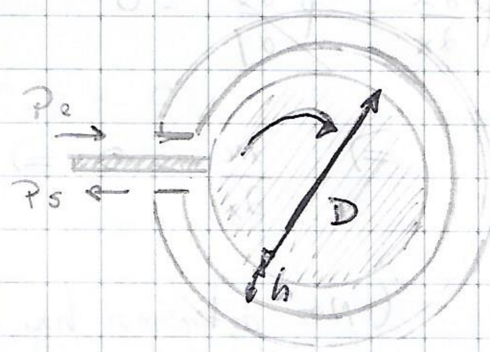
Pauta Auxiliar 7.

P2)



Como $h << D$ usaremos una aproximación de placa plana.

Escribimos las Ecuaciones de N-S en Cartesianas.

$$\hat{x} \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x .$$

$$\hat{y} \quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g_y$$

$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$ por ser un Problema estacionario.

$v \equiv 0$ por aproximación de placa infinita.

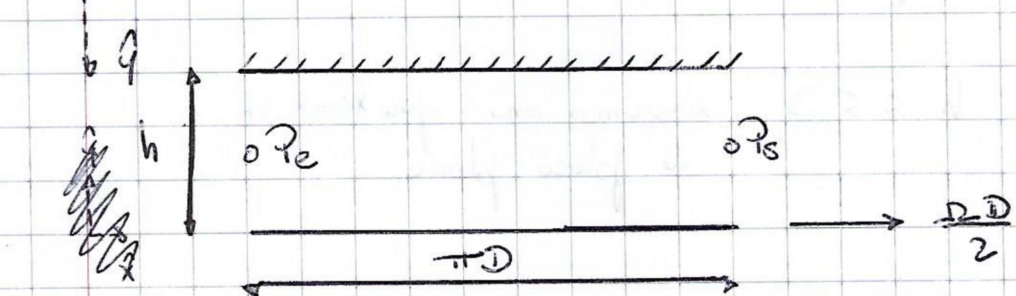
$g_x = g_y = 0.$

usamos ahora Continuidad.

$$\rho \frac{\partial u}{\partial x} + \cancel{\rho \frac{\partial v}{\partial y}}^{\,0} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0 \Rightarrow \boxed{u(x,y) = u(y)}$$

(Nos detenemos hacia el sistema pero queda más fácil !!)



$$\hat{x}\,| \quad 0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{1}{\mu}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial^2 u}{\partial y^2} \Rightarrow$$

$$u(y) = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right) y^2 + C_1 y + C_2$$

Notamos que $\frac{\partial P}{\partial x} = \frac{\Delta P}{\pi D}$.

$$u(y) = \frac{\Delta P}{2\mu\pi D} y^2 + c_1 y + c_2.$$

$$\left.\begin{array}{l} u(y=0) = 0 \\ u(y=h) = \frac{\Omega D}{2} \end{array}\right\} \text{ Condiciones de borde.}$$

$\Rightarrow c_2 = 0.$

$$\frac{\Omega D}{2} = \frac{\Delta P}{2\mu\pi D} h^2 + c_1 h$$

$$\frac{\Omega D}{2} - \frac{\Delta P}{2\mu\pi D} h^2 = c_1 h$$

$$\frac{\mu\pi\Omega D^2 - \Delta P h^2}{2\mu\pi D} = c_1 h$$

$$\boxed{c_1 = \frac{\mu\pi\Omega D^2 - \Delta P h^2}{2\mu\pi D h}}$$

$$u(y) = \frac{\Delta P\, y^2}{2\mu\pi D} + \frac{\mu\pi\Omega D^2 - \Delta P h^2}{2\mu\pi D h}\, y$$

$$u(y) = \frac{\Delta P h\,(y^2 - hy)}{2\mu\pi D h} + \frac{\Omega D y}{2h}$$

$$q = \int_0^w \int_0^h u(y)\, dy\, dz = w \int_0^h u(y)\, dy.$$

$$q = w \left\{ \frac{\Delta P h}{2\mu\pi D h}\left(\frac{h^3}{3} - \frac{h^3}{2}\right) + \frac{\Omega D}{2h}\frac{h^2}{2} \right\}$$

$$q = w \left\{ \frac{\Delta P}{2\mu\pi D}\left(-\frac{h^3}{6}\right) + \frac{\Omega D h}{4} \right\}$$

$$q = w \left\{ \frac{\Omega D h}{4} - \frac{\Delta P h^3}{12\mu\pi D} \right\}$$

$$\Rightarrow \boxed{q = \frac{w\,\Omega\,D h}{4}\left(1 - \frac{h^2 \Delta P}{3\pi\mu\Omega D^2}\right)}$$

b) Calculamos ahora el Torque.

