MA2002-1: Cálculo Avanzado y Aplicaciones
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Auxiliar 9

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P1. (I) Encuentre el disco de convergencia de las series $\sum_{n=0}^{\infty} \frac{e^{in}}{2n+1}\left(\frac{2}{3i}\right)^n (z+4i)^n$, $\sum_{n=1}^{\infty} n!(z-i)^{n!}$.

(II) Encuentre la serie de potencias de $f(z) = \frac{1}{2}\ln 1+z^2$, y su radio de convergencia.

P2. (a) Pruebe que para $b \in (-1,1)$, se tiene $\int_0^{\infty} \frac{1-b^2-x^2}{(1-b^2+x^2)^2+4b^2x^2}dx = \frac{\pi}{2}$. **Indicación:** Integre $f(z) = \frac{1}{1+z^2}$ en un contorno rectangular adecuado.

(b) Si además $b \neq 0$, pruebe que

$$\int_0^{\infty} \frac{x}{(1-b^2-x^2)^2+4b^2x^2}dx = \frac{1}{4b}\ln\frac{1+b}{1-b}.$$

P3. Pruebe que $\int_{-\infty}^{\infty} \cos(x^2)dx = \int_{-\infty}^{\infty} \sin(x^2)dx = \sqrt{\frac{\pi}{2}}$. **Hint:** Puede serle útil integrar la función $f(x) = \exp iz^2$ sobre el contorno de un octavo de circunferencia. Si lo necesita, puede usar que $\forall \alpha \in [0,\pi/2]$, $\sin(\alpha) \geq 2\alpha/\pi$.

Radios de convergencia.

i) $\sum_{n=0}^{\infty} \frac{e^{in}}{2n+1}\left(\frac{2}{3i}\right)^n (z+4i)^n$.

Las series, de manera genérica: $\sum_{k\geq 0} a_k (z-z_0)^k$

Luego, $z_0 = -4i$

$$a_k = \frac{e^{ik}}{2k+1}\left(\frac{2}{3i}\right)^k$$

$$\rightarrow \limsup_{k\to\infty} |a_k|^{1/k} = \lim \left(\frac{|e^{ik}|}{2k+1}\left(\frac{|2|}{|3i|}\right)^k\right)^{1/k}$$

$$= \lim \frac{2}{3}\cdot\left(\frac{1}{2k+1}\right)^{1/k} = \frac{2}{3} \quad \therefore \boxed{R = \frac{3}{2}}$$

$\therefore$ serie converge en $D\left(-4i, \frac{3}{2}\right)$.

ii) $\sum_{n=1}^{\infty} n!(z-i)^{n!} = \sum_{k=1}^{\infty} a_k (z-z_0)^k$,

donde $a_k = \begin{cases} k, & \text{si } k = n!,\ z_0 = i \\ 0, & \text{si } \sim \end{cases}$

$$\rightarrow \limsup_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} |k|^{1/k} = 1$$

"limsup es el límite de la subsuc. que converge al valor + alto"

$f(z) = \frac{1}{2} \ln(1+z^2)$. Recordar que

$$\frac{1}{1+z} = \sum_{n\geq 0} z^n$$

① Derivada:

$$f'(z) = \frac{z}{1+z^2} = z \sum_{n\geq 0} (z^2)^n = \sum_{n \geq 0} z^{2n+1}$$

② su primitiva debería ser: $h(z) = \sum_{n\geq 0} \frac{z^{2n+2}}{2n+2}$.

Pero $h'(z) = \sum_{n\geq 0} z^{2n+1} \Rightarrow h'(z) = f'(z)$, $\forall z \in D(0,R)$

radio de convergencia

$\Rightarrow f(z) = h(z) + C$ / $z=0$

$$\underset{0}{\overset{\shortparallel}{f(0)}} = \underset{0}{\overset{\shortparallel}{h(0)}} + C \Rightarrow C = 0 \therefore f = h \text{ en } D(0,R)$$

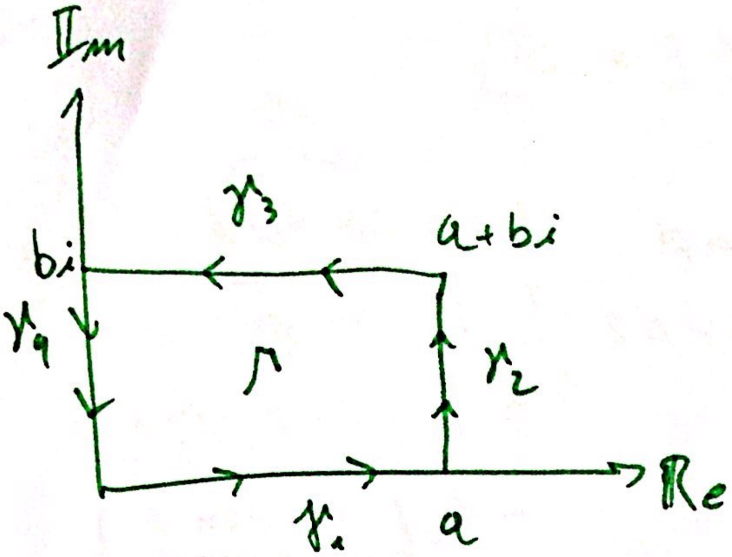
$$\therefore f(z) = \sum_{n\geq 0} \frac{z^{2n+2}}{2n+2} = \sum_{n \geq 0} a_k z^k \; ; \; k = 2n+2$$

