"RESUMEN" Sea una curva Γ parametrizada por $\vec{r}(t)$, $t \in [a,b]$

Largo de una curva

$$L(\Gamma) = \int_a^b \left\|\frac{dr}{dt}\right\| dt$$

Longitud de arco

$$s(t) = \int_a^t \left\|\frac{dr}{dt}\right\| dt$$

Parametrización en longitud de arco

$$\sigma(s) = \vec{r}(t(s))$$

Curva regular:

- $\vec{r} \in C^1$
- $\left\|\frac{dr}{dt}\right\| > 0$

$$\vec{T} = \frac{\frac{dr}{dt}}{\left\|\frac{dr}{dt}\right\|} = \frac{d\sigma}{ds}$$

$$\vec{N} = \frac{\frac{dT}{dt}}{\left\|\frac{dT}{dt}\right\|} = \frac{\frac{dT}{ds}}{\left\|\frac{dT}{ds}\right\|}$$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\tau = -N \cdot \frac{\frac{dB}{dt}}{\left\|\frac{dr}{dt}\right\|} = -N \cdot \frac{dB}{ds}$$

$$K = \frac{\left\|\frac{dT}{dt}\right\|}{\left\|\frac{dr}{dt}\right\|} = \left\|\frac{dT}{ds}\right\|$$

Triedro de Frenet

$$\frac{dT}{ds} = KN \quad \frac{dN}{ds} = -KT + \tau B \quad \frac{dB}{ds} = -\tau N$$

P1 $\vec{r}(t) = (\cos(g(t)), \operatorname{sen}(g(t)), 1)$

a) P.D.Q Γ es regular

- P.D.Q $\vec{r} \in C^1$: $\cos(x), \operatorname{sen}(x), g(t), 1 \in C^1 \Rightarrow \vec{r} \in C^1$ ✓
- P.D.Q $\left\|\frac{d\vec{r}}{dt}\right\| > 0$: $\frac{d\vec{r}}{dt} = (-\operatorname{sen}(g(t)) \cdot g'(t), \cos(g(t)) \cdot g'(t), 0)$

$$\Rightarrow \left\|\frac{d\vec{r}}{dt}\right\| = \left(\operatorname{sen}^2(g(t)) \cdot [g'(t)]^2 + \cos^2(g(t)) \cdot [g'(t)]^2\right)^{1/2} = \left([g'(t)]^2\right)^{1/2} = g'(t) > 0 \checkmark$$

$\therefore \Gamma$ es regular

b) $$L = \int_0^\infty \left\|\frac{d\vec{r}}{dt}\right\| dt = \int_0^\infty g'(t)\, dt = \lim_{b \to \infty} g(b) - g(0) = M - g(0) \quad //$$

$t \in [0, \infty)$

Para encontrar $\sigma(s)$ (parametrización en longitud de arco), tenemos que encontrar primero $s(t)$. $s(t) = \int_0^t \left\|\frac{dr}{dt}\right\| dt = \int_0^t g'(t)\, dt = g(t) - g(0)$

Recordemos que $\sigma(s) = \vec{r}(t(s))$. Entonces necesitamos $t(s)$

Sabemos que $s = g(t) - g(0) \Rightarrow g(t) = s - g(0)$ / g^{-1} (existe pq $g' > 0 \Rightarrow g$ creciente $\Rightarrow g$ inyectiva)

$\Rightarrow t(s) = g^{-1}(s - g(0))$.

$$\Rightarrow \sigma(s) = \vec{r}(t(s)) = (\cos(s - g(0)), \operatorname{sen}(s - g(0)), 1) \quad //$$

c) $$T(t) = \frac{\frac{d\vec{r}}{dt}}{\left\|\frac{d\vec{r}}{dt}\right\|} = \frac{(-\text{sen}(g(t))\cdot g'(t),\ \cos(g(t))\cdot g'(t),\ 0)}{g'(t)}$$ ← calculado en (a)