Primero $\tau_w\Big|_{y=b} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$

$$\tau_w = \mu\left(\frac{\Delta P}{2\mu\pi D}(2y - h) + \frac{\Omega D}{2h}\right).$$

$$\tau_w(y=b) = \mu\left(\frac{\Omega D}{2h} + \frac{\Delta P h}{2\mu\pi D}\right)$$

$$T = \left(\tau_w(y=0)\cdot A \cdot D/2\right) \qquad A = \pi D w$$

$$T = \pi D w \frac{D}{2}\mu\left(\frac{\Omega D}{2h} - \frac{\Delta P h}{2\mu\pi D}\right)$$

$$T = \frac{\pi\, w\, D^2 \mu}{2}\left(\frac{\Omega D}{2h} + \frac{\Delta P h}{2\mu\pi D}\right)$$

$$\boxed{T = \frac{\pi \mu w \Omega D^3}{4h}\left(1 + \frac{h^2 \Delta P}{\pi \mu \Omega D^2}\right)}$$

c) $\dot{W}_S = \Delta P \cdot Q.$

$$\dot{W}_S = \frac{w \Omega D h}{4}\left(1 - \frac{h^2 \Delta P}{3\pi \mu \Omega D^2}\right)\Delta P$$

$$\frac{\partial \dot{W}_S}{\partial \Delta p} \overset{!}{=} 0 \quad \Rightarrow$$

$$\frac{w \Omega D h}{4} - \frac{2 h^2 \Delta P}{3\pi \mu \Omega D^2} \overset{!}{=} 0$$

$$\frac{w \Omega D h}{4}\left(\frac{3\pi \mu \Omega D^2}{2h^2}\right) = \Delta P^*$$

$$\Rightarrow \quad \Delta P^* = \frac{3\pi \mu \Omega D^2}{2h^2}$$

Potencia máxima de salida.

$$\frac{w\,\Omega\,D h}{4}\left(\frac{3\pi\mu\Omega D^2}{2h^2}-\frac{h^2}{3\pi\mu\Omega D^2}\left(\frac{9\pi^2\mu^2\Omega^2 D^4}{4\,h^4}\right)\right).$$

$$\dot{W}_{max}=\frac{w\,\Omega\,D h}{4}\left(\frac{3\pi\mu\Omega D^2}{2h^2}-\frac{3\pi\,\Omega D^2}{4h^2}\right).$$

$$\Rightarrow\quad \dot{W}_{\substack{max\\ salida}}=\frac{3\pi\mu\,w\,\Omega^2 D^3}{16\,h}$$

Calculamos ahora la potencia de entrada.

$$\dot{W}_{entrada}=\Omega T$$

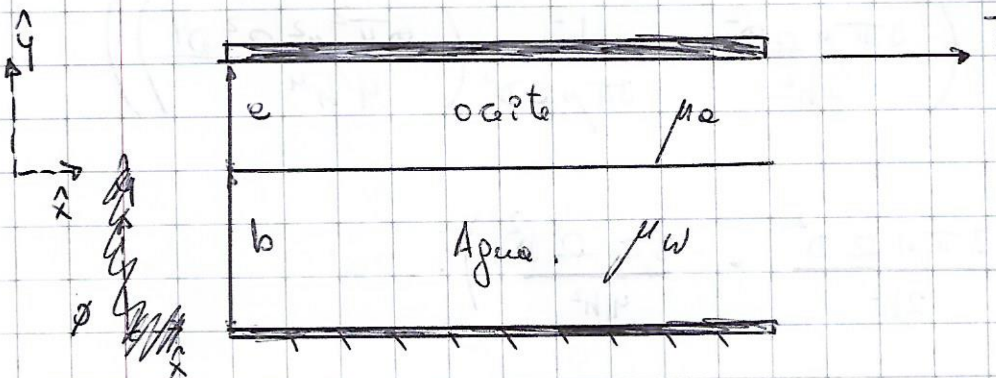
$$\dot{W}_{entrada}=\frac{\pi\mu\Omega^2 D^3}{4h}\left(1+\frac{h^2\Delta P^*}{\pi\mu\Omega D^2}\right)$$

$$\dot{W}_{entrada}=\frac{\pi\mu\Omega^2 D^3}{4h}\left(\frac{5}{2}\right)$$

$$\eta=\dot{W}_{ent}/\dot{W}_{sal}=\frac{3\times 8}{16\times 5}=0.3\sim\boxed{30\%}$$

Pauta Aux 7.

