CONTROL 3 MA2002 Otoño-2013

P1. (a) Sea $f(z)=\dfrac{z^2-3z+5}{(z-2)\left(z^2+1\right)^2}$

(1) Determine las regiones de convergencia de las series de Laurent de f centradas en: (i) $z_0=0$, (ii) $z_0=i$.

(2) Obtenga la serie de Laurent que converge en $C=\{z\colon 1<|z|<2\}$.

(b) Sea $f(z)=\dfrac{z(1-e^{z-1})}{\left(z^2-1\right)^2 sen^2(z)}$, para los z tales que $|z-1|<2.5$

(1) Clasifique las singularidades de f para los z tales que $|z-1|<2.5$

(2) Calcule los residuos en los polos de f para los z tales que $|z-1.5|<2$

P2. (a) Utilizando residuos, calcule la integral: $\displaystyle\int_{-\infty}^{\infty}\frac{sen^2(x)}{x^2(1+x^2)}\,dx$

(b) Utilizando residuos, calcule la integral: $\displaystyle\int_0^{2\pi}\frac{1}{a+cos^2(\theta)}\,d\theta$

P3. Considere el siguiente problema:

(P) : $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}+bsen(x)\quad 0<x<L,\ 0<t$

$$u(0,t)=u(L,t)=0,\ 0<t;\quad u(x,0)=\frac{\partial u}{\partial t}(x,0)=0\quad 0<x<L$$

(a) Compruebe que el MSV no permite resolver (P). Justifique.

(b) Suponga u(x,y) = U(x,y) + f(x). Aplique este supuesto a la EDP y a cada condición de (P) y determine f de modo que el problema resultante en U (con dicha función f) tenga la siguiente formulación:

(P_U): $\dfrac{\partial^2 U}{\partial t^2}=a^2\dfrac{\partial^2 U}{\partial x^2},\quad 0<x<L,\ 0<t$

$$U(0,t)=U(L,t)=0,\ 0<t;\ U(x,0)=g(x),\ \frac{\partial u}{\partial t}(x,0)=h(x),\ 0<x<1$$

(c) Resuelva (P_U) mediante el MSV y obtenga la solución de (P) a partir de la solución del problema (P_U).

Puntaje:

P1 (a) (1) 0.6 (2) 2.4 (b) (1) 2.0 (2) 1.0

P2 (a) 3.0 (b) 3.0

P3 (a) 1.0 (b) (2) (c) 3.0

Pauta P1 CONTROL 3 MA2002 Otoño-2013

P1. (a) $f(z)=\dfrac{z^2-2z+5}{(z-2)(z^2+1)^2}$ $\begin{cases}(1)\ \text{Det. regiones de convergencia de S.L. de } f \text{ entorno a } (i)\ z_0=0,\ (ii)\ z_0=i\\ (2)\ \text{Obtenga la S.L. de } f \text{ convergente en } 1<|z|<2\end{cases}$

(b) $f(z)=\dfrac{z(1-e^{z-1})}{(z^2-1)^2\operatorname{sen}^2(z)}$ $\begin{cases}(1)\ \text{Clasifique singularidades de } f \text{ para } z \text{ t.q. } |z-1|<2.5\\ (2)\ \text{Calcule residuos de los polos de } f \text{ para } z \text{ t.q. } |z-1.5|<2\end{cases}$

SOLUCION

(a) f tiene singularidades (no-reparables) en $z_1=2$, $z_2=i$, $z_3=-i$

(1) (i) Caso $z_0=0$: anillos de convergencia son: $A(0,0,1), A(0,1,2), A(0,2,+\infty)$ (0.3)

(ii) Caso $z_0=i$: anillos de convergencia son: $A(i,0,\sqrt{2}), A(i,\sqrt{2},\sqrt{5}), A(i,\sqrt{5},+\infty)$ (0.3)

(2) S.L. de f convergente en $1<|z|<2$

(Descomposicion de f) $\dfrac{z^2-2z+5}{(z-2)(z^2+1)^2}=\dfrac{A}{z-2}+\dfrac{B}{z-i}+\dfrac{C}{(z-i)^2}+\dfrac{D}{z+i}+\dfrac{E}{(z+i)^2}$ (0.3)