$$\Rightarrow T(t) = (-\text{sen}(g(t)),\ \cos(g(t)),\ 0)\ //$$

$$N(t) = \frac{\frac{dT}{dt}}{\left\|\frac{dT}{dt}\right\|} = \frac{(-\cos(g(t))\cdot g'(t),\ -\text{sen}(g(t))\cdot g'(t),\ 0)}{\left(\cos^2(g(t))\cdot[g'(t)]^2 + \text{sen}^2(g(t))\cdot[g'(t)]^2\right)^{1/2}} = \frac{(-\cos(g(t))\cdot g'(t),\ -\text{sen}(g(t))\cdot g'(t),\ 0)}{g'(t)}$$

$$\Rightarrow N(t) = (-\cos(g(t)),\ -\text{sen}(g(t)),\ 0)\ //$$

$$B(t) = T\times N = \begin{pmatrix} -\text{sen}(g) & -\cos(g) \\ \cos(g) & -\text{sen}(g) \\ 0 & 0 \\ -\text{sen}(g) & -\cos(g) \end{pmatrix} = \begin{pmatrix} \cos(g)\cdot 0 + \text{sen}(g)\cdot 0 \\ 0\cdot -\cos(g) + 0\cdot \text{sen}(g) \\ \text{sen}^2(g) - -\cos^2(g) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\ //$$

d) $$K(t) = \frac{\left\|\frac{dT}{dt}\right\|}{\left\|\frac{dr}{dt}\right\|} = \frac{g'(t)}{g'(t)} = 1\ //$$ ↙ de (c) ↙ de (a)

$$\tau(t) = -N\,\frac{\frac{dB}{dt}}{\left\|\frac{dr}{dt}\right\|} = \frac{-N\cdot\begin{pmatrix}0\\0\\0\end{pmatrix}}{\left\|\frac{dr}{dt}\right\|} = 0\ //$$

P2|

P.D.Q $\tau = \dfrac{[\sigma' \times \sigma''] \cdot \sigma'''}{\|\sigma''\|^2}$

triedro de Frenet

Recordemos que $\sigma = \sigma(s) \Rightarrow \sigma' = \dfrac{d\sigma}{ds} = T \Rightarrow T' = \dfrac{dT}{ds} = KN$

$$\Rightarrow \frac{[\sigma' \times \sigma''] \cdot \sigma'''}{\|\sigma''\|^2} \underset{\sigma' = T}{=} \frac{[T \times T'] \cdot T''}{\|T'\|^2} \underset{T' = KN}{=} \frac{[T \times KN] \cdot (KN)'}{\|KN\|^2}$$

como K es un escalar, sale del producto cruz y de la norma, luego

$$= \frac{K[T \times N](K'N + KN')}{|K|^2 \underbrace{\|N\|^2}_{1 \ (\|T\| = \|N\| = \|B\| = 1)}} \underset{\substack{T \times N = B \\ |K| = K \text{ pq } K \geq 0}}{=} \frac{KB(K'N + KN')}{K^2} = \frac{(K' \overbrace{B \cdot N}^{0} + BKN')}{K}$$

$(B \perp N) \Rightarrow B \cdot N = 0$

$(B \perp T) \Rightarrow B \cdot T = 0$

$$= \frac{BKN'}{K} \underset{N' = \frac{dN}{ds} = -KT + \tau B}{=} B(-KT + \tau B) = -K \overbrace{B \cdot T}^{0} + \tau B \cdot B \underset{\substack{x \cdot x = \|x\|^2 \\ \forall x}}{=} \tau \overbrace{\|B\|^2}^{1} = \tau \ //$$

$$\therefore \ \tau = \frac{[\sigma' \times \sigma''] \cdot \sigma'''}{\|\sigma''\|^2}$$

