- $[\sigma_{ij}] = \begin{pmatrix} 0 & 0 & -cx_2 \\ 0 & 0 & cx_1 \\ -cx_2 & cx_1 & 0 \end{pmatrix}$ en $[\underset{\sim}{x}] = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

a) En $\underset{\sim}{x}$:

$$[\sigma_{ij}] = \begin{pmatrix} 0 & 0 & -2c \\ 0 & 0 & c \\ -2c & c & 0 \end{pmatrix}$$

⟶ Calculando los invariantes

1-) $I_1 = tr(\underset{\sim}{\sigma}) = 0$

2-) $\underset{\sim}{\sigma}^2 = \underset{\sim}{\sigma} \cdot \underset{\sim}{\sigma} = \begin{pmatrix} 4c^2 & -2c^2 & 0 \\ -2c^2 & c^2 & 0 \\ 0 & 0 & 5c^2 \end{pmatrix} \wedge tr(\underset{\sim}{\sigma}^2) = 10c^2$

$$\Rightarrow I_2 = \frac{1}{2}\left((tr\,\underset{\sim}{\sigma})^2 - tr(\sigma)^2\right) = -5c^2$$

3-) $I_3 = det(\underset{\sim}{\sigma}) = 0$

⟶ La ec. característica queda como:

$$-\lambda^3 + \cancel{\lambda^2 I_1} - \lambda I_2 + \cancel{I_3} = 0$$

$$-\lambda^3 - \lambda(-5c^2) = 0$$

$$\boxed{\lambda(\lambda^2 - 5c^2) = 0}$$

$$\boxed{\therefore \lambda_1 = 0 \ ; \quad \lambda_2 = c\sqrt{5} \ ; \ \lambda_3 = -c\sqrt{5}}$$ //

b) La ecuación característica viene de $(\underset{\sim}{I} - \lambda_i \underset{\sim}{\mathbb{I}})\underset{\sim}{n}_i = 0$. Luego:

- $\underline{\lambda_1 = 0}$ $\begin{pmatrix} 0 & 0 & -2c \\ 0 & 0 & c \\ -2c & c & 0 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-2cn_3 = 0$	(1)
$cn_3 = 0$	(2)
$-2cn_1 + cn_2 = 0$	(3)

* De (1) y (2) $\rightsquigarrow n_3 = 0$. De (3) $\longrightarrow n_2 = 2n_1$

* Imponemos que los vectores sean unitarios:

$$n_1^2 + n_2^2 + \cancel{n_3^2}^{\,0} = 1 \rightsquigarrow n_1^2 + (2n_1)^2 = 1$$

$$\boxed{\therefore n_1 = \frac{1}{\sqrt{5}} \;;\; n_2 = \frac{2}{\sqrt{5}} \;;\; n_3 = 0} \Rightarrow \boxed{\overset{1}{\underset{\sim}{n}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}}$$

• $\lambda_2 = C\sqrt{5}$

$$\begin{pmatrix} -C\sqrt{5} & 0 & -2C \\ 0 & -C\sqrt{5} & C \\ -2C & C & -C\sqrt{5} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-C\sqrt{5}\, n_1 - 2C n_3 = 0 \quad (1)$$
$$-C\sqrt{5}\, n_2 + C n_3 = 0 \quad (2)$$
$$-2C n_1 + C n_2 - C\sqrt{5}\, n_3 = 0 \quad (3)$$

* De (1) $\rightsquigarrow n_3 = -\frac{\sqrt{5}}{2} n_1$

* De (2) $\rightsquigarrow -C\sqrt{5}\, n_2 + C\left(-\frac{\sqrt{5}}{2}\right) n_1 = 0 \rightsquigarrow n_2 = -\frac{1}{2} n_1$

* La ec. (3) es L.D. de las otras. Imponiendo $\hat{n}$ unitario:

$$n_1^2 + n_2^2 + n_3^2 = 1$$
$$n_1^2 - \left(-\frac{1}{2} n_1\right)^2 + \left(-\frac{\sqrt{5}}{2} n_1\right)^2 = 1$$

$$\Rightarrow \boxed{n_1 = \sqrt{\frac{2}{5}} \;;\; n_2 = -\frac{1}{\sqrt{10}} \;;\; n_3 = -\frac{\sqrt{2}}{2}} \Rightarrow \overset{2}{\underset{\sim}{n}} = \begin{pmatrix} \sqrt{2/3} \\ -1/\sqrt{10} \\ -1/\sqrt{2} \end{pmatrix}$$

• $\lambda_3 = -C\sqrt{5}$

$$\begin{pmatrix} C\sqrt{5} & 0 & -2C \\ 0 & C\sqrt{5} & C \\ -2C & C & C\sqrt{5} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{array}{ll} C\sqrt{5}\, n_1 - 2C n_3 = 0 & (1) \\ C\sqrt{5}\, n_2 + C n_3 = 0 & (2) \\ -2C n_1 + C n_2 + C\sqrt{5}\, n_3 = 0 & (3) \end{array}}$$