$\left(A=1,\ B=\frac{1}{10}(-1+7i),\ C=1/2,\ D=\frac{1}{2}(1+3i),\ E=1/2\right)$

(0.7) • $\dfrac{1}{z-2}=-\dfrac{1}{2}\cdot\dfrac{1}{1-z/2}=-\dfrac{1}{2}\sum_{n\geq 0}\left(\dfrac{z}{2}\right)^n=\sum_{n\geq 0}-\left(\dfrac{1}{2}\right)^{n+1}z^n$, para $\left|\frac{z}{2}\right|<1$ ssi $|z|<2$

(0.7) • $\dfrac{1}{z-i}=\dfrac{1}{z}\cdot\dfrac{1}{1-i/z}=\dfrac{1}{z}\sum_{n\geq 0}\left(\dfrac{i}{z}\right)^n=\sum_{n\geq 0}i^n z^{-n-1}$, para $\left|\frac{i}{z}\right|<1$ ssi $|z|>1 \Rightarrow \dfrac{1}{(z-i)^2}=-\dfrac{d}{dz}\left(\dfrac{1}{z-i}\right)=\sum_{n\geq 1}(n+1)i^n z^{-n-2}$

(0.7) • $\dfrac{1}{z+i}=\dfrac{1}{z}\cdot\dfrac{1}{1+i/z}=\dfrac{1}{z}\sum_{n\geq 0}(-1)^n\left(\dfrac{i}{z}\right)^n=\sum_{n\geq 0}(-i)^n z^{-n-1}$, para $\left|\frac{i}{z}\right|<1$ ssi $|z|>1 \Rightarrow \dfrac{1}{(z+i)^2}=-\dfrac{d}{dz}\left(\dfrac{1}{z+i}\right)=\sum_{n\geq 1}(n+1)(-i)^n z^{-n-2}$

S.L. de f en $1<|z|<2$: $\sum_{n\geq 0}-A\left(\frac{1}{2}\right)^{n+1}z^n+\sum_{n\geq 0}\left(Bi^n+D(-i)^n\right)z^{-n-1}+\sum_{n\geq 1}(n+1)\left(Ci^n+E(-i)^n\right)z^{-n-2}$

(b) Singularidades de f: $z_1=0$, $z_2=1$, $z_3=-1$, $w_k=k\pi$, $k=\pm1,\pm2,\ldots$, las singularidades que cumplen $|z-1|<2.5$ son: $z_1, z_2, z_3, w_1=\pi$

(0.5) • Caso $z_1=0$: $f(z)=\dfrac{z}{\operatorname{sen}^2(z)}g(z)$; $g(z)=\dfrac{1-e^{z-1}}{(z^2-1)^2}$ holomorfa en z_1 y $g(z_1)\neq 0$. Como $f(z)=\dfrac{z}{\operatorname{sen}^2 z}\cdot\dfrac{g(z)}{z}\Rightarrow$ polo simple

(0.5) • Caso $z_2=1$: $f(z)=\dfrac{1-e^{z-1}}{(z-1)^2}g(z)$; $g(z)=\dfrac{z}{(z+1)^2\operatorname{sen}^2 z}$ holomorfa en z_2 y $g(z_2)\neq 0$.

Como $\lim_{z\to 1}\dfrac{1-e^{z-1}}{(z-1)^2}=(LH)=\lim_{z\to 1}\dfrac{-e^{z-1}}{2(z-1)}=\dfrac{-1}{0}$ y $\lim_{z\to 1}(z-1)\dfrac{1-e^{z-1}}{(z-1)^2}=\lim_{z\to 1}\dfrac{1-e^{z-1}}{z-1}\overset{LH}{=}\lim_{z\to 1}-e^{z-1}=-1\Rightarrow z_2$ es polo simple

(0.5) • Caso $z_3=-1$: $f(z)=\dfrac{g(z)}{(z+1)^2}$, $g(z)=\dfrac{z(1-e^{z-1})}{(z-1)^2\operatorname{sen}^2(z)}$ holomorfa en z_3 y $g(-1)\neq 0\Rightarrow z_3$ polo de orden 2

(0.5) • Caso $w_1=\pi$: $f(z)=\dfrac{g(z)}{\operatorname{sen}^2 z}$, $g(z)=\dfrac{z(1-e^{z-1})}{(z^2-1)}$ holomorfa en w_1 y $g(\pi)\neq 0$