$$a_k = \begin{cases} \frac{1}{2n+2}, & k = 2n+2 \\ 0, & k = 2n+1 \end{cases}$$

$\therefore \limsup |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{2n+2}\right)^{1/2n+2}$

$= 1$

$\Rightarrow R = 1$.



$$\Gamma = \bigcup_{i=1}^{4} \gamma_i$$

luego, $$\oint_\Gamma \frac{dz}{1+z^2} = \sum_{i=1}^{4} \int_{\gamma_i} \frac{dz}{1+z^2}$$

$$\| \quad 0, \; C-G$$

Para γ_1 $\quad \gamma_1(t) = t, \; t \in [0, a]$

$dz = dt$

$$\Rightarrow \int_{\gamma_1} \frac{dz}{1+z^2} = \int_0^a \frac{dt}{1+t^2} = \operatorname{arctg} t \Big|_0^a = \operatorname{arctg}(a)$$

Para γ_2 $\quad \gamma_2(t) = a + it, \; t \in [0, b]$

$dz = i\,dt$

$$\Rightarrow \int_{\gamma_2} \frac{dz}{1+z^2} = \int_0^b \frac{i\,dt}{1+a^2-t^2+2ait} \overset{\text{Racionalizar}}{=} \int_0^b \frac{(1+a^2-t^2-2ait)\,i\,dt}{(1+a^2-t^2)^2+4a^2t^2}$$

Para γ_3 $\quad \gamma_3(t) = +t + bi$, ~~$t \in [a, 0]$~~ $t \in [a, 0]$

$dz = +dt$

$$\Rightarrow \int \frac{dz}{\ldots} = \int_a^0 \frac{+dt}{\ldots} = \int_0^a -(1+t^2-b^2 \mp 2tbi)\,dt \ldots$$

Para γ_4 $\quad \gamma_4(t) = ti \, , \; t \in [b, 0]$

$dz = i\,dt$

$$\Rightarrow \int_{\gamma_4} \frac{dz}{1+z^2} = \int_b^0 \frac{i\,dt}{1-t^2} \; ; \quad \begin{array}{l} t = \operatorname{sen}\phi \\ dt = \cos\phi \, d\phi \end{array} \Bigg\} \int \frac{\cos\phi\, d\phi}{\cos^2\phi} = \int \sec\phi$$

$$= \ln(\sec\phi + \operatorname{tg}\phi)$$

$$= \ln\left(\frac{1+\operatorname{sen}\phi}{\cos\phi}\right)$$

$$= \ln\left(\frac{1+\operatorname{sen}\phi}{\sqrt{1-\operatorname{sen}^2\phi}}\right) \; (*)$$

$$(*) = i \ln\left(\frac{1+t}{\sqrt{1-t^2}}\right)\Bigg|_b^0$$

$$= i \underbrace{\ln(1)}_{0} - i\ln\left(\frac{1+b}{\sqrt{1-b^2}}\right)$$

$$= i\ln\left(\frac{\sqrt{1-b^2}}{1+b}\right) = i\ln\left(\frac{\sqrt{(1+b)(1-b)}}{1+b}\right) = i\ln\left(\sqrt{\frac{1-b}{1+b}}\right)$$

$$= \frac{i}{2}\ln\left(\frac{1-b}{1+b}\right).$$

De esta manera:

$$\int_\Gamma \frac{dz}{1+z^2} = \operatorname{arctg}(a) + \int_0^b \frac{2at + i(1+a^2-t^2)}{(1+a^2-t^2)^2 + 4a^2t^2}\,dt - \int_0^a \frac{1+t^2-b^2-2bti}{(1-b^2+t^2)^2+4b^2t^2}\,dt$$

$$+ \frac{i}{2}\ln\left(\frac{1-b}{1+b}\right) = 0, \text{ por C-G.}$$

Pues f tiene 2 polos de orden 1, $\{i, -i\}$. Pero $b \in (-1,1) \Rightarrow \Gamma$ no contiene a ningún polo.

