Q3

Solve the problem with LOG utility first.

(a) $\mathcal{L} = \alpha \log(c_1) + (1-\alpha)\log(l_1) + \beta\left[\alpha \log(c_2) + (1-\alpha)\log(l_2)\right]$

$$+\lambda\left[-c_1 - \frac{c_2}{1+r} + w_1(1-l_1) + \frac{w_2(1-l_2)}{1+r}\right]$$

FOC's:

(c_1): $\dfrac{\alpha}{c_1} = \lambda$ $\qquad$ (l_1): $\dfrac{1-\alpha}{l_1} = \lambda w_1$

(c_2): $\dfrac{\beta\alpha}{c_2} = \dfrac{\lambda}{1+r}$ $\qquad$ (l_2): $\dfrac{\beta(1-\alpha)}{l_2} = \dfrac{\lambda w_2}{1+r}$

~~Insert~~ Insert c_1, c_2, l_1, l_2 from FOC's into the lifetime budget constraint

$$\frac{\alpha}{\lambda} + \frac{\beta\alpha}{\lambda} + \frac{1-\alpha}{\lambda} + \frac{\beta(1-\alpha)}{\lambda} = w_1 + \frac{w_2}{1+r} = I(r)$$

$$\frac{1+\beta}{\lambda} = I(r) \Rightarrow \frac{1}{\lambda} = \frac{I(r)}{1+\beta}$$

Insert $\frac{1}{\lambda}$ into FOC's

$$c_1 = \frac{\alpha}{\lambda} = \frac{\alpha}{1+\beta} I(r) \quad ; \quad c_2 = \frac{(1+r)\beta\alpha}{\lambda} = \frac{\alpha\beta(1+r)}{1+\beta} I(r).$$

$$l_1 = \frac{1}{\lambda}\frac{(1-\alpha)}{w_1} = \frac{I(r)(1-\alpha)}{(1+\beta)w_1} = \frac{(1-\alpha)}{1+\beta}\left[1 + \frac{w_2}{w_1}\frac{1}{1+r}\right]$$

$$l_2 = \frac{\beta(1-\alpha)(1+r)}{\lambda w_2} = \frac{\beta(1-\alpha)(1+r)}{(1+\beta)w_2}\left[w_1 + \frac{w_2}{1+r}\right]$$

$$l_2 = \frac{\beta(1-\alpha)}{1+\beta}\left((1+r)\frac{w_1}{w_2} + 1\right)$$

(b) Now solve the problem with ~~$u(c,l) = \alpha\log(c) + \beta\log(l)$~~

$$u(c,l) = \frac{(c^{\alpha} l^{1-\alpha})^{1-\sigma}}{1-\sigma}$$

Lagrangian:

$$\mathcal{L} = \frac{(c_1^{\alpha} l_1^{1-\alpha})^{1-\sigma}}{1-\sigma} + \beta\frac{(c_2^{\alpha} l_2^{1-\alpha})^{1-\sigma}}{1-\sigma}$$

$$+\lambda\left[-c_1 - \frac{c_2}{1+r} + w_1(1-l_1) + \frac{w_2(1-l_2)}{1+r}\right]$$

FOC's

(c_1): $\dfrac{\alpha\cancel{(1-\sigma)}\, c_1^{\alpha(1-\sigma)-1} \cdot l_1^{(1-\alpha)(1-\sigma)}}{\cancel{1-\sigma}} = \lambda$

$\Rightarrow \alpha\, c_1^{\alpha(1-\sigma)-1} \cdot l_1^{(1-\alpha)(1-\sigma)} = \lambda$

(l_1): $(1-\alpha)\, c_1^{\alpha(1-\sigma)}\, l_1^{(1-\alpha)(1-\sigma)-1} = \lambda w_1$

If we set $\sigma = 1$, we see that these FOC's become the same FOC's as in the LOG case. Therefore, solutions should be same when $\sigma = 1$.