* (3) es L.D.
* De (1) $\rightsquigarrow n_3 = \frac{\sqrt{5}}{2} n_1$
* De (2) $\rightsquigarrow n_2 = -\frac{1}{2} n_1$
* De $\hat{n}$ unitario $\rightsquigarrow n_1^2 + n_2^2 + n_3^2 = 1$

$$\boxed{n_1 = \sqrt{\tfrac{2}{5}} \;;\; n_2 = -\sqrt{\tfrac{1}{10}} \;;\; n_3 = \sqrt{\tfrac{1}{2}}} \Rightarrow \boxed{\underset{\sim}{\hat{n}}^3 = \begin{pmatrix} \sqrt{2/5} \\ -\sqrt{1/10} \\ \sqrt{1/2} \end{pmatrix}}$$

c) Esfuerzo de corte máximo

$$\tau_{max} = \frac{\lambda_{max} - \lambda_{min}}{2} \Big/ \begin{array}{l} \lambda_{max} = C\sqrt{5} \\ \lambda_{min} = -C\sqrt{5} \end{array} \Rightarrow \boxed{\tau_{max} = C\sqrt{5}}$$

- $$\underset{\sim}{T} = \begin{pmatrix} 100 & 75 & 80 \\ 75 & 56 & 150 \\ 80 & 150 & 120 \end{pmatrix} [MPa]$$

$$\rightarrow \underset{\sim}{T}^2 = \begin{pmatrix} 22025 & 23700 & 28850 \\ 23700 & 31261 & 32400 \\ 28850 & 32400 & 43300 \end{pmatrix}$$

$$I_1 = tr(\underset{\sim}{T}) = 276 [MPa]; \quad I_2 = \frac{1}{2}\left((tr\,\underset{\sim}{T})^2 - tr(\underset{\sim}{T}^2)\right) = -10205 \; [MPa^2]$$

$$I_3 = det(\underset{\sim}{T}) = -811400 \; [MPa^3]$$

$$\Rightarrow \; -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

$$-\lambda^3 + 276\lambda^2 + 10205\lambda - 811400 = 0$$

$$\boxed{\therefore \lambda_1 = 300,951 \; ; \; \lambda_2 = -65,877 \; ; \; \lambda_3 = 40,927} \; [MPa]$$

$$\boxed{\tau_{max} = \frac{\lambda_{max} - \lambda_{min}}{2} = \frac{300,951 + 65,877}{2} = 183,41} \; [MPa]$$

· Tensor de deformación $\rightarrow$ Ley de Hooke

$$\varepsilon_{ij} = \frac{(1+\nu)}{E} T_{ij} - \frac{\nu}{E} T_{kk}\,\delta_{ij}$$

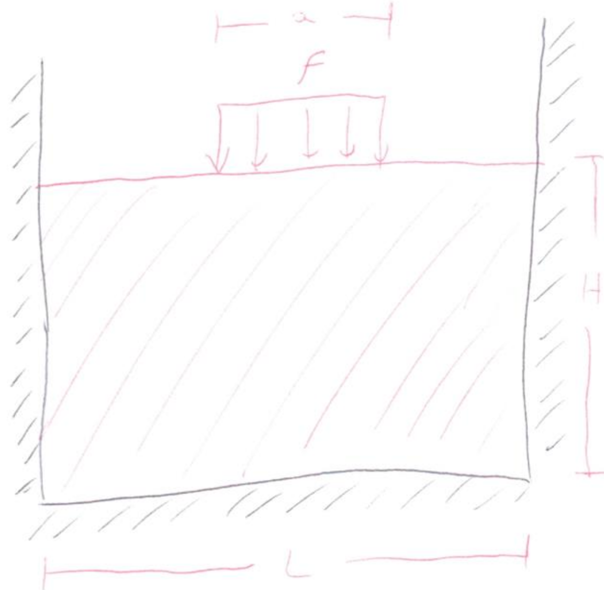
$$\hookrightarrow \varepsilon_{ii} = \frac{(1+\nu)}{E} T_{ii} - \frac{\nu}{E} tr(\underset{\sim}{T}) = 5,722 \cdot 10^{-12}\, T_{ii} - 3,286 \cdot 10^{-4}$$

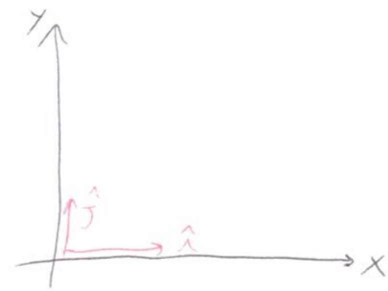
$$\boxed{\begin{aligned} \varepsilon_{11} &= 2,636 \cdot 10^{-4} \\ \varepsilon_{22} &= 3,032 \cdot 10^{-6} \\ \varepsilon_{33} &= 3,8204 \cdot 10^{-4} \end{aligned}}$$

