Aux #14

$\boxed{P1}$ Recordemos que si $z = \overline{z} \Leftrightarrow z \in \mathbb{R}$

a) $\boxed{pdq}$ $\forall n \in \mathbb{N}$ $z = (1+i)^n + (1-i)^n \in \mathbb{R}$

En efecto, $\overline{z} = \overline{(1+i)^n + (1-i)^n}$

$= \overline{(1+i)^n} + \overline{(1-i)^n}$

$= \overline{(1+i)}^n + \overline{(1-i)}^n$

$= (1-i)^n + (1+i)^n$

$= z$ $\quad$; $\quad z \in \mathbb{R}$

b) 1) $\overline{w} = \overline{(z_1\overline{z}_2 + \overline{z}_1 z_2)} = (\overline{z_1\overline{z}_2} + \overline{\overline{z}_1 z_2})$

$= (\overline{z}_1 \overline{\overline{z}}_2 + \overline{\overline{z}}_1 \overline{z}_2)$

$= (\overline{z}_1 z_2 + z_1 \overline{z}_2)$

$= (z_1 \overline{z}_2 + \overline{z}_1 z_2)$

$= w \quad \Rightarrow \quad w \in \mathbb{R}$

2) pdq $|z_1|^2 + |z_2|^2 \geq z_1\overline{z}_2 + \overline{z}_1 z_2$

En efecto, notemos que $(|z_1| - |z_2|)^2 \geq 0$ pues $|z_1|$ y $|z_2|$ viven en $\mathbb{R}$

$$\Rightarrow \left(|z_1| - |z_2|\right)^2 = |z_1|^2 - 2|z_1||z_2| + |z_2|^2 \geq 0$$

$$\Rightarrow |z_1|^2 + |z_2|^2 \geq 2|z_1||z_2|$$

$$= 2|z_1 z_2|$$

$$= 2|z_1 \overline{z_2}|$$

$z = a + bi$

$\Rightarrow |z| = \sqrt{a^2+b^2} \geq |a| = Re(z)$

$$\geq 2\, Re(z_1 \overline{z_2})$$

$\frac{w + \overline{w}}{2} = Re(w)$

$$= z_1\overline{z_2} + \overline{z_1 \overline{z_2}}$$

$$= z_1 \overline{z_2} + \overline{z_1} z_2$$

P2 Resolver $\left(\frac{\sqrt{3}-i}{2}\right)^{2n} - \left(\frac{\sqrt{3}+i}{2}\right)^{2n} = a$

Notemos que

$$\left(\frac{\sqrt{3}-i}{2}\right)^{2n} - \left(\frac{\sqrt{3}+i}{2}\right)^{2n} = \left(\frac{\sqrt{3}-i}{2}\right)^{2n} - \left(\overline{\frac{\sqrt{3}-i}{2}}\right)^{2n}$$

$$= \left(\frac{\sqrt{3}-i}{2}\right)^{2n} - \overline{\left(\frac{\sqrt{3}-i}{2}\right)^{2n}}$$

$$= 2\, \mathrm{Im}\left(\left(\frac{\sqrt{3}-i}{2}\right)^{2n}\right) \cdot i$$

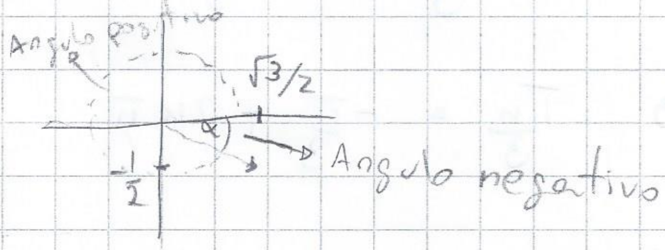
$\Rightarrow$ Si $a = \sqrt{3} \Rightarrow \underbrace{2i\, \mathrm{Im}\left(\left(\frac{\sqrt{3}-i}{2}\right)^{2n}\right)}_{\in \mathbb{C}\setminus\mathbb{R}} = \underbrace{\sqrt{3}}_{\mathbb{R}}$ no hay solución

$$\text{Si } a = i\sqrt{3} \Rightarrow 2i\,\mathrm{Im}\left(\left(\frac{\sqrt{3}-i}{2}\right)^{2n}\right) = i\sqrt{3} \quad / \ \frac{1}{i}$$

$$\Rightarrow \mathrm{Im}\left(\left(\frac{\sqrt{3}-i}{2}\right)^{2n}\right) = \frac{\sqrt{3}}{2}$$

