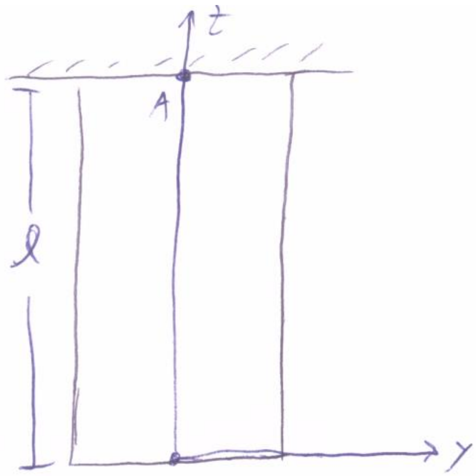


• Ec. de Movimiento en 2D (Plano y-z)

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho b_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \rho b_z = 0$$

* No hay esf. de corte $\sim$ $\tau_{yz} = 0$
* Sólo hay esf. en z $\sim$ $\sigma_y = 0$
* Sólo hay fzas. de cuerpo en z $\Rightarrow$ $b_y = 0$

$$\boxed{\therefore \frac{\partial \sigma_z}{\partial z} + \rho b_z = 0} \qquad \rho b_z = \left[\frac{N}{m^3}\right]$$

$$\therefore b_z = \left[\frac{m}{s^2}\right] = -\vec{g}$$

$$\Rightarrow \frac{\partial \sigma_z}{\partial z} = \rho \cdot g \Rightarrow \boxed{\sigma_z(z) = \rho \cdot g\, z + C}$$

* C.B: $\sigma_z(0) = 0 \Rightarrow C = 0 \Rightarrow \boxed{\sigma_z(z) = \rho \cdot g \cdot z}$ $(*)$

* Relaciones esf-deformación 2D

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \; ; \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \; ; \quad \varepsilon_{zy} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}$$

(i) (ii) (iii)

* Γ

* Ecuación constitutiva:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E} T_{ij} - \frac{\nu}{E} T_{kk} \delta_{ij}$$

$$\rightsquigarrow \varepsilon_{zz} = \frac{1+\nu}{E}\sigma_z - \frac{\nu}{E}\sigma_z \overset{(*)}{=} \frac{\sigma_z}{E} = \frac{\rho \cdot g}{E} z$$

* Reemplazando en i)

$$\hookrightarrow \frac{\partial U_z}{\partial z} = \frac{\rho \cdot g \cdot z}{E} \Rightarrow \boxed{U_z(y,z) = \frac{\rho \cdot g}{2E} z^2 + C_1(y)} \text{ (iv)}$$

$$\rightsquigarrow \varepsilon_{yy} = \frac{1+\nu}{E}\cancel{\sigma_y}^{\,0} - \frac{\nu}{E}\sigma_z \overset{(*)}{=} \frac{-\nu \rho \cdot g}{E} z$$

* Reemplazando ii)

$$\hookrightarrow \frac{\partial U_y}{\partial y} = -\frac{\nu \rho \cdot g}{E} z \Rightarrow \boxed{U_y(y,z) = -\frac{\nu \rho \cdot g}{E} y \cdot z + C_2(z)} \text{ (v)}$$

$$\rightsquigarrow \boxed{\varepsilon_{yz} = \frac{1+\nu}{E}\tau_{yz} = 0} \text{ (vi)}$$

* Derivando (iv) y (v) y (vi) en (iii)

$$\hookrightarrow \frac{\partial U_z}{\partial y} = C_1'(y) \quad ; \quad \hookrightarrow \frac{\partial U_y}{\partial z} = -\frac{\nu \rho \cdot g}{E} y + C_2'(z)$$

$$\Rightarrow \underbrace{C_1'(y) - \frac{\nu \rho \cdot g}{E} y}_{\text{Sólo depende de } y} + \underbrace{C_2'(z)}_{\text{Sólo depende de } z} = 0$$

$$\text{I)}\ C_1'(y) - \frac{\nu \rho \cdot g}{E} y = \alpha$$

$$\text{II)}\ C_2'(z) = \beta$$

$$\text{III)}\ \alpha + \beta = 0$$

* Resolvemos I)

$$C_1'(y) = \alpha + \frac{\nu \rho g}{E} y \Rightarrow \boxed{C_1(y) = \alpha y + \frac{\nu \rho g}{2E} y^2 + \gamma}$$

