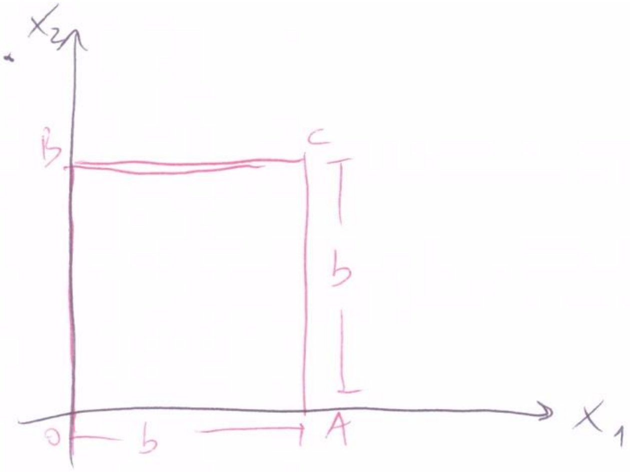


$$T_{11} = C x_2$$
$$T_{22} = C x_1$$
$$t_{12} = C$$

a) $T_{11} = \dfrac{\partial^2 \phi}{\partial x_2^2}$ (i) ; $T_{22} = \dfrac{\partial^2 \phi}{\partial x_1^2}$ (ii) ; $T_{12} = -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2}$ (iii)

- Integrando (i), (ii) y (iii)

i) $\dfrac{\partial \phi}{\partial x_2} = \dfrac{C x_2^2}{2} + \alpha_1(x_1) \rightsquigarrow \boxed{\phi(x_1, x_2) = \dfrac{C x_2^3}{6} + \alpha_1(x_1) \cdot x_2 + \alpha_2(x_1)}$

ii) $\dfrac{\partial \phi}{\partial x_1} = \dfrac{C x_1^2}{2} + \alpha_3(x_2) \rightsquigarrow \boxed{\phi(x_1, x_2) = \dfrac{C x_1^3}{6} + \alpha_3(x_2) x_1 + \alpha_4(x_2)}$

iii) $\dfrac{\partial \phi}{\partial x_1} = -C x_2 + \alpha_5(x_1) \rightsquigarrow \boxed{\phi(x_1, x_2) = -C x_2 x_1 + \int \alpha_5(x_1) dx_1 + \alpha_6(x_2)}$

- Igualando (i) con (iii)

$$\frac{C x_2^3}{6} + x_2 \cdot \alpha_1(x_1) + \alpha_2(x_1) = -C x_2 x_1 + \int \alpha_5(x_1) dx_1 + \alpha_6(x_2)$$

1-) $\boxed{\dfrac{C x_2^3}{6} = \alpha_6(x_2)}$

2-) $x_2 \cdot \alpha_1(x_1) = -C x_2 x_1 \Rightarrow \boxed{\alpha_1(x_1) = -C x_1}$

· Igualando (ii) con (iii)

$$\frac{Cx_1^3}{6} + \alpha_3(x_2)x_1 + \alpha_4(x_2) = -Cx_2x_1 + \int \alpha_5(x_1)dx_1 + \alpha_6(x_2)$$

3-) $\int \alpha_5(x_1)dx_1 = \frac{Cx_1^3}{6} \rightsquigarrow \boxed{\alpha_5(x_1) = \frac{Cx_1^2}{2}}$

4-) $\alpha_4(x_2) = \alpha_6(x_2) = \frac{Cx_2^3}{6}$

5-) $\alpha_3(x_2)x_1 = -Cx_2x_1 \rightarrow \boxed{\alpha_3(x_2) = -Cx_2}$

* Con α_3 y α_4 podemos expresar $\varnothing$ desde (ii)

$$\boxed{\varnothing(x_1, x_2) = \frac{Cx_1^3}{6} - Cx_2x_1 + \frac{Cx_2^3}{6}}$$

b) $\varepsilon_{11} = \frac{(1+\nu)}{E}T_{11} - \frac{\nu}{E}(T_{11} + T_{22})$

$$= \frac{(1+\nu)}{E}Cx_2 - \frac{\nu}{E}(Cx_2 + Cx_1)$$

$$\boxed{\varepsilon_{11} = \frac{C}{E}(x_2 - \nu x_1)}$$

$$\varepsilon_{22} = \frac{(1+\nu)}{E}T_{22} - \frac{\nu}{E}(T_{11} + T_{22})$$

$$= \frac{(1+\nu)}{E}Cx_1 - \frac{\nu}{E}(Cx_2 + Cx_1)$$

$$\boxed{\varepsilon_{22} = \frac{C}{E}(x_1 - \nu x_2)}$$

$$\boxed{\varepsilon_{12} = \frac{1+\nu}{E} t_{12} = \frac{1+\nu}{E} C}$$

· Para los desplazamientos:

$$\cdot\ \varepsilon_{11} = \frac{\partial U}{\partial x_1} = \frac{C x_2}{E} - \frac{\nu C}{E} x_1 \overset{\int}{\Longrightarrow} U(x_1, x_2) = \frac{C}{E} x_2 x_1 - \frac{\nu C x_1^2}{2E} + \alpha_1(x_2)$$

$$\cdot\ \varepsilon_{22} = \frac{\partial V}{\partial x_2} = \frac{C x_1}{E} - \frac{\nu C}{E} x_2 \overset{\int}{\Longrightarrow} V(x_1, x_2) = \frac{C}{E} x_1 x_2 - \frac{\nu C x_2^2}{2E} + \alpha_2(x_1)$$

$$\cdot\ \varepsilon_{12} = \frac{\partial U}{\partial x_2} + \frac{\partial V}{\partial x_1} = \frac{1+\nu}{E} C$$

$$\hookrightarrow \frac{\partial V}{\partial x_1} = \frac{C}{E} x_2 + \alpha_2'(x_1) \wedge \frac{\partial U}{\partial x_2} = \frac{C}{E} x_1 + \alpha_1'(x_2)$$

$$\Rightarrow \frac{C}{E} x_2 + \alpha_2'(x_1) + \frac{C}{E} x_1 + \alpha_1'(x_2) = \frac{1+\nu}{E} C$$

$$\underbrace{\left(\alpha_1'(x_2) + \frac{C}{E} x_2\right)}_{\beta} + \underbrace{\left(\alpha_2'(x_1) + \frac{C}{E} x_1\right)}_{\gamma} = \frac{1+\nu}{E} C$$

$$\Rightarrow \text{I)}\ \alpha_1'(x_2) + \frac{C}{E} x_2 = \beta$$

$$\text{II)}\ \alpha_2'(x_1) + \frac{C}{E} x_1 = \gamma$$

$$\text{III)}\ \beta + \gamma = \frac{1+\nu}{E} C$$

• De (I) $\alpha_1(x_2) = \beta x_2 - \dfrac{c x_2^2}{2E} + \delta$

• De (II) $\alpha_2(x_1) = \gamma x_1 - \dfrac{c x_1^2}{2E} + \xi$

* Reemplazando en las definiciones de los desplazamientos

$$\boxed{\begin{aligned} U(x_1,x_2) &= -\frac{\nu c}{2E}x_1^2 + \frac{c}{E}x_1x_2 - \frac{c}{2E}x_2^2 + \beta x_2 + \delta \\ V(x_1,x_2) &= -\frac{\nu c}{2E}x_2^2 + \frac{c}{E}x_1x_2 - \frac{c}{2E}x_1^2 + \gamma x_1 + \xi \end{aligned}}$$

* Nos quedan las constantes β, γ, δ y ξ, las cuales se pueden despejar con condiciones de borde adecuadas sobre U, V, sus derivadas y la ecuación III)

$\bullet \quad \nabla^2 \nabla^2 \phi = 0 \quad$ con $\quad \nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)$

$$\Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \right) = 0$$

$*$ Con $\phi(r,\theta) = C r \theta \operatorname{sen}\theta$:

$\hookrightarrow \frac{\partial^2 \phi}{\partial r^2} = 0 \; ; \; \frac{\partial \phi}{\partial r} = C\theta \cdot \operatorname{sen}\theta \; ; \; \frac{\partial^2 \phi}{\partial \theta^2} = Cr(2\cos\theta - \theta \operatorname{sen}\theta)$

$$\therefore \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \right) = \left(\cancel{\frac{C\theta \cdot \operatorname{sen}\theta}{r}} + \frac{2C\cos\theta}{r} - \cancel{\frac{C\theta \operatorname{sen}\theta}{r}} \right)$$

$$= \frac{2C\cos\theta}{r}$$

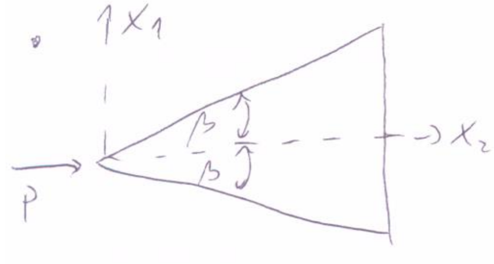
$$\rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left(\frac{2C\cos\theta}{r} \right)$$

$\hookrightarrow \frac{\partial^2}{\partial r^2}\left(\frac{2C\cos\theta}{r} \right) = \frac{4C\cos\theta}{r^3}$

$\hookrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{2C \cdot \cos\theta}{r} \right) = \frac{-2C\cos\theta}{r^3}$