Como $\lim_{z\to\pi}\dfrac{z-\pi}{\operatorname{sen}^2 z}\overset{LH}{=}\lim_{z\to\pi}\dfrac{1}{2\operatorname{sen} z\cos z}=\dfrac{1}{0}$ y $\lim_{z\to\pi}\dfrac{(z-\pi)^2}{\operatorname{sen}^2 z}\overset{LH}{=}\lim_{z\to\pi}\dfrac{2(z-\pi)}{2\operatorname{sen} z\cos z}\overset{LH}{=}\lim_{z\to\pi}\dfrac{2}{2\cos(2z)}=1\Rightarrow w_1$ es polo de orden 2

Conclusion: $z_1=0$ y $z_2=1$ son polos de orden 1 y $z_3=-1$, $w_1=\pi$ son polos de orden 2

Calculo de residuos para los polos z t.q. $|z-1.5|<2\Rightarrow z_1=0$, $z_2=1$ y $w_1=\pi$

(0.3) • $z_1=0\rightarrow \operatorname{Res}(f,0)=\lim_{z\to 0}z\cdot f(z)=\lim_{z\to 0}\dfrac{z^2(1-e^{z-1})}{(z^2-1)^2\operatorname{sen}^2 z}=\lim_{z\to 0}\left(\dfrac{z}{\operatorname{sen} z}\right)^2\left(\dfrac{1-e^{z-1}}{(z^2-1)^2}\right)=1-e^{-1}$

(0.3) • $z_2=1\rightarrow \operatorname{Res}(f,1)=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}\dfrac{z(1-e^{z-1})}{(z-1)(z+1)^2\operatorname{sen}^2 z}\overset{LH}{=}\lim_{z\to 1}\dfrac{(1-e^{z-1})-ze^{z-1}}{(z+1)^2\operatorname{sen}^2 z+(z-1)[(z+1)^2\operatorname{sen}^2 z]'}=-\dfrac{1}{4\operatorname{sen}^2(1)}$

(0.4) • $w_1=\pi\rightarrow \operatorname{sen}(z)=-\operatorname{sen}(\pi-z)\Rightarrow \operatorname{sen}(\pi-z)=(z-\pi)+\frac{1}{3!}(z-\pi)^3+\cdots\Rightarrow \operatorname{sen}^2(z-\pi)=(z-\pi)^2-\frac{2}{3!}(z-\pi)^4+\cdots$

$\Rightarrow \dfrac{(z-\pi)^2}{\operatorname{sen}^2(z)}=\dfrac{1}{h(z)}$, con $h(z)=1-\frac{2}{3!}(z-\pi)^2+\cdots\Rightarrow h(\pi)=1$, $h'(\pi)=0$

$\Rightarrow (z-\pi)^2f(z)=\dfrac{z(1-e^{z-1})}{(z^2-1)^2}\cdot\dfrac{1}{h(z)}\Rightarrow \dfrac{d}{dz}\left((z-\pi)^2f(z)\right)=\left(\dfrac{z(1-e^{z-1})}{(z^2-1)^2}\right)'\dfrac{1}{h(z)}+\left(\dfrac{z(1-e^{z-1})}{(z^2-1)^2}\right)\left(\dfrac{1}{h(z)}\right)'$

$\Rightarrow \operatorname{Res}(f,\pi)=\left(\dfrac{z(1-e^{z-1})}{(z^2-1)^2}\right)'\Big/_{\pi}$, ya que $\left(\dfrac{1}{h(z)}\right)'\Big/_{\pi}=-\dfrac{h'(z)}{h^2(z)}\Big/_{\pi}=0$

$$\int_{-\infty}^{+\infty} \frac{\operatorname{sen}^2(x)}{x^2(1+x^2)}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1-\cos(2x)}{x^2(1+x^2)}\,dx \longrightarrow \int_\gamma \underbrace{\frac{1-e^{2iz}}{z^2(1+z^2)}}_{f(z)}\,dz$$

singularidades de $f(z)$ $\to z_1=0$, $\to z_2=i$, $\to z_3=-i$.

