Resolución Auxiliar 8

$\boxed{P1}$ a) $\sum_{k=3}^{n-1} (k-2)(k+1)$

traslación de índice

$$= \sum_{k=1}^{n-3} ((k-2)+2)((k+1)+2)$$

$$= \sum_{k=1}^{n-3} (k)(k+3)$$

$$= \sum_{k=1}^{n-3} k^2+3k$$

$$= \sum_{k=1}^{n-3} k^2 + 3\sum_{k=1}^{n-3} k$$

$$= \frac{(n-3)(n-2)(2(n-3)+1)}{6} + 3\frac{(n-3)(n-2)}{2}$$

b) $\sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} \cdot \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$ Racionalizar

$$= \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\cancel{k+1} - \cancel{k}}$$

$$= \sum_{k=1}^{n} \sqrt{k+1}-\sqrt{k} = \sqrt{n+1}-\sqrt{1}$$ } Telescopica

$$= \sqrt{n+1}-1$$

c) $\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right)$

$= \sum_{k=1}^{n} \ln(k+1) - \ln(k)$ telescopica

$= \ln(n+1) - \cancel{\ln(1)}^{0}$

$= \ln(n+1)$

d) $\sum_{k=1}^{n} k \cdot \ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} k(\ln(k+1) - \ln(k))$

$= \sum_{k=1}^{n} k\ln(k+1) - k\ln(k)$

$= \sum_{k=1}^{n} (k+1-1)\ln(k+1) - k\ln(k)$

$= \sum_{k=1}^{n} (k+1)\ln(k+1) - \ln(k+1) - k\ln(k)$

$= \sum_{k=1}^{n} (k+1)\ln(k+1) - k\ln(k) - \ln(k+1)$

$= \sum_{k=1}^{n} (k+1)\ln(k+1) - k\ln(k) - \sum_{k=1}^{n} \ln(k+1)$

$= (n+1)\ln(n+1) - \cancel{1\ln(1)}^{0} - \ln((n+1)!)$

$= (n+1)\ln(n+1) - \ln((n+1)!)$

$\boxed{P2}$ Escribimos la suma como sumatoria

$$\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(2n-1)2n}$$

Los elementos son de la forma $\frac{1}{k(k+1)}$

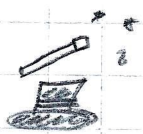
usemos como referencia

$\Rightarrow$ la suma va de n y llega a $(2n-1)$

$$\Rightarrow \sum_{k=n}^{2n-1} \frac{1}{k(k+1)}$$

Importante! ver ultima plana, si les cuesta separar, pueden usar nikita nipone

Notemos que $\frac{1}{k} - \frac{1}{k+1} = \frac{\cancel{k}+1-\cancel{k}}{k(k+1)}$ (*)

(Magia )

$$= \frac{1}{k(k+1)}$$

$$\Rightarrow \sum_{k=n}^{2n-1} \frac{1}{k(k+1)} = \sum_{k=n}^{2n-1} \frac{1}{k} - \frac{1}{k+1}$$

telescopica

$$= \frac{1}{n} - \frac{1}{(2n-\cancel{1})\cancel{+1}}$$

$$= \frac{1}{n} - \frac{1}{2n}$$

(*) Esto es el borrador, para efectos de una prueba no es relevante

$$\frac{1}{K(K+1)} = \frac{A}{K} + \frac{B}{K+1}$$

$$= \frac{A(K+1) + BK}{K(K+1)} = \frac{AK + A + BK}{K(K+1)}$$

$$= \frac{(A+B)K + A}{K(K+1)}$$

$$\Rightarrow \frac{1}{K(K+1)} = \frac{(A+B)K + A}{K(K+1)}$$

$$\Rightarrow 1 = (A+B)K + A$$

Por correspondencia

$$\Rightarrow 0K^1 + 1K^0 = (A+B)K^1 + A \cdot K^0$$

$$\Rightarrow \begin{array}{l} A+B=0 \\ A=1 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \end{array}$$

$$\Rightarrow \frac{1}{K(K+1)} = \frac{A}{K} + \frac{B}{K+1} = \frac{1}{K} + \frac{-1}{K+1} = \frac{1}{K} - \frac{1}{K+1}$$

(Con esto ya no es necesaria la Magia)

