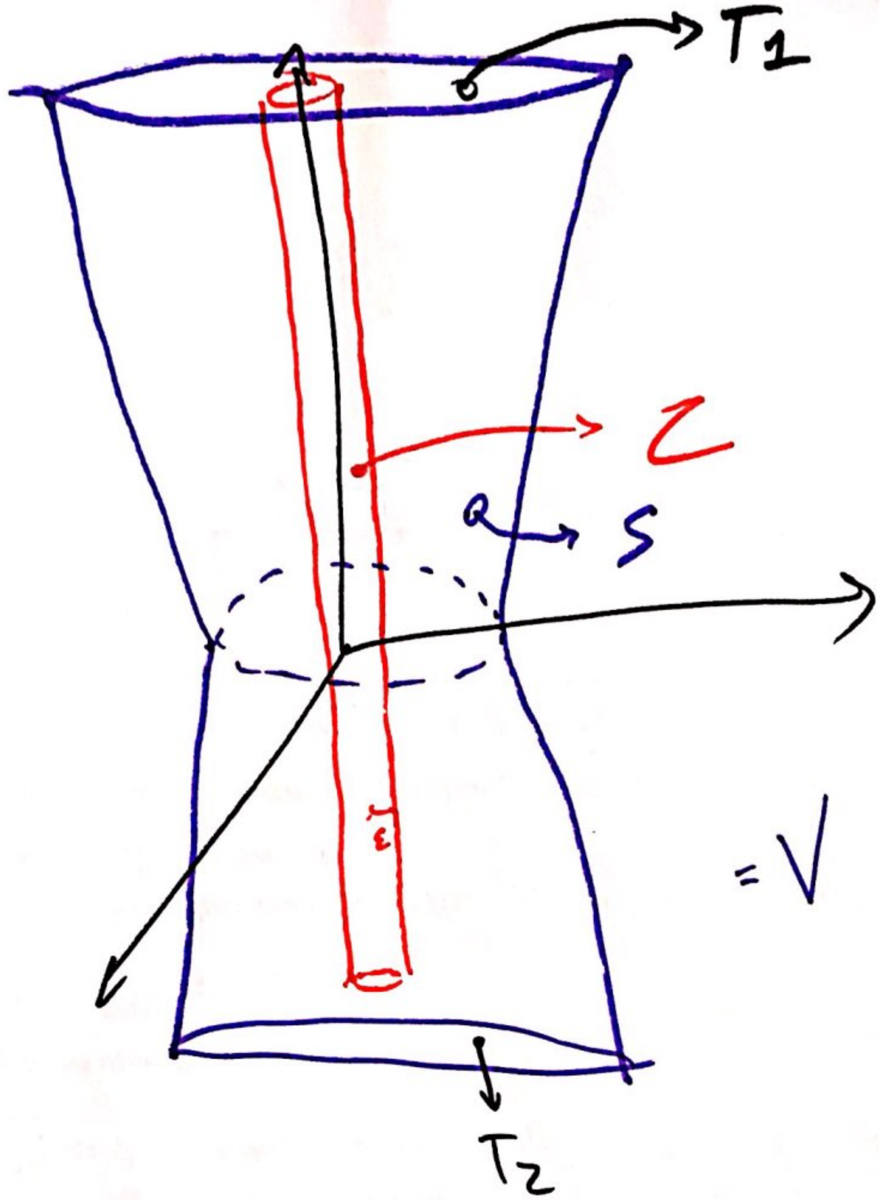
$$\vec{F} = \left(\frac{5}{\rho}\right)\hat{\rho} + e^{-z^2}\,\hat{\theta} + 3z\,\hat{k}$$

$$\vec{F}(\sigma(\theta, z)) = \frac{5}{1+|z|}\,\hat{\rho} + e^{-z^2}\,\hat{\theta} + 3z\,\hat{k}$$

$$\Rightarrow \vec{F}(\sigma(\theta, z)) \cdot \left(\frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial z}\right) = 5 - \frac{3z^2}{|z|} - 3z^2$$

$$\Rightarrow \iint_S \vec{F} \cdot dS = \int_0^{2\pi} \int_{-h}^{h} 5 - 3|z| - 3z^2 \, d\theta \, dz$$

$$= 2\pi\left(10h - 2h^3 - 3h^2\right)$$



Notemos que $\vec{F}$ es no diff en $\rho = 0$ (i.e., el eje z)

$$\Rightarrow S \cup C \cup T_1 \cup T_2 = \partial V .$$

· Para C | $\sigma(\theta, z) = -2\hat{\rho} + z\hat{k}$. ($\hat{n}$ apunta hacia adentro)

$$\vec{F}(\sigma(\theta, z)) = -\frac{5}{2}\hat{\rho} + 3z\hat{k} .$$

$$\therefore \iint_C \vec{F} \cdot d\vec{s} = -20h\pi$$

Para T_1 | $\sigma(\rho,\theta) = \rho\hat{\rho} + h\hat{k}$, $\rho \in [\varepsilon, h+1]$

Para T_2 | $\sigma(\rho,\theta) = -\rho\hat{\rho} - h\hat{k}$,

$$\Rightarrow \iint_{T_1} \vec{F}\cdot d\vec{S} + \iint_{T_2} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_{\varepsilon}^{h+1} 3h\rho\, d\rho\, d\theta$$

$$+ \int_0^{2\pi}\int_{\varepsilon}^{h+1} 3(-h)(-\rho)\, d\rho\, d\theta$$

$$= 6\pi h\left((h+1)^2 - \varepsilon^2\right).$$

Por otro lado,

$$div(\vec{F}) = \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\frac{5}{\rho}\rho\right) + \frac{\partial}{\partial\theta}\left(e^{-z^2}\right) + \frac{\partial}{\partial z}(3z\rho)\right)$$

$$= 3 \quad 2\cdot\int_0^{2\pi}\int_0^{h}\int_0^{1+z} \rho\, d\rho\, dz\, d\theta$$

$$\Rightarrow \iiint_V div(\vec{F}) = 3\cdot vol(V)$$

$$= 3\left(\frac{2}{3}\pi\left((1+h)^3 - 1^3\right) - 2\pi h\varepsilon^2\right),$$

$$\int_0^h \pi\frac{(1+z)^2}{\cancel{4}}\,dz = \pi\int 1+2z+z^2\,dz$$

$$= \pi\left(h + h^2 + \frac{h^3}{3}\right) = \frac{\pi}{3}\left[(1+h)^3 - 1\right]$$

Ainsi, $\iint_S \vec{F}\cdot\vec{dS} = \iiint_V \operatorname{div}\vec{F}\,dV - \iint_{\Gamma_1} F\cdot dS - \iint_{\Gamma_2} F\,dS - \iint_C F$

$$= 2\pi\left((1+h)^3 - 1\right) - 6\pi h\varepsilon^2 + 20\pi h - 6\pi h\left(h^2 + 2h + 1 - \varepsilon^2\right)$$

$$= 2\pi\left(h^3 + 3h^2 + 3h\right) - 6\pi h\varepsilon^2 + 20\pi h - 6\pi h^3 - 12\pi h^2 - 6\pi h + 6\pi h\varepsilon^2$$

$$= 2\pi h + 6\pi h^2 + 6\pi h + 20\pi h - 6\pi h^3 - 12\pi h^2 - 6\pi h$$

$$= -4\pi h^3 - 6\pi h^2 + 20h\pi$$

$$= 2\pi\left(10h - 2h^3 - 3h^2\right)$$