— Veamos si son reparables y calculemos los residuos

<u>Caso $z_1=0$</u> polo orden 1

$$\lim_{z\to 0} f(z) \overset{L'H}{=} \lim_{z\to 0} \frac{-2ie^{2iz}}{2z+4z^3} = \frac{-2i}{0} \longrightarrow \boxed{z_1=0 \text{ NO Reparable}}$$

0,2

Residuo $\longrightarrow$ $$\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{1-e^{2iz}}{z(1+z^2)} \overset{L'H}{=} \lim_{z\to 0} \boxed{\frac{-2ie^{2iz}}{1+3z^2} = -2i}$$

0,4+0,6

<u>Caso $z_2=i$</u> polo orden 1.

$$\lim_{z\to i} f(z) = \lim_{z\to i}\frac{1-e^{2iz}}{z^2(1+z^2)} = \frac{1-e^{-2}}{0} \longrightarrow \boxed{z_2=i \text{ NO Reparable}}$$

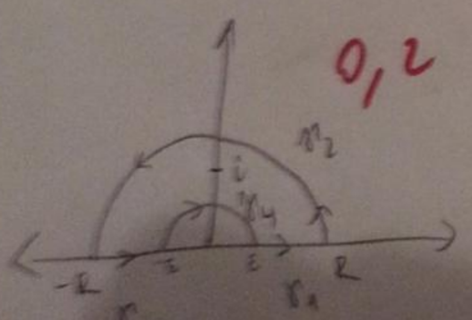
0,2

Residuo $\to$ $$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{\cancel{(z-i)}(1-e^{2iz})}{z^2\cancel{(z-i)}(z+i)} = \lim_{z\to i}\frac{1-e^{2iz}}{z^2(z+i)} = \boxed{\frac{1-e^{-2}}{-2i} = \frac{i}{2}(1-e^{-2})}$$

0,4+0,6

Para la integral se utiliza $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$

0,2



(encierra solo polos con $Im(z)>0$) no finito

La funcion $H(z) = \dfrac{1}{z^2(1+z^2)} \to \boxed{|H(z)| \approx \dfrac{1}{|z|^4}}$ 0,2

esto implica

$$\lim_{R\to\infty} \boxed{\int_{\gamma_2} \frac{1e^{2iz}}{z^2(1+z^2)}\,dz = 0}$$ 0,2

, por T. Cauchy y T. Residuos

(cuando $R\to\infty$, $\varepsilon\to 0$)

$$\int_{-\infty}^{\infty} \frac{sen^2(x)}{x^2(1+x^2)}dx \longrightarrow \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1-e^{2ix}}{x^2(1+x^2)} = \frac{1}{2}\left[\pi i \, Res(f,0) + 2\pi i \, Res(f,i)\right]$$

$$= \frac{1}{2}\left[\pi i(-2i) + 2\pi i\left(\frac{i}{2}(1-e^{-2})\right)\right]$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{sen^2(x)}{x^2(1+x^2)}dx = \pi - \frac{\pi}{2}(1-e^{-2}) = \pi - \frac{\pi}{2} + \frac{\pi}{2}e^{-2}$$

0,6 + 0,4

$$= \frac{\pi}{2} + \frac{\pi}{2}e^{-2} = \frac{\pi}{2}(1+e^{-2})$$

(b)

$$I = \int_0^{2\pi} \frac{1}{a+\cos^2\theta}d\theta = \int_0^{2\pi} \frac{1}{a+\frac{1}{2}(1+\cos 2\theta)}d\theta = \int_0^{2\pi} \frac{2}{2a+1\cos(2\theta)}d\theta$$

$z = e^{i\theta}$, $dz = e^{i\theta}i\,d\theta$

$$sen\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

cambio de variable $t = 2\theta \longrightarrow dt = 2d\theta$

$$I = \int_0^{4\pi} \frac{dt}{2a+1+\cos t} = 2\int_0^{2\pi} \frac{1}{2a+1+\cos t}dt$$

cos es par

$$= 2\int_0^{2\pi} \frac{\frac{dz}{iz}}{2a+1+\frac{z+z^{-1}}{2}} = -2i\int_0^{2\pi} \frac{2\,dz}{4az+2z+z^2+1} = -4i\int_0^{2\pi} \frac{dz}{z^2+(4a+2)z+1}$$

1.0

$\Rightarrow I = -4i(2\pi i \sum_k Res(f, z_k)$, z_k polos encerrados por $\gamma = |z| = 1$

$$= 8\pi \sum_k Res(f, z_k)$$

...amos las singularidades:

$$z^2+(4a+2)z+1=0 \implies z = \frac{-(4a+2)\pm\sqrt{(4a+2)^2-4}}{2} = -(2a+1)\pm\frac{\sqrt{16a^2+16a+4-4}}{2}$$

$$= -(2a+1)\pm 2\sqrt{a(a+1)}$$

0,4

$z_1 = -(2a+1)+2\sqrt{a(a+1)}$ y $z_2 = -(2a+1)-2\sqrt{a(a+1)}$

Veamos si z_1 y z_2 están dentro o fuera de $\gamma=|z|=1$

- $|z_2| = 2a+1+2\sqrt{a(a+1)} > 1$ pues $a>0$.