P3 recordemos coordenadas cilíndricas

$$\vec{r} = (x,y,z) = (\rho\cos\theta, \rho\,\text{sen}\,\theta, z) = \rho\hat{\rho} + z\hat{z}$$

$$\rho = \sqrt{x^2+y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

$\hat{\rho} = (\cos\theta, \text{sen}\,\theta, 0)$

$\hat{\theta} = (-\text{sen}\,\theta, \cos\theta, 0)$

$\hat{z} = (0, 0, 1)$

$\dfrac{d\hat{\rho}}{d\theta} = \hat{\theta} \qquad \dfrac{d\hat{\theta}}{d\theta} = -\hat{\rho}$

a) Tenemos que encontrar ρ, ya sabemos que $z = a\theta$

$$\underbrace{x^2+y^2}_{\rho^2} + z = \pi \qquad \Rightarrow \rho^2 = \pi - z = \pi - a\theta$$

$$\Rightarrow \rho = \sqrt{\pi - a\theta}$$

luego $\vec{r}(\theta) = \underbrace{(\sqrt{\pi-a\theta}\cos\theta, \sqrt{\pi-a\theta}\,\text{sen}\,\theta, a\theta)}_{\text{cartesianas}} = \underbrace{\sqrt{\pi-a\theta}\,\hat{\rho} + a\theta\hat{z}}_{\text{cilíndricas (o "polares")}}$

$x = \sqrt{\pi-a\theta}\cos\theta$

$y = \sqrt{\pi-a\theta}\,\text{sen}\,\theta$ $\Big\}$ cartesianas

$z = a\theta$

$= x\hat{x} + y\hat{y} + z\hat{z}$

b) $T(\theta) = \dfrac{\frac{dr}{d\theta}}{\left\|\frac{dr}{d\theta}\right\|}$

Forma 1: en cilíndricas

$$\vec{r}(\theta) = \sqrt{\pi-a\theta}\,\hat{\rho} + a\theta\hat{z}$$

$\hat{z}$ es constante

$$\frac{d\vec{r}}{d\theta} = \underbrace{\frac{1}{2\sqrt{\pi-a\theta}}\cdot -a\,\hat{\rho} + \sqrt{\pi-a\theta}\,\hat{\theta}}_{\substack{\text{regla del producto} \\ (\hat{\rho} = \hat{\rho}(\theta))}} + a\hat{z}$$

$$\frac{d\vec{r}}{dt} = \frac{-a}{2\sqrt{\pi-a\theta}}\hat{\rho} + \sqrt{\pi-a\theta}\,\hat{\theta} + a\hat{z} \Rightarrow \left\|\frac{dr}{dt}\right\| = \left(\frac{a^2}{4(\pi-a\theta)} + (\pi-a\theta) + a^2\right)^{1/2}$$

$$T(t) = \frac{\frac{d\vec{r}}{dt}}{\left\|\frac{d\vec{r}}{dt}\right\|} \quad //$$

Forma 2: en cartesianas

$$\vec{r}(t) = (\sqrt{\pi - a\theta}\cos\theta, \sqrt{\pi - a\theta}\,sen\,\theta, a\theta)$$

$$\frac{d\vec{r}}{dt} = \left(\frac{1}{2\sqrt{\pi - a\theta}} \cdot -a\cos\theta - \sqrt{\pi - a\theta}\,sen\,\theta, \frac{-a}{2\sqrt{\pi - a\theta}}\,sen\,\theta + \sqrt{\pi - a\theta}\cos\theta, a\right)$$

$$\left\|\frac{d\vec{r}}{dt}\right\| = \left[\left(\frac{-a}{2\sqrt{\pi - a\theta}}\cos\theta - \sqrt{\pi - a\theta}\,sen\,\theta\right)^2 + \left(\frac{-a}{2\sqrt{\pi - a\theta}}\,sen\,\theta + \sqrt{\pi - a\theta}\cos\theta\right)^2 + a^2\right]^{1/2}$$

$$= \frac{a^2}{4(\pi - a\theta)}\cos^2\theta + (\pi - a\theta)sen^2\theta + \cancel{a\cos\theta\,sen\,\theta} + \frac{a^2}{4(\pi - a\theta)}sen^2\theta + (\pi - a\theta)\cos^2\theta - \cancel{a\cos\theta\,sen\,\theta} + a^2$$

$$= \frac{a^2}{4(\pi - a\theta)} + (\pi - a\theta) + a^2$$

$$T(t) = \frac{\frac{d\vec{r}}{dt}}{\left\|\frac{d\vec{r}}{dt}\right\|} \quad //$$

Conclusión: cilíndricas ahorra MUCHA matraca :)