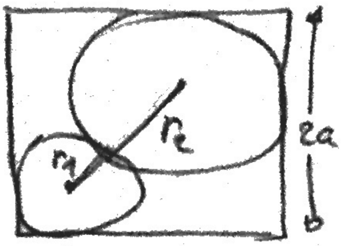
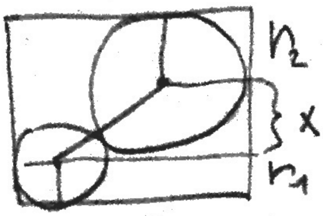
P1



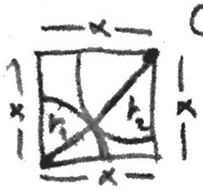
a) Una relación entre r_1 y r_2



Queremos calcular x.

$$2a = r_1 + r_2 + x$$

Sin embargo sabemos que los centros de los círculos pasan por la diagonal, si hacemos zoom



$$\Rightarrow (r_1+r_2) = x\sqrt{2} \Rightarrow x = \frac{(r_1+r_2)}{\sqrt{2}}$$

Entonces $2a = r_1+r_2 + \frac{(r_1+r_2)}{\sqrt{2}} = \left(1+\frac{1}{\sqrt{2}}\right)(r_1+r_2)$

$$\frac{2a}{\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)} = r_1+r_2 \Rightarrow \frac{2\sqrt{2}a}{\sqrt{2}+1} = r_1+r_2$$

b) ahora queremos optimizar la función

$$A_O = r_1^2\pi + r_2^2\pi$$

Pero $r_2 = \frac{2\sqrt{2}a}{\sqrt{2}+1} - r_1$

$$A_O = r_1^2\pi + \pi\left(\frac{2\sqrt{2}a}{\sqrt{2}+1} - r_1\right)^2 = 2r_1^2\pi - \frac{4\sqrt{2}a}{\sqrt{2}+1}r_1\pi + \frac{8a^2}{(\sqrt{2}+1)^2}$$

$A_O' = 4r_1\pi - \frac{4\sqrt{2}a}{\sqrt{2}+1}\pi$ impongo $A_O' = 0$

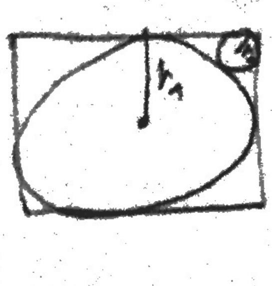
$$\frac{4\sqrt{2}a}{\sqrt{2}+1} = 4r_1^* \Rightarrow r_1^* = \frac{\sqrt{2}a}{\sqrt{2}+1}$$

Ahora debemos chequear condiciones de 2° orden

$A_O'' = 4\pi \Rightarrow r_1^*$ es un mínimo

Sin embargo r_1^* es el único punto en el que $A_O' = 0$, ¿Entonces que hacemos?

A_O es continua, y r_1 no puede tomar cualquier valor, está acotado por $[0, a]$, como A_O es creciente hacia los extremos, vemos que su máximo debe estar en $r_1 = a$ $r_2 = \frac{2\sqrt{2}a}{\sqrt{2}+1} - a = \frac{\sqrt{2}a - a}{\sqrt{2}+1}$



Ese es el dibujo de la situación de máxima area.

P2

a) $\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{2x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2)$

b) $\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+\frac{x^2}{a^2}}$ $\quad u = \frac{x}{a} \Rightarrow du = \frac{1}{a}dx$

$\Rightarrow \frac{1}{a^2}\int \frac{a\,du}{1+u^2} = \frac{1}{a}\int \frac{du}{1+u^2} = \frac{\arctan(u)}{a}$

$= \frac{\arctan(\frac{x}{a})}{a}$

c) $\int e^{\sin^2(x)} \sin(2x)\,dx$

$\cos(2x) = 2\cos^2(x) - 1$

$\cos(2x) = 2 - 2\sin^2(x) - 1$

$\frac{\cos(2x) - 1}{2} = -\sin^2(x)$

$\frac{1-\cos(2x)}{2} = \sin^2(x)$

$\int e^{\frac{1-\cos(2x)}{2}} \sin(2x)\,dx$

$u = \frac{1-\cos(2x)}{2}$; $du = \sin(2x)\,dx$

$\Rightarrow \int e^u du = e^u = e^{\frac{1-\cos(2x)}{2}}$

d) $\int \frac{dx}{x\sqrt{x^2-1}}$ $\quad x = \sec u \quad dx = \tan u \cdot \sec u$