P1)



Escribimos las Ecuaciones de N-S. en Cartesianas.

$$\hat{x}) \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g x$$

$$\hat{y}) \quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g y$$

$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$ por ser un problema estacionario.

$v \equiv 0$ por ser una aproximación de Placa infinita.

$g x = g y = 0$.

$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0$ por NO poseer gradientes de Presión.

$$u(y) = \begin{cases} u_a(y) \rightarrow \text{Si } y \in [0, a] \\ u_w(y) \rightarrow \text{Si } y \in [-b, 0] \end{cases}$$

de N-S Tenemos.

$$\boxed{\begin{aligned} &\frac{\partial^2 u_a}{\partial y^2} = 0 \quad \Rightarrow \quad u_a = C_1 y + C_2 \\ &\frac{\partial^2 u_w}{\partial y^2} = 0 \quad \Rightarrow \quad u_w = \tilde{C}_1 y + \tilde{C}_2 \end{aligned}}$$

Condiciones de borde.

$$u_a(y = a) = U \Rightarrow \boxed{C_1 a + C_2 = U}$$

$$u_w(y = -b) = 0 \Rightarrow \boxed{-\tilde{C}_1 b + \tilde{C}_2 = 0}$$

$$u_a(y=0) = u_w(y=0) \Rightarrow \boxed{C_2 = \tilde{C}_2.}$$

$$\tau_w(y=0) = \tau_a(y=0)$$

$$\boxed{\mu_w \tilde{C}_1 = \mu_a C_1}$$

$$c_1 a + c_2 = U$$

$$-\frac{\mu_a}{\mu_w} c_1 b + c_2 = 0$$

$$c_1 \left(a + \frac{\mu_a}{\mu_w} b \right) = U$$

$$\boxed{c_1 = \frac{\mu_w U}{\mu_w a + \mu_a b}}$$

$$\boxed{c_2 = U - \left(\frac{\mu_w U a}{\mu_w a + \mu_a b}\right)} \Rightarrow \boxed{\tilde{c}_2 = \left(-\frac{\mu_w U a}{\mu_w a + \mu_a b}\right) + U}$$

$$\tilde{c}_1 = \frac{\mu_a}{\mu_w}\left(\frac{\mu_w U}{\mu_w a + \mu_a b}\right) = \boxed{\frac{\mu_a U}{\mu_w a + \mu_a b}}$$

ahora calculamos la velocidad en la interface.

$$u_{1a}(y=0) = u_w(y=0) = u_{interface}$$

$$\Rightarrow u_{interface} = c_2 = U - \frac{\mu_w U a}{\mu_w a + \mu_a b}$$

$$C_2 = U - \frac{\mu_w U e}{\mu_w e + \mu_e b}$$

$$\Rightarrow C_2 = \frac{U \mu_e b}{\mu_w e + \mu_e b}$$

$$q_e = \int_0^w \int_0^e \mu_e(y)\, dy\, dz = w\left(C_1 \frac{y^2}{2} + C_2 y\right).$$

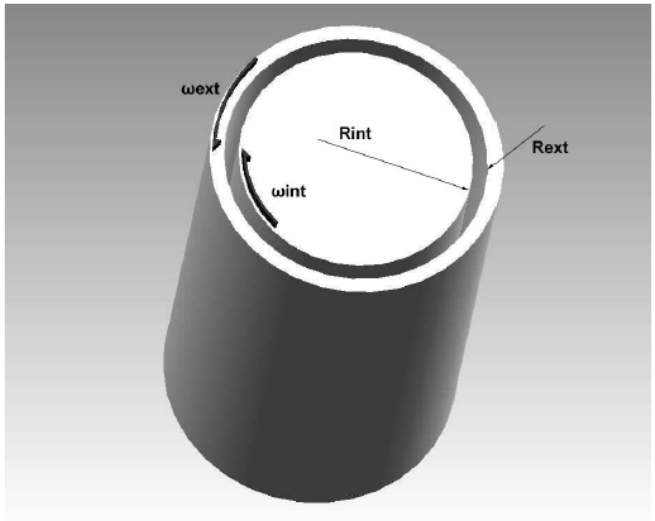
$$q_e = w\left(\frac{\mu_w U e^2}{2(\mu_w e + \mu_e b)} + \frac{U \mu_e b e}{\mu_w e + \mu_e b}\right).$$