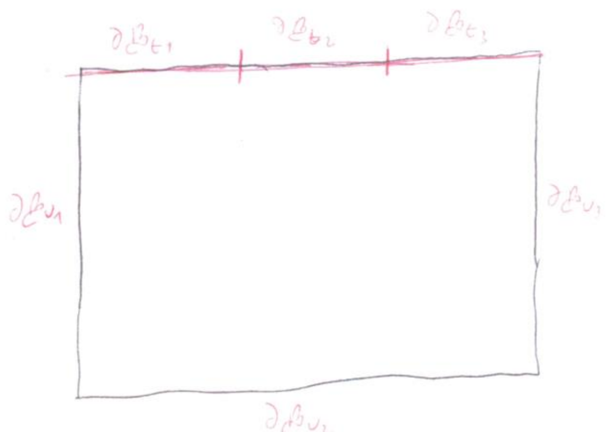
$$\hookrightarrow \varepsilon_{ij} = \frac{(1+\nu)}{E} T_{ij} = 5,922 \cdot 10^{-12}\, T_{ij}$$

$$\boxed{\begin{aligned} \varepsilon_{12} = \varepsilon_{21} &= 4,44 \cdot 10^{-4} \\ \varepsilon_{13} = \varepsilon_{31} &= 4,738 \cdot 10^{-4} \\ \varepsilon_{23} = \varepsilon_{32} &= 8,883 \cdot 10^{-4} \end{aligned}}$$

$$\boxed{\therefore \underset{\sim}{\varepsilon} = \begin{pmatrix} 2,636 & 4,44 & 4,738 \\ 4,44 & 0,0303 & 8,883 \\ 4,738 & 8,883 & 3,82 \end{pmatrix} \cdot 10^{-}}$$







$$\partial B = \partial B_u \cup \partial B_t$$

- $\partial B_u = \partial B_{u1} \cup \partial B_{u2} \cup \partial B_{u3}$
- $\partial B_t = \partial B_{t1} \cup \partial B_{t2} \cup \partial B_{t3}$

i) $\partial B_{u1}: \ \hat{n} = -\hat{\imath} \ ; \ \hat{u} = 0$

$\partial B_{u2}: \ \hat{n} = -\hat{j} \ ; \ \hat{u} = 0$

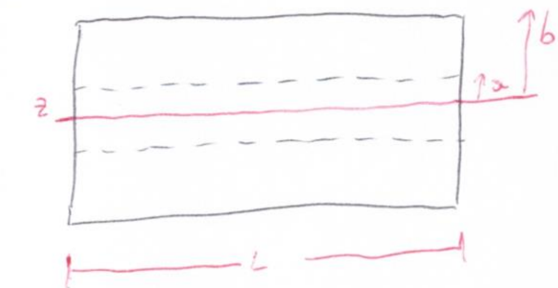
$\partial B_{u3}: \ \hat{n} = \hat{\imath} \ ; \ \hat{u} = 0$

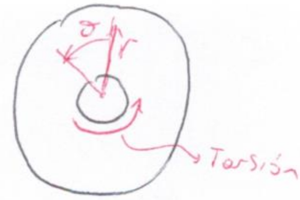
ii) $\partial B_{t1}: \ \hat{n} = \hat{j} \ ; \ \hat{t} = 0$

$\partial B_{t2}: \ \hat{n} = \hat{j} \ ; \ \hat{t} = -F\hat{j}$

$\partial B_{t3}: \ \hat{n} = \hat{j} \ ; \ \hat{t} = 0$

- Problema de def. plana: Bloque largo en la dir z en comparación al resto de las dimensiones ($M >> H$; $M >> L$), por lo que la deformación en z se puede despreciar, pero existe esfuerzo en z





1-) Ecs. de equilibrio (Coordenadas cilíndricas)

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + \rho b_r = 0$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2\,T_{r\theta}}{r} + \rho b_\theta = 0$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r} + \rho b_z = 0$$

* Suposiciones: — No hay esfuerzos en $z \Rightarrow T_{zi} = T_{iz} = 0$

- Simetría Axial, sólo hay dependencia en $r \Rightarrow \frac{\partial T_{ij}}{\partial \theta} = \frac{\partial T_{ij}}{\partial z} = 0$

- Fzas. de cuerpo nulas $\Rightarrow \rho b_i = 0$

$$\Rightarrow \boxed{\begin{array}{l} \text{i) } \dfrac{\partial T_{rr}}{\partial r} + \dfrac{1}{r}(T_{rr} - T_{\theta\theta}) = 0 \\ \hline \text{ii) } \dfrac{\partial T_{r\theta}}{\partial r} + \dfrac{2}{r} T_{r\theta} = 0 \end{array}}$$