$\cos^2 u + \sin^2 u = 1 \quad / \frac{1}{\cos^2 u}$

$1 + \tan^2 u = \sec^2 u$

$= \int \frac{\tan u \cdot \sec u\,du}{\sec u \cdot \sqrt{\sec^2 u - 1}}$

$= \int \frac{\tan u \cdot \sec u\,du}{\sec u \cdot \tan u} = \int du = u = \operatorname{arcsec}(x)$

e) $\displaystyle\int \frac{\sin(x)dx}{\sin(x)-\cos(x)+1}$ usamos $\sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$

$\cos(x) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})$

$$= \int \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})\,dx}{2\sin(\frac{x}{2})\cos(\frac{x}{2}) - \cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2}) + 1} = \int \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})dx}{2\sin(\frac{x}{2})\cos(\frac{x}{2}) + 2\sin^2(\frac{x}{2})}$$

$$= \int \frac{2\cos(\frac{x}{2})dx}{2\cos(\frac{x}{2}) + 2\sin(\frac{x}{2})} = \int \frac{\cos(\frac{x}{2})\ dx}{\cos(\frac{x}{2}) + \sin(\frac{x}{2})}$$

uso mikita milona $+\frac{\sin(\frac{x}{2})}{2} - \frac{\sin(\frac{x}{2})}{2}$

$$= \int \frac{\frac{\cos(\frac{x}{2})}{2} + \frac{\sin(\frac{x}{2})}{2}}{\cos(\frac{x}{2}) + \sin(\frac{x}{2})} + \frac{\frac{\cos(\frac{x}{2})}{2} - \frac{\sin(\frac{x}{2})}{2}}{\cos(\frac{x}{2}) + \sin(\frac{x}{2})}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos(\frac{x}{2}) - \sin(\frac{x}{2})}{\cos(\frac{x}{2}) + \sin(\frac{x}{2})}$$

claramente $\left(\cos(\frac{x}{2}) + \sin(\frac{x}{2})\right)' = \frac{-\sin(\frac{x}{2})}{2} + \frac{\cos(\frac{x}{2})}{2}$

$$\Rightarrow \frac{1}{2}\int dx + \frac{1}{2}\left(2\log\left(\cos(\tfrac{x}{2}) + \sin(\tfrac{x}{2})\right)\right)$$

$$= \frac{x}{2} + \log\left(\cos(\tfrac{x}{2}) + \sin(\tfrac{x}{2})\right)$$

f) $\int x\sqrt{a^2+x^2}\,dx$

$u = a^2 + x^2$
$du = 2x\,dx$

$$\frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} = \frac{1}{3}(a^2+x^2)^{\frac{3}{2}} + C$$

g) $\int \frac{\cot(x)}{\ln(\sin(x))}dx$

$u = \ln(\sin(x))$
$du = \frac{\cos(x)}{\sin(x)}dx = \cot(x)\,dx$

$$= \int \frac{1}{u}du = \ln(u) = \ln(\cancel{\ln}(\sin(x))) + C$$

h) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}$

$$\Rightarrow \int 2e^u du = 2e^u = 2e^{\sqrt{x}} + C$$

i) $\int \frac{1}{1-\cos(x)} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \int \frac{1+\cos(x)}{1-\cos^2(x)}$

$$\Rightarrow \int \frac{1}{1-\cos^2(x)} + \int \frac{\cos(x)}{1-\cos^2(x)} = \int \frac{1}{\sin^2(x)} + \int \frac{\cos(x)}{\sin^2(x)}$$

$$= \int \csc^2(x) + \int \cot(x)\csc(x)$$

$$= -\cot(x) + \csc(x)$$

P3

$$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x} = \frac{1}{a^2}\int \frac{\sec^2 x\, dx}{1 + \frac{b^2}{a^2}\tan^2 x}$$

$$\frac{b}{a}\tan x = u$$

$$dx\,\frac{b}{a}\sec^2 x = du$$

$$\Rightarrow \frac{1}{a^2}\int \frac{\frac{a}{b}\,du}{1+u^2} = \frac{1}{ab}\int \frac{du}{1+u^2} = \frac{\arctan(u)}{ab}$$

$$\frac{b}{a}\tan x = u$$

$$\Rightarrow \int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\arctan\left(\frac{b\tan x}{a}\right)}{b \cdot a}$$