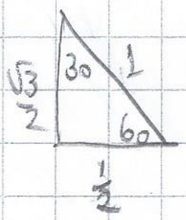
Pasemos a forma polar $\frac{\sqrt{3}}{2} - \frac{1}{2}i = z$

$$|z| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$



$$\theta = -\arctan\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= -\arctan\left(\frac{1}{\sqrt{3}}\right) \Leftrightarrow$$



$$= -\frac{\pi}{6}$$

$$\Rightarrow \frac{\sqrt{3}}{2} - \frac{1}{2}i = |z|e^{i\theta} = 1e^{i\left(-\frac{\pi}{6}\right)}$$

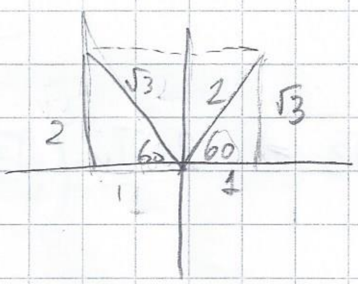
$$\Rightarrow \mathrm{Im}\left(\left(\frac{\sqrt{3}-i}{2}\right)^{2n}\right) = \mathrm{Im}\left(\left(e^{i\left(-\frac{\pi}{6}\right)}\right)^{2n}\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \mathrm{Im}\left(e^{i\left(-\frac{\pi}{3}n\right)}\right) = \sqrt{3}/2$$

$$\Rightarrow \mathrm{Im}\left(\cos\left(-\frac{\pi}{3}n\right) + i\,\mathrm{sen}\left(-\frac{\pi}{3}n\right)\right) = \sqrt{3}/2$$

$$\Rightarrow \mathrm{Sen}\left(-\frac{\pi}{3}n\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \operatorname{sen}\left(-\frac{\pi n}{3}\right) = \frac{\sqrt{3}}{2}$$



$$\Rightarrow 1) \quad -\frac{\pi n}{3} = \frac{\pi}{3} + 2k\pi$$

$$k \in \mathbb{Z}$$

$$2) \vee \quad -\frac{\pi n}{3} = \pi - \frac{\pi}{3} + 2k\pi$$

$$\Rightarrow \quad \frac{\pi n}{3} = -\frac{\pi}{3} + 2k\pi \quad \vee \quad \frac{\pi n}{3} = \frac{\pi}{3} - \pi + 2k\pi$$

$$\Rightarrow \quad n = -1 + 6k \quad \vee \quad n = 1 - 3 + 6k$$

$$= -2 + 6k \quad k \in \mathbb{Z}$$

$\boxtimes$

P3 $w \in \mathbb{C}$, $w^3 = 1$ $w \neq 1$ pruebe que

$$(1+w)^3 + (1+w^2)^9 + (1+w^3)^6 = 62$$

En efecto, notemos que en el caso de raíces n-ésimas

$$\circledast \quad \forall k \in [1..n-1] \; w_k = e^{i\frac{2k\pi}{n}} = \left(e^{i\frac{2\pi}{n}}\right)^k = w_1^k$$

Luego $w^3 = 1 \Rightarrow w = w_0, w_1, w_2$ con $n = 3$

$$w_0 = e^{i\frac{2\cdot 0\pi}{3}} = e^0 = 1$$

$$w_1 = e^{i\frac{2\cdot 1\pi}{3}} = e^{i\frac{2\pi}{3}} \quad \Rightarrow \quad w_1^2 = w_2$$

$$w_2 = e^{i\frac{2\cdot 2\pi}{3}} = e^{i\frac{4\pi}{3}} \quad \Rightarrow \quad w_2^2 = e^{i\frac{8\pi}{3}} = e^{i\left(\frac{2\pi}{3} + 2\pi\right)}$$

$$= w_1$$

Luego $\omega_1^2 = \omega_2$ y $\omega_2^2 = 1$ y $\omega_0 = 1$

$\Rightarrow$ Si $\omega \in \mathbb{C}$ tq $\omega^3 = 1$ tq $\omega \neq 1$ podemos tomar cualquiera ω_1 o ω_2 pues en la expresión aparece ω y ω^2

$\omega = \omega_1 = e^{\frac{i2\pi}{3}}$ y recordemos que $1 + \omega_1 + \omega_2 = 0$

$\Rightarrow -\omega_2 = 1 + \omega_1$

$\Rightarrow -\omega_1 = 1 + \omega_2$

$$\Rightarrow (1+\omega)^3 + (1+\omega^2)^9 + (1+\omega^3)^6$$

es cogí $w = w_1$

$$= (1+\omega_1)^3 + (1+\omega_1^2)^9 + (1+\omega_1^3)^6$$

$\omega_1^3 = 1$

$$= (-\omega_2)^3 + (1+\omega_2)^9 + (1+1)^6$$

$$= -(\omega_2)^3 + -(\omega_1)^9 + (2)^6$$

$$= -1 + (-1) + 64$$

$$= 62$$

$\boxed{P4}$ Si $x^2+x+1=0$

$\Rightarrow$ Si z es solucion, $z^3=1$

En efecto, Como 1) $x^2+x+1=0 \quad / \cdot x$

$$2) \Rightarrow x^3+x^2+x=0$$

$$\Rightarrow (2)-(1) \Rightarrow x^3+x^2+x-(x^2+x+1)=0$$

$$\Rightarrow x^3-1=0$$

$$\Rightarrow x^3=1 \quad ⊛$$

Luego toda solución cumple ⊛ por lo que son raices de la unidad.