Vamos a tomar $\lim_{a \to \infty}$!!

$$\operatorname{arctg}(a) + \frac{i}{2}\ln\left(\frac{1-b}{1+b}\right) = \int_0^a \frac{1+t^2-b^2-2bti}{(1+t^2-b^2)^2+4b^2t^2}\,dt - \underbrace{\int_0^b \frac{2at+i(1+a^2-t^2)\,dt}{(1+a^2-t^2)^2+4a^2t^2}}$$

(*)

$$-\frac{a^2}{a^4}\int_0^b \frac{\frac{2t}{a}+i\left(\frac{1}{a^2}+1-\frac{t^2}{a^2}\right)}{\left(\frac{1}{a^2}+1-\frac{t^2}{a^2}\right)^2+\frac{4t^2}{a^2}}$$

Tomando $a \to \infty$

$\operatorname{arctg}(a) \longrightarrow \pi/2$

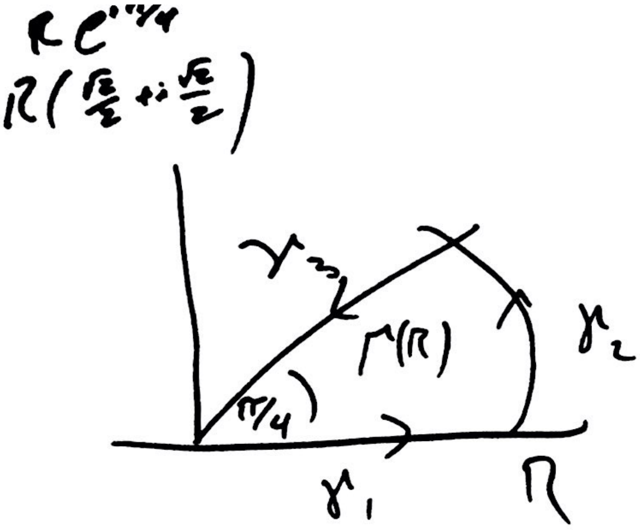
(*)

$$\lim_{a\to\infty} \underbrace{-\frac{1}{a^2}}_{\Downarrow\ \text{nula}} \underbrace{\int_0^b \overbrace{\frac{\frac{2t}{a}+i\left(\frac{1}{a^2}+1-\frac{t^2}{a^2}\right)}{\left(\frac{1}{a^2}+1-\frac{t^2}{a^2}\right)^2+\frac{4t^2}{a^2}}}^{h(t)}\,dt}_{\text{acotada para } t} = 0$$

$= G'$

acotada para t, pues $\left|\int_0^b h(t)\,dt\right| \leq b \cdot \sup_{t\in[0,b]}|h(t)|$

Así, $$\frac{\pi}{2}+\frac{i}{2}\ln\left(\frac{1-b}{1+b}\right) = \int_0^\infty \frac{1+t^2-b^2-2bti}{(1+t^2-b^2)^2+4b^2t^2}\,dt$$



$\gamma_1(t) = t, \ t \in [0,R]$

$\gamma_2(t) = R e^{it}, \ t \in [0, \frac{\pi}{4}]$

$\gamma_3(t) = -t e^{i\pi/4}, \ t \in [-R, 0]$

$= -t\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$$\Rightarrow \int_{\Gamma(R)} e^{iz^2} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = 0 \ / \ C\text{-}G$$

① $$\int_{\gamma_1} e^{iz^2}\,dz = \int_0^R e^{it^2}\,dt = \int_0^R \cos(t^2)\,dt + i\int_0^R \sin(t^2)\,dt$$

$\tau = -t$

③ $$\int_{\gamma_3} e^{iz^2}\,dz = \int_{-R}^0 e^{-it^2 e^{i\pi/2}} \cdot -e^{i\pi/4}\,dt = -\int_0^R e^{it^2 e^{i\pi/2}} e^{i\pi/4}\,d\tau$$

$$= -e^{i\frac{\pi}{4}} \int_0^R e^{-t^2}\,dt \xrightarrow[R\to\infty]{} -e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}$$

$$= -\frac{1}{2}\left(\sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}\right)$$

②
$$\left|\int_{\Gamma_2} \exp(iz^2)\,dz\right| = \left|\int_0^{\pi/4} \exp(iR^2 e^{2it}) Rie^{it}\,dt\right|$$

$$= R\left|\int_0^{\pi/4} \exp(iR^2(\cos(2t)+i\sin(2t)))e^{it}\,dt\right|$$

$$\leq R\int_0^{\pi/4} \exp(-R^2\sin(2t))\,dt \qquad / \ \sin\alpha \geq \frac{2\alpha}{\pi} \ \forall \alpha\in[0,\pi/2]$$

$$\leq R\int_0^{\pi/4} \exp\left(-R^2\cdot\frac{4t}{\pi}\right)dt$$

$$= R\cdot\left(-\frac{\pi}{4R^2}\right)\exp\left(\frac{-R^2 4t}{\pi}\right)\Big|_0^{\pi/4}$$

$$= -\frac{\pi}{4R}\left(\exp(-R^2)-1\right) \xrightarrow[R\to\infty]{} 0$$

$\therefore\ R\to\infty$

$$\Rightarrow \int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\left(\sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}\right)$$

$\blacksquare$