$$z_1\cdot z_2 = (2a+1)^2-(2\sqrt{a(a+1)})^2$$
$$= 4a^2+4a+1-4(a^2+a)$$
$$= \cancel{4a^2}+\cancel{4a}+1-\cancel{4a^2}-\cancel{4a}$$
$$= 1$$

$\implies$ si $|z_2|>1 \implies |z_1|<1$ 0,2

- luego solo z_2 esta dentro de $\gamma=|z|=1$

Calculemos $Res(f, z_1) = \lim_{z\to z_1} \frac{z-z_1}{z^2+(4a+2)z+1} = \lim_{z\to z_1} \frac{\cancel{(z-z_1)}}{\cancel{(z-z_1)}(z-z_2)}$

$$= \lim_{z\to z_1} \frac{1}{z-z_2} = \frac{1}{z_1-z_2} = \frac{1}{4\sqrt{a(a+1)}}$$

Por lo tanto $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+\cos^2\theta} = 8\pi\frac{1}{4\sqrt{a(a+1)}} = \frac{2\pi}{\sqrt{a(a+1)}}$

0,4

$f = \dfrac{1}{z^2(1+z^2)}$

0, i, $-i$ $\rightarrow$ NO REPARABLES 0,2 0,2

$\lim\limits_{z\to 2i} \dfrac{1}{z^2(1+z^2)}$

$\lim\limits_{z\to i} \dfrac{(z-i)}{z^2(1+z^2)} = \lim\limits_{z\to i} \dfrac{1}{2z+4z^3} = \dfrac{1}{2i-4i} = -\dfrac{1}{2i} = \dfrac{i}{2}$ 0,3 +0,2 residuo polo i

$\lim\limits_{z\to 0} \dfrac{z}{z^2(1+z^2)} = \lim\limits_{z\to 0} \dfrac{1}{2z+4z^3} = \dfrac{1}{0}$

$\lim\limits_{z\to 0} \dfrac{1}{(2-1)!} \dfrac{d^{2-1}}{dz^{2-1}} \left[(z-0)^2 \cdot \dfrac{1}{z^2(1+z^2)}\right]$

$\lim\limits_{z\to 0} \dfrac{d}{dz}\left(\dfrac{1}{1+z^2}\right) = \lim\limits_{z\to 0} \left(\dfrac{-2z}{(1+z^2)^2}\right) = 0$ 0,3 +0,2 residuo polo 0

$f(z) = \dfrac{e^{2iz}}{z^2(1+z^2)}$

$\lim\limits_{z\to i} \dfrac{(z-i)e^{2iz}}{z^2(1+z^2)} = \lim\limits_{z\to i} \dfrac{e^{2iz} + 2i(z-i)e^{2iz}}{2z+4z^3} = \dfrac{e^{-2}}{2i-4i} = \dfrac{e^{-2}}{-2i} = \dfrac{ie^{-2}}{2}$ +0,2 0,3 residuo polo i

$\lim\limits_{z\to 0} \dfrac{z e^{2iz}}{z^2(1+z^2)} = \lim\limits_{z\to 0} \dfrac{e^{2iz} + 2ize^{2iz}}{2z+4z^3} = \dfrac{1}{0}$

$\lim\limits_{z\to 0} \dfrac{1}{(2-1)!} \dfrac{d^{2-1}}{dz^{2-1}} \left[(z-0)^2 \dfrac{e^{2iz}}{z^2(1+z^2)}\right] = \lim\limits_{z\to 0} \dfrac{d}{dz}\left(\dfrac{e^{2iz}}{1+z^2}\right)$

$= \lim\limits_{z\to 0} \dfrac{2i(1+z^2)e^{2iz} - 2ze^{2iz}}{(1+z^2)^2} = 2i$ 0,3 +0,2 residuo polo 0

Cover Page

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2(1+x^2)}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1-\cos 2x}{x^2(1+x^2)} \longrightarrow \int_{\gamma} \frac{1-e^{2iz}}{z^2(1+z^2)}\,dz$$