Take the ratio of two FOC's.

$$\frac{1-\alpha}{\alpha}\frac{c_1}{l_1} = w_1 \Rightarrow w_1 l_1 = \frac{1-\alpha}{\alpha} c_1 \Rightarrow l_1 = \frac{1-\alpha}{\alpha}\frac{c_1}{w_1}$$

(c_2): $\beta\alpha\, c_2^{\alpha(1-\sigma)-1}\, l_2^{(1-\alpha)(1-\sigma)} = \dfrac{\lambda}{1+r}$

(l_2): $\beta(1-\alpha)\, c_2^{\alpha(1-\sigma)}\, l_2^{(1-\alpha)(1-\sigma)-1} = \dfrac{\lambda w_2}{1+r}$

Take ratio of these FOC's to obtain

$$w_2 l_2 = \frac{1-\alpha}{\alpha} c_2 \Rightarrow l_2 = \frac{1-\alpha}{\alpha}\frac{c_2}{w_2}$$

* Use the FOCs for c_1 & c_2 to obtain the following Euler Equation (take the ratio of the two FOCs):

$$c_1^{\alpha(1-\sigma)-1}\, l_1^{(1-\alpha)(1-\sigma)} = \beta(1+r)\, c_2^{\alpha(1-\sigma)-1}\, l_2^{(1-\alpha)(1-\sigma)}$$

Insert l_1 as a fnc. of c_1 & l_2 as a fnc of c_2 obtained in the previous page to the Euler Equation above.

$$c_1^{\alpha(1-\sigma)-1}\left(\frac{1-\alpha}{\alpha}\frac{c_1}{w_1}\right)^{(1-\alpha)(1-\sigma)} = \beta(1+r)\, c_2^{\alpha(1-\sigma)-1}\cdot\left(\frac{1-\alpha}{\alpha}\frac{c_2}{w_2}\right)^{(1-\alpha)(1-\sigma)}$$

$$c_1^{-\sigma}\cdot\frac{1}{w_1^{(1-\alpha)(1-\sigma)}} = \beta(1+r)\, c_2^{-\sigma}\,\frac{1}{w_2^{(1-\alpha)(1-\sigma)}}$$

$$\left(\frac{c_2}{c_1}\right)^{\sigma} = \beta(1+r)\left(\frac{w_1}{w_2}\right)^{(1-\alpha)(1-\sigma)}$$

$$c_2 = (\beta(1+r))^{1/\sigma}\left(\frac{w_1}{w_2}\right)^{\frac{(1-\alpha)(1-\sigma)}{\sigma}}\cdot c_1$$

Note that LBC reads

$$c_1 + \frac{c_2}{1+r} + w_1 l_1 + \frac{w_2 l_2}{1+r} = w_1 + \frac{w_2}{1+r} = I(r)$$

$$c_1 + \frac{c_2}{1+r} + \frac{1-\alpha}{\alpha}c_1 + \frac{1}{1+r}\frac{1-\alpha}{\alpha}c_2 = I(r)$$

* $\Rightarrow$ $c_1 + \dfrac{c_2}{1+r} = \alpha\cdot I(r)$

Set $w_1 = w_2 \Rightarrow$ $c_2 = (\beta(1+r))^{1/\sigma}c_1$

$$c_1 + \frac{1}{1+r}c_1(\beta(1+r))^{1/\sigma} = \alpha I(r)$$

$$c_1\left(1+\beta^{1/\sigma}(1+r)^{\frac{1}{\sigma}-1}\right) = \alpha I(r)$$

$$c_1 = \frac{\alpha I(r)}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}$$

$$c_2 = \frac{\alpha(\beta(1+r))^{1/\sigma}}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}\, I(r)$$

$$l_1 = \frac{1-\alpha}{\alpha}\frac{c_1}{w_1} = \frac{(1-\alpha)}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}\left(1+\frac{1}{1+r}\right)$$

$$l_2 = \frac{1-\alpha}{\alpha}\frac{c_2}{w_2} = \frac{(1-\alpha)\beta(1+r)^{1/\sigma}}{1+\beta^{1/\sigma}(1+r)^{\frac{1-\sigma}{\sigma}}}\left(1+\frac{1}{1+r}\right)$$