$\hookrightarrow \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\left(\frac{2C\cos\theta}{r} \right) = \frac{-2C\cos\theta}{r^3}$

$$\boxed{\therefore \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left(\frac{2C\cos\theta}{r} \right) = 0}$$



$$\boxed{\phi(r,\theta) = C r \theta \cdot \text{sen}(\theta)}$$

$$* \quad \sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\frac{\partial \phi}{\partial r} = C\theta \cdot \text{sen}\,\theta$$

$$\frac{\partial \phi}{\partial \theta} = Cr(\text{sen}\,\theta + \theta\cos\theta)$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = Cr(2\cos\theta - \theta\,\text{sen}\,\theta)$$

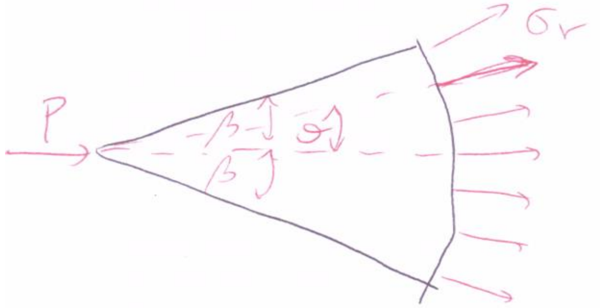
$$\Rightarrow \sigma_r = \frac{C\theta \cdot \text{sen}\,\theta}{r} + \frac{2C\cos\theta}{r} - \frac{C\theta \cdot \text{sen}\,\theta}{r} \Rightarrow \boxed{\sigma_r = \frac{2C\cos\theta}{r}}$$

$$* \quad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = 0$$

$$* \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{\cancel{r}} C\cancel{r}(\text{sen}\,\theta + \theta\cos\theta)\right) = 0$$

$$\boxed{\therefore \sigma_r = \frac{2C\cos\theta}{r} \; ; \; \sigma_\theta = \tau_{r\theta} = 0}$$

• Para determinar la constante C, hacemos un corte a un radio r:



$$P = -\int_{-\beta}^{\beta} \sigma_r \cdot \cos\theta \, r \, d\theta$$

$$= -2C \int_{-\beta}^{\beta} \cos^2\theta \, d\theta$$

$$\Rightarrow P = -2C\left(\theta + \text{sen}(\theta)\cdot\cos(\theta)\right)\Big|_{-\beta}^{\beta}$$

$$= -2C\left(2\beta + \text{sen}(\beta)\cdot\cos(\beta) - \text{sen}(-\beta)\cdot\cos(-\beta)\right)\cdot\frac{1}{2}$$

$$= -C\left(2\beta + \text{sen}(\beta)\cdot\cos(\beta) + \text{sen}(\beta)\cdot\cos(\beta)\right)$$

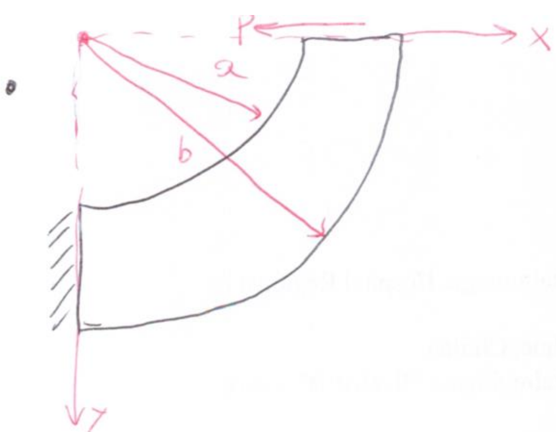
$$= -C\left(2\beta + 2\,\text{sen}(\beta)\cos(\beta)\right) = -C\left(2\beta + \text{sen}(2\beta)\right)$$

$$\therefore\quad C = -\frac{P}{2\beta + \text{sen}(2\beta)}$$

$$\boxed{\sigma_r = \frac{-2P\cos\theta}{r\left(2\beta + \text{sen}(2\beta)\right)}}$$

- Para simular una placa semiinfinita, $\beta = \frac{\pi}{2}$

$$\boxed{\therefore\ \sigma_r = \frac{-2P\cos\theta}{\pi r}}$$



$$\phi(r,\theta) = f(r)\,\mathrm{Sen}(\theta)$$

* Para que $\phi(r,\theta)$ satisfaga las ecs. de equilibrio, ésta debe cumplir con la ec. biharmónica (coordenadas Polares)

$$\nabla^2\nabla^2\phi = 0 \quad \text{con} \quad \nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right)\mathrm{Sen}(\theta) = 0$$

$$\Rightarrow \left.\nabla^2\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right)\mathrm{Sen}(\theta)\right\}(1)$$

$$+ \underbrace{\nabla^2(\mathrm{Sen}(\theta))\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right)}_{(2)} = 0$$

$$(1): \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right)\mathrm{Sen}(\theta)$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right)\mathrm{Sen}(\theta)$$