$f_1(z) = \frac{1}{z^2(1+z^2)}$

$f_2(z) = \frac{e^{2iz}}{z^2(1+z^2)}$

$$\frac{1}{2}\left[2\pi i \operatorname{Res}(f_1, i) + \pi i \operatorname{Res}(f_1, 0) - 2\pi i \operatorname{Res}(f_2, i) - \pi i \operatorname{Res}(f_2, 0)\right]$$

$$= \frac{1}{2}\left[\cancel{2}\pi i \frac{i}{\cancel{2}} + \pi i \cdot 0 - \cancel{2}\pi i \frac{ie^{-2}}{\cancel{2}} - \pi i\, 2i\right] = \frac{1}{2}\left[-\pi + \pi e^{-2} + 2\pi\right]$$

$$= \frac{1}{2}\left[\pi + \pi e^{-2}\right] = \boxed{\frac{\pi}{2}(1+e^{-2})}$$

0,6 + 0,4

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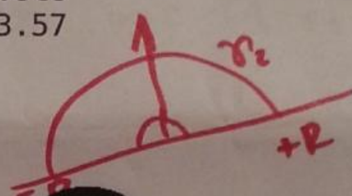

+0,2 por dibujo

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+0,2 por $H(z) = \frac{1}{z^2(z^2+1)} \rightarrow |H(z)| \approx \frac{1}{|z|^4}$

10 20 30 40 50 60

+0,2 por $\lim\limits_{R\to\infty} \int_{\gamma_2} \frac{1-e^{2iz}}{z^2(1+z^2)}\,dz = 0$

Cover Page

ESTA PAUTA ES PARA LOS QUE EN LA PARTE (a) SEPARARON LA INTEGRAL EN DOS Y TUVIERON QUE CALCULAR 4 POLOS.

EL PUNTAJE ES EL MISMO QUE SI NO HUBIERAN SEPARADO LA INTEGRAL.

SIN SEPRAR LA INTEGRAL, SON DOS POLOS CADA UNO VALE UN 1PTO.

AL SEPARAR LA INTEGRAL, SON DOS POLOS POR CADA INTEGRAL, CADA UNO DE LOS 4 POLOS VALEN 0,5 PTOS.

LOS DEMAS PUNTAJES SON LO MISMO.

Pauta P3 CONTROL 3 Ma2002 Otoño-2013

P1. (P) $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + bsen(x) \quad 0 < x < L,\ 0 < t$

$u(0,t) = u(L,t) = 0,\ 0 < t; \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$ 0<x<L

SOLUCION

(a) El MSV no permute resolver (P)

(1.0) $u(x,t) = X(x)T(t)$ en la EDP resulta: $XT'' = a^2X''T + bsen(x)$ la que no se puede separar.

(b) Reemplazar condición (C):$u(x,t) = U(x,t) + f(x)$ en (P) para obtener el problema

(P_U): $\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2},\ 0 < x < L,\ 0 < t$

$U(0,t) = U(L,t) = 0,\ 0 < t; \quad U(x,0) = g(x); \ \frac{\partial U}{\partial t}(x,0) = h(x)$ 0<x<L

(0.5) (C) en la EDP $\rightarrow \frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2} + a^2 f''(x) + bsen(x) \rightarrow f$ debe cumplir: $f''(x) = -\frac{b}{a^2} sen(x)$

La función f resulta ser $f(x) = \frac{b}{a^2} sen(x) + cx + d$

(0.8) (C) en $u(0,t) = u(L,t) = 0 \rightarrow U(0,t) + f(0) = U(L,t) + f(L) = 0$, y si $f(0) = f(L) = 0$,

la definición de f implica $d = 0$ y $c = -\frac{b}{a^2 L} sen(L) \rightarrow f(x) = \frac{b}{a^2}(sen(x) - \frac{1}{L} sen(L))$

(0.5) (C) en $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \rightarrow U(x,0) = -f(x)$ y $\frac{\partial U}{\partial t}(x,0) = 0$

El problema resultante es

(0.2) (P_U): $\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2},\ 0 < x < L,\ 0 < t$

$U(0,t) = U(L,t) = 0,\ 0 < t; \quad U(x,0) = -f(x); \ \frac{\partial U}{\partial t}(x,0) = 0, \quad 0 < x < L$

(c) Solucion de (P_U) mediante el MSV

(0.3) (1) $U(x,t) = X(x)T(t)$ en la EDP implica: $XT'' = a^2X''T \rightarrow X''/X = T''/a^2T = -\lambda \rightarrow$

$X'' + \lambda X = 0$ y $T'' + \lambda a^2 T = 0$.