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$$\max_{c_1,c_2} \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\sigma}}{1-\sigma}$$

s.t: $c_1 + \frac{c_2}{1+r} = w_1$

$$\mathcal{L} = \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\sigma}}{1-\sigma} + \lambda\left[w_1 - c_1 - \frac{c_2}{1+r}\right]$$

c_1: $c_1^{-\sigma} = \lambda$ $\quad$ c_2: $c_2^{-\sigma} = \frac{\lambda}{1+r} = \frac{c_1^{-\sigma}}{1+r}$ $\quad$ $c_1 + \frac{c_2}{1+r} = w$.

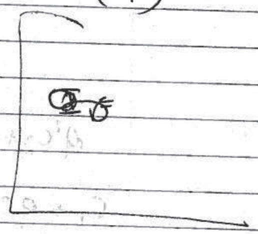
Then $c_1^{-\sigma} = (1+r)c_2^{-\sigma} \Rightarrow \left(\frac{c_1}{c_2}\right)^{-\sigma} = 1+r \Rightarrow \left(\frac{c_1}{c_2}\right)^{\sigma} = \frac{1}{1+r} \Rightarrow \left(\frac{c_2}{c_1}\right)^{\sigma} = 1+r$

$\Rightarrow \frac{c_2}{c_1} = (1+r)^{1/\sigma} \Rightarrow \boxed{c_2 = c_1(1+r)^{1/\sigma}}$

Budget: $c_1 + c_1(1+r)^{1/\sigma - 1} = w_1$

$c_1\left[1+(1+r)^{1/\sigma-1}\right] = w_1$

$\boxed{c_1 = \frac{w_1}{1+(1+r)^{1/\sigma-1}}}$ $\quad$ $\boxed{c_2 = \frac{w_1(1+r)^{1/\sigma}}{1+(1+r)^{1/\sigma-1}}}$



σ

) i) $\sigma<1$ $\quad r\uparrow$

$\sigma=1$ $\quad c_1\downarrow$.

$\sigma>1$.

	c_1	c_2
IE	+	+
SE	−	+
WE	0	0
TE	$\sigma<1 \Rightarrow c_1-$; $\sigma=1 \Rightarrow 0$; $\sigma>1 \Rightarrow c_1+$	+

2. Consider a three period economy. The household's preference is given by $u(c_1) + u(c_2) + u(c_3)$, where c_t is consumption in period t. The real interest is $r = 0$. Before tax incomes are given by $e_1 = 200$, $e_2 = 1000$, and $e_3 = 200$. The consumer is not allowed to borrow, i.e. there is a constraint on borrowing. The government uses lump sum taxes to finance government expenditures.

   (a) [15] Assuming that lump-sum taxes are given by $T_1 = 100$, $T_2 = 100$, and $T_3 = 100$, solve the consumer's optimal consumption and end-of-period net asset position in each period.

   (b) [15] Assume that the government reduces current tax T_1 to zero with out changing its expenditures. T_3 also remains at 100. Solve the consumer's optimal consumption and end-of-period net asset position in each period.

   (c) [5] Explain why Ricardian Equivalence fails when the taxes are changed from the levels in part (a) to the levels in part (b) (in doing so, focus on which assumption of Ricardian Equivalence is not satisfied in this environment).

(a) First solve the unconstrained problem. Note S_3 should be zero since it does not deliver any utility.