* Resolvemos II)

$$C_2'(z) = \beta \Rightarrow \boxed{C_2(z) = \beta z + \delta}$$

* Reemplazando estos resultados en (iv) y (v)

$$U_z(y,z) = \frac{\rho \cdot g}{2E} z^2 + \alpha y + \frac{\nu \rho g}{2E} y^2 + \gamma$$

$$U_y(y,z) = -\frac{\nu \rho g}{E} y \cdot z + \beta z + \delta$$

* Las condiciones de borde para el desplazamiento nos servirán para encontrar las constantes α, β, γ y δ:

i) $U_z(0,\ell) = 0 \Rightarrow \frac{\rho g}{2E}\ell^2 + \gamma = 0 \Rightarrow \boxed{\gamma = -\frac{\rho g}{2E}\ell^2}$

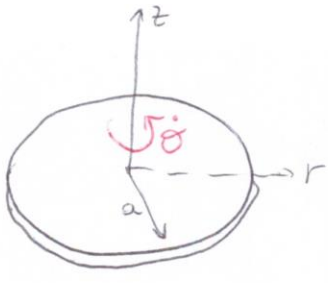
ii) $U_y(0,\ell) = 0 \Rightarrow \beta\ell + \delta = 0$

iii) $\frac{\partial U_y}{\partial z}(0,\ell) = 0 \Rightarrow \boxed{\beta = 0} \rightsquigarrow \boxed{\delta = 0}$

iv) Desde la ecuación III) $\rightsquigarrow \boxed{\alpha = -\beta = 0}$

* Finalmente, el campo de desplazamientos queda como:

$$\boxed{\begin{aligned} U_y(y,z) &= -\frac{\nu\rho\cdot g}{E}\, y\cdot z \\ U_z(y,z) &= \frac{\rho\cdot g}{2E} z^2 + \frac{\nu\rho\cdot g}{2E} y^2 - \frac{\rho\cdot g}{2E}\ell^2 \end{aligned}}$$



* Suposiciones: - Esfuerzo plano en $r-\theta \Rightarrow$ [illegible]

- Fza. de cuerpo sólo en $\hat{e}_r \Rightarrow b_\theta = b_z = 0$

$$\boxed{i)\ \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + \rho b_r = 0}$$

$$\boxed{ii)\ \frac{\partial T_{r\theta}}{\partial r} + 2\frac{T_{r\theta}}{r} = 0}$$

* Suposiciones: $U_r = U(r)\ ;\ U_\theta = 0$

$$\boxed{iii)\ \varepsilon_{rr} = \frac{\partial U(r)}{\partial r}\ ;\ iv)\ \varepsilon_{\theta\theta} = \frac{U(r)}{r}\ ;\ v)\ \varepsilon_{r\theta} = 0}$$

* Ec. Constitutivas: Esf. Plano

$$T_{rr} = \frac{E}{1-\nu^2}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}) \quad \text{(Aplicando iii) y iv))}$$

$$\boxed{vi)\ T_{rr} = \frac{E}{1-\nu^2}\left(\frac{\partial U}{\partial r} + \nu\frac{U}{r}\right)}$$

$$T_{\theta\theta} = \frac{E}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr})$$

$$\boxed{vii)\ T_{\theta\theta} = \frac{E}{1-\nu^2}\left(\frac{U}{r} + \nu\frac{\partial U}{\partial r}\right)} \wedge \boxed{viii)\ T_{r\theta} = 0}$$