Si $x=1 \Rightarrow x^2+x+1=1+1+1=3\neq 0$

$\Rightarrow$ Son raices de la unidad distinta de 1

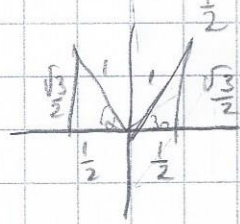
P5 Resolver $w^4 = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$

$$= \frac{1+i\sqrt{3}}{1-i\sqrt{3}} \cdot \frac{1+i\sqrt{3}}{1+i\sqrt{3}}$$

$$= \frac{1+2\sqrt{3}i+3i^2}{4}$$

$$= \frac{-2+2\sqrt{3}i}{4}$$

$$= \frac{-1}{2}+\frac{\sqrt{3}}{2}i \qquad \arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \arctan(\sqrt{3})$$



$$= e^{i\frac{2\pi}{3}}$$

$\Rightarrow \quad w^4 = e^{i\frac{2\pi}{3}}, \quad w = re^{i\varphi}$

$\Rightarrow \quad r^4 = 1 \quad \wedge \quad e^{i4} = e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad r = \sqrt[4]{1} \quad \wedge \quad 4\varphi = \frac{2\pi}{3} + 2k\pi \qquad k=0,1,2,3$

$\Rightarrow \quad r = 1 \quad \wedge \quad \varphi = \frac{2\pi}{12} + \frac{k\pi}{2}$

$\Rightarrow \quad w_0 = re^{i\varphi_0} = 1e^{i\left(\frac{\pi}{6}\right)}$

$w_1 = e^{i\varphi_1} = e^{i\left(\frac{\pi}{6}+\frac{\pi}{2}\right)}$

$w_2 = e^{i\varphi_2} = e^{i\left(\frac{\pi}{6}+\pi\right)}$

$w_3 = e^{i\varphi_3} = e^{i\left(\frac{\pi}{6}+\frac{3\pi}{2}\right)}$

P6 a) Recordemos que

$$\omega_k = (\omega_1)^k \quad \text{para raíces de la unidad}$$

$$\omega_0, \omega_1, \omega_2, \dots, \omega_{n-1} \quad y \quad \omega_n = \omega_0$$

$$\Rightarrow \quad \omega_0\omega_1 + \omega_1\omega_2 + \dots + \omega_{n-1}\omega_0 = \omega_0\omega_1 + \dots + \omega_{n-1}\omega_n$$

$$= \sum_{k=0}^{n-1} \omega_k \omega_{k+1}$$

$$= \sum_{k=0}^{n-1} (\omega_1)^k (\omega_1)^{k+1}$$

$$= \omega_1 \sum_{k=0}^{n-1} ((\omega_1)^2)^k$$

geometrica

$$= \omega_1 \left[\frac{(\omega_1^2)^n - 1}{\omega_1^2 - 1}\right]$$

$$= \omega_1 \frac{(\omega_1^n)^2 - 1}{\omega_1^2 - 1}$$

$$= \omega_1 \frac{1^2 - 1}{\omega_1^2 - 1}$$

$$= \omega_1 \cdot 0 = 0$$

b) Idem parte a)

$$\sum_{j=0}^{n-1} (w_j)^k = \sum_{j=0}^{n-1} ((w_1)^j)^k \quad \text{geometrica}$$

c) $$\sum_{j=0}^{n-1}(z+w_j)^n = \sum_{j=0}^{n-1}\sum_{k=0}^{n}\binom{n}{k}(w_j)^k z^{n-k}$$

$$= \sum_{j=0}^{n-1}\sum_{k=0}^{n}\binom{n}{k}(w_1)^{jk} z^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k} z^{n-k}\sum_{j=0}^{n-1}((w_1)^j)^k$$

b)
0

$$= \left(\binom{n}{0}z^n + \binom{n}{n}z^0 + \sum_{k=1}^{n-1}\binom{n}{k}z^{n-k}\right)\sum_{j=0}^{n-1}((w_1)^j)^k$$

$$= z^n\sum_{j=0}^{n-1}((w_1)^j)^0 + 1\sum_{j=0}^{n-1}((w_1)^n)^j + \cancel{\sum_{k=1}^{n-1}\binom{n}{k}z^{n-k}\overset{n \neq k \; 0}{\sum_{j=0}^{n-1}(w_j)^k}}$$

$$= z^n \cdot n + n \quad \blacksquare$$