$$(2): \left(\cancelto{0}{\frac{\partial^2}{\partial r^2}} + \cancelto{0}{\frac{1}{r}\frac{\partial}{\partial r}} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\mathrm{Sen}(\theta))\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right)$$

$$= -\frac{1}{r^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right)\mathrm{Sen}(\theta)$$

$$\therefore \nabla^2\nabla^2\phi = \left[\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{1}{r^2}f\right) - \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{1}{r^2}f\right)\right]\cancel{sen(\theta)} = 0 \quad \forall\theta$$

$$\boxed{\therefore \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{1}{r^2}f\right) = 0}$$

b) $f(r) = Ar^3 + B\cdot\frac{1}{r} + Cr + Dr\ln(r)$

$$\therefore \phi(r,\theta) = \left(Ar^3 + B\cdot\frac{1}{r} + Cr + Dr\ln(r)\right)Sen(\theta)$$

* Reemplazando en la definición de esfuerzos (Coordenadas polares)

$$\rightarrow \sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \left(3Ar - \frac{B}{r^3} + \frac{C}{r} + D\left(\frac{\ln(r)}{r}+\frac{1}{r}\right)\right)Sen(\theta) - \left(Ar + \frac{B}{r^3} + \frac{C}{r} + \frac{D\ln(r)}{r}\right)sen(\theta)$$

$$\boxed{\therefore \sigma_r = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)sen(\theta)}$$

$$\rightarrow \sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \Longrightarrow \boxed{\sigma_\theta = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)Sen(\theta)}$$

$$\rightarrow \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -\frac{\partial}{\partial r}\left(Ar^2 + \frac{B}{r^2} + C + D\ln(r)\right)\cos(\theta)$$

$$\boxed{\therefore \tau_{r\theta} = \left(-2Ar + \frac{2B}{r^3} - \frac{D}{r}\right)\cos(\theta)}$$

* Los constantes se calculan a partir de las condiciones de borde del problema.

i) $\sigma_r(a,\theta) = 0 \Rightarrow \left(2Aa - \frac{2B}{a^3} + \frac{D}{a}\right) \cancel{sen(\theta)} = 0$ $\forall\theta$

$$\boxed{\therefore \left(2Aa - \frac{2B}{a^3} + \frac{D}{a}\right) = 0} \; (i)$$

ii) $\sigma_r(b,\theta) = 0$

$$\boxed{\therefore 2Ab - \frac{2B}{b^3} + \frac{D}{b} = 0} \; (ii)$$

iii) $\left.\int_a^b \tau_{r\theta}\, dr\right|_{\theta=0} = P \quad \rightsquigarrow -2A\int_a^b r\,dr - 2B\int_a^b + D\int_a^b \frac{1}{r}dr = \left.\frac{P}{\cos\theta}\right|_{\theta=0}$

$$\boxed{\therefore -A(b^2-a^2) + B\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + D\ln\left(\frac{b}{a}\right) = P} \; (iii)$$

* Despejando las constantes A, B y D de (i), (ii) y (iii) y definiendo $\gamma = (a^2-b^2) + (a^2+b^2)\ln(b/a)$, se tiene:

$$\boxed{\begin{aligned} \sigma_r &= \left(\frac{P}{\gamma}r + \frac{Pa^2b^2}{\gamma}\cdot\frac{1}{r^3} - \frac{P}{\gamma}(a^2+b^2)\cdot\frac{1}{r}\right)sen(\theta) \\ \sigma_\theta &= \left(\frac{3P}{\gamma}r - \frac{Pa^2b^2}{\gamma}\cdot\frac{1}{r^3} - \frac{P}{\gamma}(a^2+b^2)\frac{1}{r}\right)sen(\theta) \\ \tau_{r\theta} &= \left(-\frac{P}{\gamma}r - \frac{Pa^2b^2}{\gamma}\cdot\frac{1}{r^3} + \frac{P}{\gamma}(a^2+b^2)\cdot\frac{1}{r}\right)\cos(\theta) \end{aligned}}$$

* En el extremo superior ($\theta = 0$)

$$\boxed{\sigma_r = \sigma_\theta = 0 \;,\; \tau_{r\theta} = -\frac{P}{\gamma}\left(r + \frac{a^2b^2}{r^3} - \frac{1}{r}(a^2+b^2)\right)}$$

* En el extremo inferior ($\theta = \frac{\pi}{2}$)

$\tau_{r\theta} = 0$	$\sigma_r = \frac{P}{\gamma}\left(r + \frac{a^2b^2}{r^3} - \frac{1}{r}(a^2+b^2)\right)$
	$\sigma_\theta = \frac{P}{\gamma}\left(3r - \frac{a^2b^2}{r^3} - \frac{1}{r}(a^2+b^2)\right)$