P3 Muestre que

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

Notemos que

$$\star \quad \frac{1/2}{k} - \frac{1}{k+1} + \frac{1/2}{k+2} = \frac{\frac{1}{2}(k+1)(k+2) - k(k+2) + \frac{1}{2}(k)(k+1)}{k(k+1)(k+2)}$$

$$= \frac{\cancel{\frac{1}{2}k^2} + \cancel{\frac{3}{2}k} + 1 \cancel{-k^2} \cancel{-2k} + \cancel{\frac{1}{2}k^2} + \cancel{\frac{1}{2}k}}{k(k+1)(k+2)}$$

$$= \frac{1}{k(k+1)(k+2)}$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{n} \frac{1/2}{k} - \frac{1}{k+1} + \frac{1/2}{k+2}$$

$$= \sum_{k=1}^{n} \frac{1/2}{k} - \frac{1/2}{k+1} - \frac{1/2}{k+1} + \frac{1/2}{k+2}$$

$$= \frac{1}{2}\left(\sum_{k=1}^{n} \left(\frac{1}{k} + \frac{-1}{k+1}\right) + \left(\frac{-1}{k+1} + \frac{1}{k+2}\right)\right)$$

$$= \frac{1}{2}\left[\left(\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}\right) + \left(\sum_{k=1}^{n} \frac{-1}{k+1} + \frac{1}{k+2}\right)\right]$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{n+1}\right) + \left(-\frac{1}{2} + \frac{1}{n+2}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1}{n+2} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{(n+1)-(n+2)}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{-1}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] \blacksquare$$

Veamos ☆

$$\frac{1}{K(K+1)(K+2)} = \frac{A}{K} + \frac{B}{K+1} + \frac{C}{K+2}$$

$$= \frac{A(K+1)(K+2) + B(K)(K+2) + C(K)(K+1)}{K(K+1)(K+2)}$$

Reordenando y factorizando

$$= \frac{A(K^2+3K+2) + B(K^2+2K) + C(K^2+K)}{K(K+1)(K+2)}$$

$$= \frac{K^2(A+B+C) + K(3A+2B+C) + 2A}{K(K+1)(K+2)}$$

$$\Rightarrow \frac{1}{K(K+1)(K+2)} = \frac{K^2(A+B+C) + K(3A+2B+C) + 2A}{K(K+1)(K+2)}$$

$$\Rightarrow \quad 1 = k^2(A+B+C) + k(3A+2B+C) + 2A$$

por correspondencia

$$\Rightarrow \quad 0k^2 + 0k^1 + 1k^0 = k^2(A+B+C) + k^1(3A+2B+C) + 2A\,k^0$$

$$\left.\begin{array}{ll} Ec_1 & A+B+C = 0 \\ Ec_2 & 3A+2B+C = 0 \\ Ec_3 & 2A = 1 \end{array}\right\} \quad \text{de } (3) \quad \boxed{A = \frac{1}{2}}$$

$$-1 \cdot Ec_1 + Ec_2 \Rightarrow 2A + B = 0 \qquad A = \frac{1}{2}$$

$$\Rightarrow 2 \cdot \frac{1}{2} + B = 0$$

$$\Rightarrow \boxed{B = -1}$$

de Ec_1 $\quad A + B + C = 0$

$$\Rightarrow \frac{1}{2} + (-1) + C = 0$$

$$\Rightarrow \boxed{C = \frac{1}{2}}$$

$$\Rightarrow \frac{1}{k(k+1)(k+2)} = \frac{A}{k} + \frac{B}{k+1} + \frac{C}{k+2}$$

$$= \frac{1/2}{k} + \frac{-1}{k+1} + \frac{1/2}{k+2}$$

P4 sea $S_n = \sum_{k=1}^{n} kr^k$

a) pdq $S_n = r(S_n - nr^n) + \sum_{k=0}^{n-1} r^{k+1}$

tomemos S_n

$$S_n = \sum_{k=1}^{n} kr^k = \sum_{k=0}^{n-1} (k+1)r^{k+1}$$ (traslación de) indice

