a) The equation describing the evolution of the capital stock per unit of effective labor is given by :

(1) $\dot{k} = sf(k) - (n + g + \delta)k$

Substituting in for the intensive form of the Cobb-Douglas -- $f(k) = k^{\alpha}$ -- yields:

$$\dot{k} = sk^{\alpha} - (n + g + \delta)k$$

On the balanced growth path, $\dot{k}$ is zero -- investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k*, we have $sk^{*\alpha} = (n + g + \delta)k^*$. Rearranging to solve for k* yields:

(2) $k^* = \left[s/(n + g + \delta)\right]^{1/(1-\alpha)}$

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (2) into the intensive form of the production function -- $y = k^{\alpha}$:

(3) $y^* = \left[s/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1 - s)y^*$. Substituting equation (3) into this expression yields:

(4) $c^* = (1 - s)\left[s/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$

b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k, take equation (2) -- the balanced-growth-path level of k -- and rearrange it to solve for s:

(5) $s = (n + g + \delta)k^{*1-\alpha}$

Now substitute equation (5) into equation (4):

$$c^* = \left[1 - (n + g + \delta)k^{*1-\alpha}\right] \cdot \left[(n + g + \delta)k^{*1-\alpha}/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$$

After some straightforward algebraic manipulation, this simplifies to:

(6) $c^* = k^{*\alpha} - (n + g + \delta)k^*$

Equation (6) can be easily interpreted. Consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor, which on the balanced growth path is the same as break-even investment per unit of effective labor, $(n + g + \delta)k^*$.

Now use equation (6) to maximize c* with respect to k*. The first-order condition is given by :

$$\partial c^*/\partial k^* = \alpha k^{*\alpha-1} - (n + g + \delta) = 0$$

or simply:

(7) $\alpha k^{*\alpha-1} = (n + g + \delta)$

Note that equation (7) is just a specific form of $f'(k^*) = (n + g + \delta)$, which is the general condition that implicitly defines the golden-rule level of capital per unit of effective labor. It has a graphical interpretation as the level of k where the slope of the intensive form of the production function is equal to the slope of the break-even investment line.

c) To get the saving rate that will yield the golden-rule level of k, substitute equation (8) into (5):

$$s_{GR} = (n + g + \delta) \cdot [\alpha/(n + g + \delta)]^{(1-\alpha)/(1-\alpha)}$$

which simplifies to:

(9) $s_{GR} = \alpha$

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

What happens during the transition? Look at the production function $Y = F(K,AL)$. On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and of AL (which is growing at n_{NEW}). Thus during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally, in the end, K, AL and thus Y are all growing at the new lower n_{NEW}.

The derivative of $y^* = f(k^*)$ with respect to n is given by:

(1) $\partial y^*/\partial n = f'(k^*)[\partial k^*/\partial n]$

To find $\partial k^*/\partial n$, use the equation for the evolution of the capital stock per unit of effective labor, $\dot{k} = sf(k) - (n + g + \delta)k$. In addition, use the fact that on a balanced growth path, $\dot{k} = 0$, $k = k^*$ and thus $sf(k^*) = (n + g + \delta)k^*$. Taking the derivative of both sides of this expression with respect to n yields:

$$sf'(k^*) \cdot \frac{\partial k^*}{\partial n} = (n + g + \delta) \cdot \frac{\partial k^*}{\partial n} + k^*$$

and rearranging yields:

(2) $$\frac{\partial k^*}{\partial n} = \frac{k^*}{sf'(k^*) - (n + g + \delta)}$$

Substituting equation (2) into equation (1) gives us:

(3) $$\frac{\partial y^*}{\partial n} = f'(k^*) \cdot \left[\frac{k^*}{sf'(k^*) - (n + g + \delta)} \right]$$

Rearrange the condition that implicitly defines k^* – $sf(k^*) = (n + g + \delta)k^*$ – and solve for s yielding:

(4) $s = (n + g + \delta)k^*/f(k^*)$

Substitute equation (4) into equation (3):

(5) $$\frac{\partial y^*}{\partial n} = \frac{f'(k^*)k^*}{[(n + g + \delta)f'(k^*)k^*/f(k^*)] - (n + g + \delta)}$$