$c_1 + S_1 = e_1 - T_1$, $c_2 + S_2 = e_2 - T_2 + (1+r)S_1$, $c_3 = e_3 - T_3 + (1+r)S_2$

Lifetime B.C. is (LBC) $\quad c_1 + c_2 + c_3 = e_1 - T_1 + e_2 - T_2 + e_3 - T_3$

$= 200 - 100 + 1000 - 100 + 200 - 100$

$= 1100.$

$$\mathcal{L} = u(c_1) + u(c_2) + u(c_3) + \lambda\left[-c_1 - c_2 - c_3 + 1100\right]$$

$\Rightarrow u'(c_1) = \lambda;\ u'(c_2) = \lambda;\ u'(c_3) = \lambda \Rightarrow c_1 = c_2 = c_3$

Substituting into LBC, we obtain $c_t = \frac{1100}{3}\ \forall t$

Check whether unconstrained solution violates borrowing constraints

$c_1 + \underline{S_1} = e_1 - T_1 = 100 \Rightarrow \frac{1100}{3} + S_1 = 100 \Rightarrow \underline{S_1} = \frac{-800}{3} < 0$

net asset position

So $S_1 \geq 0$ is violated.

The consumer then chooses $S_1 = 0$ & $c_1 = e_1 - T_1 = 100$.

The consumer still wants to set $c_2 = c_3$.

From lifetime B.C.

$c_1 + c_2 + c_3 = e_1 - T_1 + e_2 - T_2 + e_3 - T_3$

& from $c_1 = e_1 - T_1$ (from the last result) we have

$c_2 + c_3 = e_2 - T_2 + e_3 - T_3 = 1000 - 100 + 200 - 100$

$\Rightarrow \quad c_2 + c_3 = 1000$

Together with the optimality condition $c_2 = c_3$

we have $c_2 = c_3 = 500$

Now check whether borrowing constraint is violated.

$c_2 + S_2 = e_2 - T_2$

$\cancel{c_2}\ 500 + S_2 = 1000 \quad -100 \quad S_2 = 400 \quad \Rightarrow S_2 = 500 > 0$

Since this solution does not violate the borrowing constraint, It is optimal. Thus, we have $c_1 = 100$, $c_2 = 500$, $c_3 = 500$

$S_1 = 0$, $S_2 = \cancel{500}\ 400$, $S_3 = 0$.

(b) $\underset{T_1}{\underset{\|}{0}} + T_2 + 100 = 100 + 100 + 100 \Rightarrow T_2 = 200$

Note that PV taxes ~~does~~ not Δ since G's are fixed. as a result, unconstrained solution will be the same as in part (a). since lifetime income is the same. ~~Since~~ the unconstrained solution $c_1 = c_2 = c_3 = \frac{1100}{3}$ violates the 1st period B.C. since $c_1 + S_1 = 200 \Rightarrow S = -\frac{500}{3} < 0$.

Thus, $c_1 = e_1 - T_1 = 200$ & $S_1 = 0$.

Using $c_1+c_2+c_3 = e_1-T_1+e_2-T_2+e_3-T_3$

& $c_1 = e_1 - T_1$

we have $c_2+c_3 = e_2-T_2+e_3-T_3$

$= 1000-200+200-100$

$c_2+c_3 = 900$

Using the optimality condition $c_2=c_3$ we obtain

$c_2 = c_3 = 450$.

Using 2nd period B.C. $c_2+S_2 = e_2-T_2$

$S_2 = 1000-200-450 = \underline{350} > 0 \Rightarrow$ So, $c_2=c_3$ is optimal since borrowing constraint is not violated.

~~So~~

Then, the optimal solution is.

$c_1 = 200$, $c_2 = 450$, $c_3 = 450$

$S_1 = 0$, $S_2 = 350$, $S_3 = 0$.

(c). The consumer is borrowing constrained in the 1st period, i.e. he wants to consume MORE but he can't. As a result, when the govnt reduces taxes in the 1st period, he has more income in the 1st period ~~he increases~~ his 1st period consumption because this is what he wanted to do anyways. Thus, Consumption allocations change with this tax change.