* Sabiendo que $\rho \underset{\sim}{b_r} = \rho r \dot{\theta}^2 \underset{\sim}{\hat{e}_r}$, reemplazando vi) y vii) en (i):

$$\frac{\partial}{\partial r}\left(\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial r} + \nu \frac{u}{r}\right)\right) + \frac{1}{r}\left(\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial r} + \nu\frac{u}{r}\right) - \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu \frac{\partial u}{\partial r}\right)\right)$$

$$+ \rho r \dot{\theta}^2 = 0$$

* Reordenando

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2} u + \frac{\rho \dot{\theta}^2 (1-\nu^2)}{E} r = 0 \quad / \cdot r^2$$

$$\boxed{r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} - u + \frac{\rho \dot{\theta}^2 (1-\nu^2)}{E} r^3 = 0}$$

Ecuación de Cauchy-Euler no homogénea (2º orden)

Soln/
$$u(r) = \alpha_1 r + \frac{\alpha_2}{r} - \frac{\rho \dot{\theta}^2 (1-\nu^2)}{8E} r^3$$

* Condiciones de borde: $\cdot\, r = 0 \rightsquigarrow u < \infty \Rightarrow \boxed{\alpha_2 = 0}$

$\cdot$ El esfuerzo T_{rr} en $a = 0$ (Traction-Free)

↳ En vi):

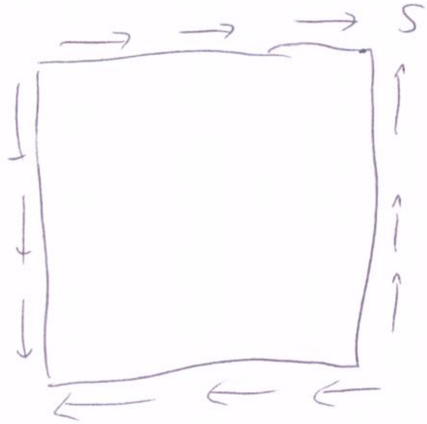
$$\rightsquigarrow \frac{\partial u}{\partial r} = \alpha_1 - \frac{3}{8}\frac{\rho \dot{\theta}^2 (1-\nu^2)}{E} r^2$$

$$\Rightarrow T_{rr}(a) = \frac{E}{1-\nu^2}\left(\alpha_1 - \frac{3}{8}\frac{\rho\dot{\theta}^2(1-\nu^2)}{E} a^2 + \frac{\nu \alpha_1 \cancel{a}}{\cancel{a}} - \frac{\nu \rho \dot{\theta}^2 (1-\nu^2)}{8E\cancel{a}} a^{\cancel{3}}\right) = 0$$

$$\alpha_1 (1+\nu) = \frac{\rho \dot{\theta}^2 (1-\nu^2)}{8E}(3a^2 + a^2)$$

$$\boxed{\therefore \alpha_1 = \frac{\rho \dot{\theta}^2 (1-\nu)}{2E} a^2}$$

$$\boxed{\therefore u(r) = \frac{\rho \dot{\theta}^2 (1-\nu) a^2}{2E} r - \frac{\rho \dot{\theta}^2 (1-\nu^2)}{8E} r^3}$$



$$\boxed{\phi(x,y) = Cxy}$$

$* \ \sigma_x = \dfrac{\partial^2 \phi}{\partial y^2} = 0$

$* \ \sigma_y = \dfrac{\partial^2 \phi}{\partial x^2} = 0$

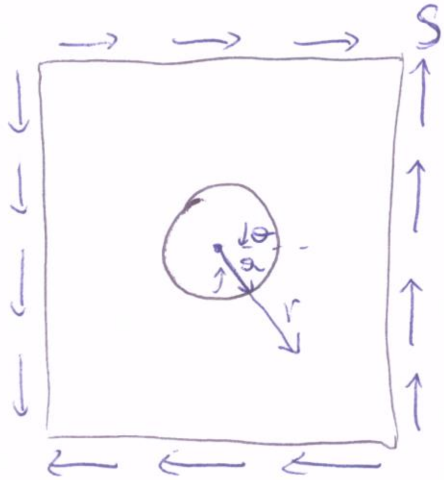
$* \ \tau_{xy} = \dfrac{-\partial^2 \phi}{\partial x \partial y} = -C$