$$q_w = w \int_{-b}^{0} u_w(y)\, dy = w\left(\tilde{C}_2 b - \tilde{C}_1 \frac{b^2}{2}\right).$$

$$q_w = w\left(\frac{U \mu_e b^2}{\mu_w e + \mu_e b} - \frac{\mu_w U b^2}{2(\mu_w e + \mu_e b)}\right)$$

35.1 Enunciado

Sean dos cilindros concéntricos de longitud unitaria, con radios R_{ext} y R_{int}, respectivamente, separados por una película de aceite de viscosidad μ. El cilindro exterior gira a una velocidad angular ω_{int} (sentido horario), mientras que el exterior gira a una velocidad angular ω_{ext} (sentido antihorario).



Halle las ecuaciones que definen:

1. La distribución de velocidades entre cilindros.
2. La distribución de presiones entre cilindros.
3. El par necesario en el cilindro exterior para que se produzca el giro.

35.2 Resolución

Cálculos previos

- Las condiciones de contorno que definen este problema són:

$$r = R_{ext} \Rightarrow V_\theta = \omega_{ext}\, R_{ext}$$

$$r = R_{int} \Rightarrow V_\theta = \omega_{int} R_{int} (-1)$$

- La ecuación de continuidad, en coordenadas cilíndricas, establece:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \times \frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r} \times \frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

- La ecuación de Navier-Stokes, en cilíndricas, se enuncia:

$$\rho \left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_\theta \frac{1}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} \right) =$$
$$\rho\, g_r - \frac{\partial P}{\partial r} + \mu \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, V_r) \right) + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{\partial^2 V_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} \right]$$

$$\rho \left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + V_\theta \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} - \frac{V_\theta\, V_r}{r} \right) =$$
$$\rho\, g_\theta - \frac{\partial P}{r\partial \theta} + \mu \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, V_\theta) \right) + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} \right]$$

$$\rho \left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_\theta \frac{1}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} \right) =$$
$$\rho\ g_z - \frac{\partial P}{\partial z} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\ \frac{\partial V_z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2} \right]$$

Únicamente existe variación de velocidad V_θ en dirección radial, con lo que se tiene:

- La ecuación de continuidad:

$$\frac{1}{r}\ \frac{\partial}{\partial \theta}(\rho V_\theta) = 0 \Rightarrow \frac{\partial}{\partial \theta}(\rho V_\theta) = 0 \Rightarrow \rho \frac{\partial V_\theta}{\partial \theta} = 0 \Rightarrow \frac{\partial V_\theta}{\partial \theta} = 0$$
$$\rho = \text{constante}$$

- La ecuación de Navier-Stokes:

  La presión reducida variará únicamente en la dirección radial

$$-\rho\ \frac{V_\theta^2}{r} = \rho\ g_r - \frac{\partial P}{\partial r} = -\rho\ g\, \frac{\partial h}{\partial r} - \frac{\partial P}{\partial r} = -\frac{\partial P^*}{\partial r}$$

$$0 = \mu\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\ \ V_\theta) \right) \Rightarrow \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\ \ V_\theta) \right) = 0$$

Se considera que no existen fuerzas másicas en la dirección z.

$$0 = \rho\ g_z - \frac{\partial P}{\partial z} \Rightarrow \frac{\partial P^*}{\partial z} = 0$$

No hay gradiente de presión reducida en la dirección z.

1. Así, se tiene que:

De la primera ecuación de Navier-Stokes:

$$P^* = \int \rho \frac{V_\theta^2}{r} dr$$

Será necesario conocer la distribución de velocidades en la dirección θ, ya que esta dependerá de r.