(0.7) (2) (condiciones traspasables)

$U(0,t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$ (SI) ó $T(t) = 0$ (NO, porque implica $U = 0$)

$U(L,t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$ (SI) ó $T(t) = 0$ (NO, porque implica $U = 0$)

$U(x,0) = -f(x) \rightarrow X(x)T(0) = f(x)$ (NO, porque X, f dependen solo de x y T de t)

$\frac{\partial U}{\partial t}(x,0) = 0 \rightarrow X(x)T'(0) = 0 \rightarrow X(x) = 0$ (NO, implica $U = 0$) ó $T'(0) = 0$ (SI)

(P_X): $X'' + \lambda X = 0;\ X(0) = X(L) = 0$; (P_T): $T'' + \lambda a^2 T = 0,\ T'(0) = 0$

(3) Solucion de (P_X) y (P_T) para valores comunes de λ. Primero se resuelve (P_X)

(0.3) **Caso** $\lambda = 0$. $X'' = 0 \rightarrow X(x) = \alpha x + \beta$, y $X(0) = X(L) = 0 \rightarrow X = 0$, se descarta $\lambda = 0$

(0.5) **Caso** $\lambda < 0$, $\lambda = -k^2, k > 0$. $X'' - k^2 X = 0 \rightarrow X(x) = \alpha e^x + \beta e^{-x}$, $X(0) = 0\ \alpha + \beta = 0$

y $X(L) = 0 \rightarrow \alpha(e^L - e^{-L}) = 0 \rightarrow \alpha = 0 \rightarrow X = 0$, se descarta $\lambda < 0$.

(0.7) **Caso** $\lambda > 0$, $\lambda = k^2, k > 0$. $X'' + k^2 X = 0 \rightarrow X(x) = \alpha\cos(kx) + \beta sen(kx)$,

$X(0) = 0 \rightarrow \alpha = 0,\ X(L) = 0 \rightarrow sen(kl) = 0 \rightarrow k = \frac{n\pi}{L}, n \in \mathbb{N} \rightarrow X_n(x) = \alpha_n sen(\frac{n\pi}{L}x)$

Solucion de (P_T) para $\lambda = k^2$, $k = \frac{n\pi}{L}$. $T'' + k^2 a^2 T = 0 \rightarrow T(t) = \gamma\cos(kx) + \delta sen(kx)$,

y $T'(0) = 0 \rightarrow \delta = 0 \rightarrow T_n(t) = \gamma_n \cos(\frac{n\pi a}{L}t)$.

$U_n = X_n T_n = A_n\, sen(\frac{n\pi}{L}x)\cos(\frac{n\pi a}{L}t)$ es solucion de $(P_U)'=(P_U) - (\ U(x,0) = -f(x))$

(0.5) (4) (Principio de superposición): $U(x,t) = \sum_{n\ge1} U_n(x,t)$ es solucion de $(P_U)'$.

$U(x,0) = -f(x) \rightarrow -f(x) = \sum_{n\ge1} U_n(x,0) = \sum_{n\ge1} A_n\, sen(\frac{n\pi}{L}x)$ (s. Fourier en senos)

$A_n = \int_0^L f(x) sen(\frac{n\pi}{L}x)dx = \frac{2b}{a^2 L}\int_0^L (sen(x) - \frac{1}{L}sen(L))\, sen(\frac{n\pi}{L}x)dx \quad (\alpha = \frac{n\pi}{L})$

$= \frac{2b}{a^2 L}\int_0^L sen(x)\, sen(\alpha x)dx - \frac{2b}{a^2 L^2}\int_0^L sen(L)\, sen(\alpha x)dx = \frac{2b}{a^2 L}B_n - \frac{2b}{a^2 L^2}C_n$

$B_n = \frac{1}{2}\int_0^L (cos((1-\alpha)x) - cos((1+\alpha)x)dx = \cdot\cdot\cdot = \frac{1}{1-\alpha^2}(-1)^n sen(L)$

$C_n = -\frac{1}{1-\alpha^2}(\cos(\alpha L) - 1) = \frac{1}{1-\alpha^2}(1 - (-1)^n)$