Δ Cond. de borde $\tau_{xy} = S$ cuando $x, y \to \infty \Rightarrow \boxed{C = -S}$

$$\boxed{\begin{array}{c} \therefore \ \phi(x,y) = -Sxy \\ \sigma_x = \sigma_y = 0 \ ; \ \tau_{xy} = S \end{array}}$$

$*$ En coordenadas polares $\rightsquigarrow \phi(r,\theta) = -Sr^2 \operatorname{sen}\theta \cdot \cos\theta$

$$\boxed{\phi(r,\theta) = \frac{-Sr^2 \operatorname{sen}(2\theta)}{2}}$$



$$\phi_2(r,\theta) = -\frac{Sr^2 \operatorname{sen}(2\theta)}{2} + A \operatorname{sen}(2\theta) + \frac{B \operatorname{sen}(2\theta)}{r^2}$$

$*\ \sigma_r = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$

$= -S \cdot \operatorname{sen}(2\theta) - 2B \cdot \operatorname{sen}(2\theta) + S \cdot \operatorname{sen}(2\theta)$

$- \dfrac{6B \cdot \operatorname{sen}(2\theta)}{r^4} - \dfrac{4A \cdot \operatorname{sen}(2\theta)}{r^2}$

$$\boxed{\therefore \sigma_r = \left(S - \frac{4A}{r^2} - \frac{6B}{r^4}\right) \cdot \operatorname{sen}(2\theta)}$$

$$* \ \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = -S \cdot sen(2\theta) + \frac{6B\, sen(2\theta)}{r^4}$$

$$\boxed{\therefore \sigma_\theta = \left(-S + \frac{6B}{r^4}\right) \cdot sen(2\theta)}$$

$$* \ \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\left(-Sr^2\cos(2\theta) + 2A\cos(2\theta) + \frac{2B\cdot\cos(2\theta)}{r^2}\right)\right)$$

$$= -\frac{\partial}{\partial r}\left(-Sr\cos(2\theta) + \frac{2A\cdot\cos(2\theta)}{r} + \frac{2B\cdot\cos(2\theta)}{r^3}\right)$$

$$= S\cdot\cos(2\theta) + \frac{2A\cdot\cos(2\theta)}{r^2} + \frac{6B\cdot\cos(2\theta)}{r^4}$$

$$\boxed{\therefore \tau_{r\theta} = \left(S + \frac{2A}{r^2} + \frac{6B}{r^4}\right)\cos(2\theta)}$$

* Condiciones de Borde: En la superficie del agujero no hay esfuerzos aplicados.

i) $\sigma_r(r=a) = 0 \Rightarrow \left(S - \frac{4A}{a^2} - \frac{6B}{a^4}\right)\cdot sen(2\theta) = 0$

$$\Rightarrow \boxed{\frac{4A}{a^2} + \frac{6B}{a^4} = S} \ (*)$$

ii) $\tau_{r\theta}(r=a) = 0 \Rightarrow \boxed{\frac{2A}{a^2} + \frac{6B}{a^4} = -S} \ (**)$

* Resolviendo para A y B:

$$\boxed{A = S \cdot a^2}$$

$$\boxed{B = -\frac{S \cdot a^2}{2}}$$

* Por lo tanto, el estado de esfuerzos en la placa es:

$$\boxed{\begin{aligned}
\sigma_r &= S\left(1 - 4\left(\frac{a}{r}\right)^2 + 3\left(\frac{a}{r}\right)^4\right) \cdot \operatorname{sen}(2\theta) \\
\sigma_\theta &= -S\left(1 + 3\left(\frac{a}{r}\right)^4\right) \cdot \operatorname{sen}(2\theta) \\
\tau_{r\theta} &= S\left(1 + 2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4\right) \cdot \cos(2\theta)
\end{aligned}}$$