De la segunda ecuación de Navier-Stokes:

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\ V_\theta)\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\ V_\theta) = C_1$$

Las constantes C1 y C2 son constantes de integración

$$\frac{\partial}{\partial r}(r\ V_\theta) = r\ C_1$$

$$r\ V_\theta = C_1\ \frac{r^2}{2} + C_2$$

$$V_\theta = C_1\ \frac{r}{2} + \frac{C_2}{r}$$

Con las condiciones de contorno:

$$r = R_{ext} \Rightarrow V_\theta = \omega_{ext}\ R_{ext}$$

$$r = R_{int} \Rightarrow V_\theta = \omega_{int}\ R_{int}\ (-1)$$

$$(1) \qquad \omega_{ext}\ R_{ext} = C_1\ \frac{R_{ext}}{2} + \frac{C_2}{R_{ext}}$$

$$(2) \qquad -\omega_{int}\ R_{int} = C_1\ \frac{R_{int}}{2} + \frac{C_2}{R_{int}}$$

$$C_2 = \omega_{ext}\ R_{ext}^2 - C_1\ \frac{R_{ext}^2}{2} \Rightarrow -\omega_{int}\ R_{int} = C_1\ \frac{R_{int}}{2} + \frac{1}{R_{int}}\left[\omega_{ext}\ R_{ext}^2 - C_1\ \frac{R_{ext}^2}{2}\right]$$

$$-\omega_{ext}\ R_{ext}^2 - \omega_{int}\ R_{int}^2 = C_1\ \left(\frac{R_{int}^2}{2} - \frac{R_{ext}^2}{2}\right) \Rightarrow C_1 = \frac{2\left(\omega_{int}\ R_{int}^2 + \omega_{ext}\ R_{ext}^2\right)}{R_{ext}^2 - R_{int}^2}$$

$$C_2 = \omega_{ext}\ R_{ext}^2 - \frac{2\ \left(\omega_{int}\ R_{int}^2 + \omega_{ext}\ R_{ext}^2\right)}{R_{ext}^2 - R_{int}^2}\ \frac{R_{ext}^2}{2}$$

$$C_2 = \omega_{ext}\ R_{ext}^2 - R_{ext}^2\ \frac{\left(\omega_{int}\ R_{int}^2 + \omega_{ext}\ R_{ext}^2\right)}{R_{ext}^2 - R_{int}^2}$$

Entonces:

$$V_\theta = \frac{2\ \left(\omega_{int}\ R_{int}^2 + \omega_{ext}\ R_{ext}^2\right)}{R_{ext}^2 - R_{int}^2}\ \frac{r}{2} + \frac{1}{r}\left(\omega_{ext}\ R_{ext}^2 - R_{ext}^2\ \frac{\left(\omega_{int}\ R_{int}^2 + \omega_{ext}\ R_{ext}^2\right)}{R_{ext}^2 - R_{int}^2}\right)$$

2. De la primera ecuación de Navier-Stokes:

$$P^* = \int \rho \frac{V_\theta^{\,2}}{r} dr$$

Introduciendo la ecuación de V_θ en la integral, se tiene:

$$P^* = \int_{R_{int}}^{R_{ext}} \rho \; \frac{1}{r} \left(C_1^{\,2} \frac{r^2}{4} + \frac{C_2^{\,2}}{r^2} + C_1 \; C_2 \right) dr$$

y se halla:

$$P^* = \rho \left(C_1^{\,2} \frac{R_{ext}^{\;2} - R_{int}^{\;2}}{8} + \frac{C_2^{\,2}}{2} \; \frac{1}{R_{int}^{\;2} - R_{ext}^{\;2}} + C_1 \; C_2 \; Ln \frac{R_{ext}}{R_{int}} \right)$$

3. Los esfuerzos cortantes en cilíndricas se pueden dar:

$$\tau_{r\theta} = r \;\; \mu \;\; \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\Bigg|_{r=R_{ext}}$$

puesto que

$$V_\theta = C_1 \; \frac{r}{2} + \frac{C_2}{r}$$

$$\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) = \; - \frac{C_2}{r}$$

$$\tau_{r\theta} = \; - \mu \; C_2$$

$$M_{R_{ext}} = \tau_{r\theta} \;\; 2 \;\; \pi \;\; R_{ext} \;\; R_{ext}$$

así, queda:

$$M_{R_{ext}} = -2 \;\; \pi \;\; \mu \;\; C_2 \; R_{ext}^2$$