$$= \sum_{k=0}^{n-1} kr^{k+1} + r^{k+1}$$

lo encontramos, falta la otra parte

$$= \sum_{k=0}^{n-1} kr^{k+1} + \sum_{k=0}^{n-1} r^{k+1}$$

r es cte

$$= \sum_{k=0}^{n-1} kr^k \cdot r + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r\sum_{k=0}^{n-1} k \cdot r^k + \sum_{k=0}^{n-1} r^{k+1}$$

entro nr^n a la sumatoria

$$= r\left(\sum_{k=0}^{n-1} k \cdot r^k + nr^n - nr^n\right) + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r\left(\sum_{k=0}^{n} k \cdot r^k - n \cdot r^n\right) + \sum_{k=0}^{n-1} r^{k+1}$$

en $k=0$

$kr^k = 0r^0$

$= 0$

No aporta nada a la sumatoria

$$= r\left(\sum_{k=1}^{n} k \cdot r^k - nr^n\right) + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r\Big(\underbrace{\sum_{k=1}^{n} k r^k}_{S_n} - n r^n\Big) + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r\left(S_n - n r^n\right) + \sum_{k=0}^{n-1} r^{k+1}$$

que era lo que debiamos probar

b) demostrar que

$$S_n = \frac{r - (n+1) r^{n+1} + n r^{n+2}}{(1-r)^2}$$

En efecto, por (a)

$$S_n = r(S_n - n r^n) + \sum_{k=0}^{n-1} r^{k+1}$$

$$\Rightarrow \quad S_n = r S_n - n r^{n+1} + \sum_{k=0}^{n-1} r \cdot r^k \qquad / - r S_n$$

$$\Rightarrow \quad S_n - r S_n = -n r^{n+1} + r \sum_{k=0}^{n-1} r^k$$

$$S_n (1-r) = -n r^{n+1} + r\left(\frac{1 - r^{n-1+1}}{1-r}\right)$$

$$\Rightarrow S_n(1-r) = \frac{(1-r)(-nr^{n+1}) + r(1-r^n)}{(1-r)} \quad / \cdot \frac{1}{(1-r)}$$

$$\Rightarrow S_n = \frac{-nr^{n+1} + nr^{n+2} + r - r^{n+1}}{(1-r)^2}$$

$$\Rightarrow S_n = \frac{r-(n+1)r^{n+1} + nr^{n+2}}{(1-r)^2}$$

que era lo que queríamos probar

P5 propuestos

a) $$\sum_{k=p}^{q} k(k+1) = \sum_{k=0}^{q-p} (k+p)(k+p+1)$$

$$= \sum_{k=0}^{q-p} k^2 + (2p+1)k + p(p+1)$$

$$= \sum_{k=0}^{q-p} k^2 + \sum_{k=0}^{q-p} (2p+1)k + \sum_{k=0}^{q-p} p(p+1)$$

$$= \frac{(q-p)(q-p+1)(2(q-p)+1)}{6} + (2p+1)\sum_{k=0}^{q-p} k + p(p+1)\sum_{k=0}^{q-p} 1$$

$$= \frac{(q-p)(q-p+1)(2(q-p)+1)}{6} + (2p+1)\frac{(q-p)(q-p+1)}{2} + p(p+1)(q-p+1)$$

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b) $$\sum_{k=3}^{n} b_k - b_{k-3} = \sum_{k=3}^{n} (b_k - \overbrace{b_{k-1}) + (b_{k-1} - b_{k-2}) + (b_{k-2}}^{} - b_{k-3})$$

Telescopico

$$= \sum_{k=3}^{n} b_k - b_{k-1} + \sum_{k=3}^{n} b_{k-1} - b_{k-2} + \sum_{k=3}^{n} b_{k-2} - b_{k-3}$$

$$= b_n - b_2 + b_{n-1} - b_1 + b_{n-2} - b_0$$

$$= b_n + b_{n-1} + b_{n-2} - b_2 - b_1 - b_0$$

c) $$\sum_{k=1}^{n} \frac{1}{\sqrt{k}(k+1) + k\sqrt{k+1}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k+1} + \sqrt{k})} \cdot \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k+1} - \sqrt{k}}$$

$$= \sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k}\sqrt{k+1}(\cancel{k}+1-\cancel{k})}$$

$$= \sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k}\sqrt{k+1}}$$

$$= \sum_{k=1}^{n} \frac{\cancel{\sqrt{k+1}}}{\sqrt{k}\cancel{\sqrt{k+1}}} - \frac{\cancel{\sqrt{k}}}{\cancel{\sqrt{k}}\sqrt{k+1}}$$