To turn this into the elasticity that we want, multiply both sides of equation (5) by n/y^*:

$$\frac{n}{y^*} \cdot \frac{\partial y^*}{\partial n} = \frac{n}{(n + g + \delta)} \cdot \frac{f'(k^*)k^*/f(k^*)}{[f'(k^*)k^*/f(k^*)] - 1}$$

Using the definition that $\alpha_K(k^*) \equiv f'(k^*)k^*/f(k^*)$ gives us:

(6) $$\frac{n}{y^*} \cdot \frac{\partial y^*}{\partial n} = -\frac{n}{(n + g + \delta)} \cdot \left[\frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)} \right]$$

Now, with $\alpha_K(k^*) = 1/3$, $g = 2\%$ and $\delta = 3\%$, we need to calculate the effect on y^* of a fall in n from 2% to 1%. Using the midpoint of $n = 0.015$ to calculate the elasticity gives us:

$$\frac{n}{y^*} \cdot \frac{\partial y^*}{\partial n} = -\frac{0.015}{(0.015 + 0.02 + 0.03)} \cdot \left(\frac{1/3}{1 - 1/3} \right) \cong -0.12$$

So this 50% drop in the population growth rate (from 2% to 1%) will lead to a (-0.50) (-0.12) ≅ 0.06 or 6% increase in the level of output per unit of effective labor. This illustrates the point that observed differences in population growth rates across countries are not nearly enough to account for differences in y that we see.

a) Assuming that investment rises by the full amount of the fall in the deficit, the share of output that is devoted to investment -- the saving rate -- should rise from s = 0.15 to s_{NEW} = 0.18. Note that this is a 20% increase in the saving rate (0.03 is 20% of 0.15). From equation (1.22) in the text, the elasticity of output with respect to the saving rate is:

$$(1)\quad \frac{s}{y^*}\cdot\frac{\partial y^*}{\partial s}=\frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}$$

where $\alpha_K(k^*)$ is the share of income paid to capital (assuming that capital is paid its marginal product).

We are told to assume that $\alpha_K(k^*) = 1/3$. Substituting this in gives us:

$$\frac{s}{y^*}\cdot\frac{\partial y^*}{\partial s}=\frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}=\frac{1/3}{1-1/3}=\frac{1}{2}$$

Thus the elasticity of output with respect to the saving rate is 1/2. So this 20% increase in the saving rate -- from s = 0.15 to s_{NEW} = 0.18 -- will cause output to rise relative to what it would have been by about 10%. For such a huge policy change -- total elimination of the budget deficit -- this appears to be a rather modest benefit. [Note that the analysis has really been carried out in terms of output per unit of effective labor. Since the paths of A and L are not affected, however, if output per unit of effective labor rises by 10%, output itself is also 10% higher than what it would have been.]

b) Consumption will rise even less than output. Although output winds up 10% higher than what it would have been, the fact that the saving rate is higher means that we are consuming a smaller fraction of output than before the deficit reduction. Thus we do not get to enjoy the entire 10% increase in output as increased consumption. Note that by consumption, we are implicitly including government purchases. We can calculate the elasticity of consumption with respect to the saving rate. On the balanced growth path, consumption is given by:

(2) $c^* = (1 - s)y^*$

Taking the derivative with respect to s yields:

$$(3)\quad \frac{\partial c^*}{\partial s} = -y^* + (1-s)\cdot\frac{\partial y^*}{\partial s}$$

To turn this into an elasticity, multiply both sides of equation (3) by s/c^*:

$$\frac{\partial c^*}{\partial s}\cdot\frac{s}{c^*} = \frac{-y^* s}{(1-s)y^*} + (1-s)\cdot\frac{\partial y^*}{\partial s}\cdot\frac{s}{(1-s)y^*}$$

where we have substituted $c^* = (1 - s)y^*$ on the right-hand side. Simplifying gives us:

$$(4)\quad \frac{\partial c^*}{\partial s}\cdot\frac{s}{c^*} = \frac{-s}{(1-s)} + \frac{\partial y^*}{\partial s}\cdot\frac{s}{(1-s)y^*}$$

From part a, the second term -- the elasticity of output with respect to the saving rate – is equal to 1/2. We can use the midpoint between $s = 0.15$ and $s_{NEW} = 0.18$ to calculate the elasticity:

$$\frac{\partial c^*}{\partial s}\cdot\frac{s}{c^*} = \frac{-0.165}{(1-0.165)} + 0.5 \cong 0.30$$