2-) Relaciones entre deformación y desplazamiento

$$\varepsilon_{rr} = \frac{\partial U_r}{\partial r}\;;\quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\;;\quad \varepsilon_{zz} = \frac{\partial U_z}{\partial z}$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z}\right);\quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial U_\theta}{\partial z} + \frac{1}{r}\frac{\partial U_z}{\partial \theta}\right)$$

* Suposiciones: - Tubo no se alarga $\Rightarrow U_z = 0$

- Simetría axial por torsión homogénea $\Rightarrow U_\theta = U_\theta(r)$
- Simetría axial sobre la deformación radial $\Rightarrow U_r = U_r(r)$

$$\Rightarrow \boxed{iii)\, \varepsilon_{rr} = \frac{\partial U_r}{\partial r} \;;\; iv)\, \varepsilon_{\theta\theta} = \frac{U_r}{r} \;;\; v)\, \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)}$$

3-) Ecuaciones Constitutivas

$$T_{ij} = 2\mu\, \varepsilon_{ij} + \lambda\, \varepsilon_{kk}\, \delta_{ij}$$

$$\rightsquigarrow T_{rr} = 2\mu\, \varepsilon_{rr} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta}) \quad (\text{Usando iii) y iv)})$$

$$\boxed{vi)\; T_{rr} = 2\mu \frac{\partial U_r}{\partial r} + \lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)}$$

$$\rightsquigarrow T_{\theta\theta} = 2\mu\, \varepsilon_{\theta\theta} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta})$$

$$\boxed{vii)\; T_{\theta\theta} = 2\mu \cdot \frac{U_r}{r} + \lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)}$$

$$\rightsquigarrow T_{r\theta} = 2\mu\, \varepsilon_{r\theta}\,,$$

$$\boxed{viii)\; T_{r\theta} = \mu\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)}$$

• Ahora tenemos todo lo necesario para resolver el campo de desplazamientos:

$$\rightarrow \text{De (ii)}\; \frac{\partial T_{r\theta}}{\partial r} + \frac{2}{r} T_{r\theta} = 0 \quad \Rightarrow \boxed{T_{r\theta} = \frac{C_0}{r^2}}$$

$\rightarrow$ Reemplazando viii) en la ec. anterior: $\mu\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right) = \frac{C_0}{r^2}$

$$\frac{\partial U_\theta}{\partial r} - \frac{1}{r} U_\theta = \frac{C_0}{\mu} \cdot \frac{1}{r^2}$$

·Soln] $\boxed{U_\theta(r) = C_1 r - \frac{C_0}{2\mu}\cdot\frac{1}{r}}$

· Condiciones de Borde:

a) $U_\theta(b) = 0$ (Empotramiento) $\Rightarrow C_1 b - \frac{C_0}{2\mu b} = 0 \rightsquigarrow C_1 = \frac{C_0}{2\mu b^2}$

b) $U_\theta(a) = U_0$ (Despl. conocido) $\Rightarrow U_0 = C_1 a - \frac{C_0}{2\mu a} = \frac{C_0 a}{2\mu b^2} - \frac{C_0}{2\mu a}$

$$\therefore C_0 = \frac{2\mu U_0 a b^2}{a^2 - b^2} \wedge C_1 = \frac{U_0 a}{a^2 - b^2}$$

$$\boxed{\therefore U_\theta(r) = \frac{U_0 a}{a^2 - b^2} r - \frac{U_0 b^2}{a^2 - b^2}\cdot\frac{1}{r}}$$

→ De (i) $\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0$

→ Reemplazando vi) y vii) en (i)

$$\frac{\partial}{\partial r}\left(2\mu \frac{\partial U_r}{\partial r} + \lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)\right) + \frac{1}{r}\left[2\mu \frac{\partial U_r}{\partial r} + \cancel{\lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)} - 2\mu\frac{U_r}{r} - \cancel{\lambda\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r}\right)}\right] = 0$$

→ Reordenando, llegamos a la siguiente EDO:

$$\frac{d^2 U_r}{dr^2} + \frac{1}{r}\frac{dU_r}{dr} - \frac{U_r}{r^2} = 0$$

·Soln] $U_r(r) = \frac{C_3}{r} + C_4 r$

·Condiciones de Borde:

$$\left.\begin{aligned} &a)\ U_r(b) = 0 \Rightarrow \frac{C_3}{b} + C_4 b = 0 \\ &b)\ U_r(a) = 0 \Rightarrow \frac{C_3}{a} + C_4 a = 0 \end{aligned}\right\} C_3 = C_4 = 0 \Rightarrow \boxed{U_r(r) = 0}$$