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$$= \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

$$= \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}}$$

d) $\sum_{k=0}^{n} k k!$

notemos que $(k+1)! = (k+1)k!$

$$= kk! + k!$$

$$\Rightarrow (k+1)! - k! = kk!$$

$$\Rightarrow \sum_{k=0}^{n} kk! = \sum_{k=0}^{n} (k+1)! - k! \qquad \text{telescopica}$$

$$= (n+1)! - 0!$$

$$= (n+1)! - 1$$

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e) $$\sum_{k=1}^{n} \cos((2k-1)x) = \sum_{k=1}^{n} \cos((2k-1)x)\,\frac{2\operatorname{sen}(x)}{2\operatorname{sen}(x)} \quad \text{Hint}$$

$$= \sum_{k=1}^{n} \frac{2\cos((2k-1)x)\operatorname{sen}(x)}{2\operatorname{sen}(x)} \quad \text{ordenamos}$$

Hint

$$= \sum_{k=1}^{n} \frac{\operatorname{sen}([(2k-1)x]+x) - \operatorname{sen}([(2k-1)x]-x)}{2\operatorname{sen}(x)}$$

$$= \sum_{k=1}^{n} \frac{\operatorname{sen}(2kx \cancel{-x} \cancel{+x}) - \operatorname{sen}(2kx - x - x)}{2\operatorname{sen}(x)}$$

$$= \sum_{k=1}^{n} \frac{\operatorname{sen}(2kx) - \operatorname{sen}(2kx - 2x)}{2\operatorname{sen}(x)}$$

$$= \sum_{k=1}^{n} \frac{\operatorname{sen}(2xk) - \operatorname{sen}(2x(k-1))}{2\operatorname{sen}(x)}$$

$$= \frac{1}{\operatorname{sen}(x)} \sum_{k=1}^{n} \operatorname{sen}(2x \cdot k) - \operatorname{sen}(2x(k-1))$$

$$= \frac{1}{\operatorname{sen}(x)} \cdot \left[\operatorname{sen}(2xn) - \cancel{\operatorname{sen}(2x(0))}^{0} \right]$$

$$= \frac{\operatorname{sen}(2xn)}{\operatorname{sen}(x)}$$

Obs: Solo las expresiones $\frac{P(x)}{q(x)}$ tal que el grado del polinomio "q" es mayor estricto que el grado del polinomio "p", y además que

$$q(x) = (x - \alpha_0)(x - \alpha_1) \dots (x - \alpha_n)$$

donde $\alpha_i \neq \alpha_j \ \forall i, j$

Si no lo cumple no se puede descomponer así

Se pueden separar en

$$\frac{P(x)}{q(x)} = \frac{A_0}{x - \alpha_0} + \frac{A_1}{x - \alpha_1} + \dots + \frac{A_n}{x - \alpha_n}$$

Forma alternativa para separar $\frac{1}{k(k+1)}$

$$\frac{1}{k(k+1)} = \frac{1+k-k}{k(k+1)} = \frac{(1+k)-k}{k(k+1)}$$

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$$= \frac{\cancel{(1+k)}}{k\cancel{(k+1)}} - \frac{\cancel{k}}{\cancel{k}(k+1)}$$

$$= \frac{1}{k} - \frac{1}{k+1}$$

Forma alternativa para separar $\frac{1}{k(k+1)(k+2)}$

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{k(k+1)(k+2)} \cdot \frac{2}{2}$$

$$= \frac{1}{2} \cdot \frac{2}{k(k+1)(k+2)}$$

$$= \frac{1}{2} \cdot \frac{(2+k)-k}{k(k+1)(k+2)}$$

$$= \frac{1}{2}\left[\frac{\cancel{(2+k)}}{k(k+1)\cancel{(k+2)}} - \frac{\cancel{k}}{\cancel{k}(k+1)(k+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right]$$

Si $a_k = \frac{1}{k(k+1)}$

$\Rightarrow a_{k+1} = \frac{1}{(k+1)(k+2)}$

$$a_k = \frac{1}{2}\left[a_k - a_{k+1}\right]$$