Thus the elasticity of consumption with respect to the saving rate is approximately 0.3. So this 20% increase in the saving rate -- from $s = 0.15$ to $s_{NEW} = 0.18$ -- will lead to consumption being approximately 6% above what it would have been. Again, the Solow model yields only modest benefits from a rather large change in government policy.

c) The immediate effect of the deficit reduction is that consumption falls. Although y^* does not jump immediately -- it only begins to move towards its new, higher balanced-growth-path level -- we are now saving a greater fraction and thus consuming a smaller fraction of this same y^*. At the moment of the rise in s by 3 percentage points -- since $c = (1 - s)y^*$ and y^* is unchanged -- c falls. In fact, the percentage change in c will be the percentage change in $(1 - s)$. Now, $(1 - s)$ falls from 0.85 to 0.82, which is approximately a 3.5% drop -- 0.03 is about 3.5% of 0.85. Thus at the moment of the rise in s, consumption falls by about three and a half percent.

We can use some results from the text on the speed of convergence to determine how long it takes for consumption to return to what it would have been without the deficit reduction. After the initial rise in s, s remains constant throughout. Since $c = (1 - s)y$, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In the text it is shown that the rate of convergence of k and y -- after a linear approximation -- is given by $\lambda = (1 - \alpha_K)(n + g + \delta)$. With $(n + g + \delta) = 6\%$ per year and $\alpha_K = 1/3$, this yields $\lambda \cong 4\%$. This means that k and y move 4% of the remaining distance toward their balanced-growth-path values of k^* and y^* each year. Since c is proportional to y -- $c = (1 - s)y$ -- it also approaches its new balanced-growth-path value at that same constant rate. That is, analogous to equation (1.26) in the text, we could write:

$$(5)\quad c(t) - c^* \cong e^{-(1-\alpha_K)(n+g+\delta)t}[c(0) - c^*]$$

or equivalently:

(6) $$e^{-\lambda t} = \frac{c(t) - c^*}{c(0) - c^*}$$

The term on the right-hand side of equation (6) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 3.5% and will eventually be 6% higher than it would have been. Thus it must change by 9.5% on the way to the balanced growth path. It will therefore be equal to what it would have been, 3.5%/9.5% ≅ 36.8% of the way to the new balanced growth path. Equivalently, this is when the remaining distance to the new balanced growth path is 63.2% of the original distance. In order to find out how long this will take, we need to find a t* that solves:

(7) $$e^{-\lambda \cdot t^*} = 0.632$$

Taking the natural log of both sides of equation (7) yields:

$$-\lambda \cdot t^* = \ln(0.632) \qquad \Rightarrow \qquad t^* = 0.459/0.04$$

and thus:

(8) $$t^* \cong 11.5 \text{ years}$$

It will take a fairly long time -- over a decade -- for consumption to return to what it would have been in the absence of the deficit reduction.

a) Define the marginal product of labor as $w \equiv \partial F(K,AL)/\partial L$. Then write the production function as $Y = ALf(k) = ALf(K/AL)$. Taking the partial derivative of output with respect to L yields:

(1) $w \equiv \partial Y/\partial L = ALf'(k)[-K/AL^2] + Af(k) = A[(-K/AL)f'(k) + f(k)] = A[f(k) - kf'(k)]$

So we do have $w = A[f(k) - kf'(k)]$.

b) Define the marginal product of capital as $r \equiv \partial F(K,AL)/\partial K$. Again, writing the production function as $Y = ALf(k) = ALf(K/AL)$ and now taking the partial derivative of output with respect to K yields:

(2) $r \equiv \partial Y/\partial K = ALf'(k)[1/AL] = f'(k)$

Substitute equations (1) and (2) into $wL + rK$:

$$wL + rK = A[f(k) - kf'(k)]\,L + f'(k)K = ALf(k) - f'(k)[K/AL]AL + f'(k)K$$

Simplifying gives us:

(3) $wL + rK = ALf(k) - f'(k)K + f'(k)K = ALf(k) \equiv ALF(K/AL, 1)$

Finally, since F is constant returns to scale, equation (3) can be rewritten as:

(4) $wL + rK = F(ALK/AL, AL) = F(K, AL)$

c) As shown above, $r = f'(k)$. Since k is constant on a balanced growth path, so is $f'(k)$ and thus so is r. In other words, on a balanced growth path, $\dot{r}/r = 0$. This fits one of Kaldor's stylized facts about growth which is that the return to capital is approximately constant. Since capital is paid its marginal product, the share of output going to capital is rK/Y. On a balanced growth path:

(5) $$\frac{\dot{(rK/Y)}}{(rK/Y)} = \dot{r}/r + \dot{K}/K - \dot{Y}/Y = 0 + (n+g) - (n+g) = 0$$

Thus on a balanced growth path, the share of output going to capital is constant. Since the shares of output going to capital and labor sum to 1, this implies that the share of output going to labor is also constant on the balanced growth path.

d) We need to determine what is happening to the growth rate of w as k rises toward k*. As shown above, w = A[f(k) - kf '(k)]. Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

$$(6)\quad \frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{\dot{[f(k) - kf'(k)]}}{[f(k) - kf'(k)]} = g + \frac{[f'(k)\dot{k} - \dot{k}f'(k) - kf''(k)\dot{k}]}{f(k) - kf'(k)} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)} > g$$

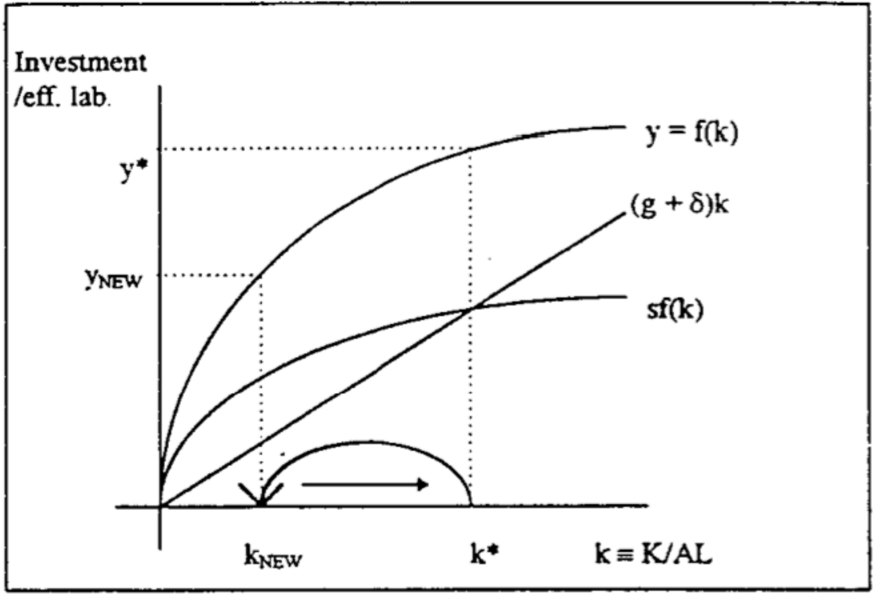
This is true because the denominator is positive since f(k) is a concave function, while the numerator is positive because k and $\dot{k}$ are positive while f " (k) is negative. Thus the marginal product of labor is growing faster than on the balanced growth path. Intuitively, the marginal product of labor rises by the rate of growth of the effectiveness of labor on the balanced growth path since that is what is increasing the marginal product of labor. As we move from k to k*, however the amount of capital per unit of effective labor is also rising which also makes labor more productive and this increases the marginal product of labor even more.

The growth rate of the marginal product of capital, r, is:

$$(7)\quad \frac{\dot{r}}{r} = \frac{\dot{[f'(k)]}}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)}$$

This growth rate is negative since f ' (k) > 0, f " (k) < 0 and $\dot{k}$ > 0 as k rises towards k*. Thus as the economy moves from k to k*, the marginal product of capital falls. That is, it grows at a rate less than on the balanced growth path where its growth rate is 0.

a) At some time -- call it t_0 -- there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k^* to k_{NEW}. We can see this by simply looking at the definition -- $k \equiv K/AL$ -- and we have had an increase in L without a jump in K or A. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the diagram, it falls from y^* to y_{NEW}.



b) Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g + \delta)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k^*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y^*.

c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k^*. At k^*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k^* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.