SOLUCIÓN APUNTE MA1002
CÁLCULO DIFERENCIAL E INTEGRAL
SEMANA 6

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Use este material con responsabilidad.

E3.

a)

$$\begin{aligned} I_n &= \int x^n \sin(x)dx \qquad ; \begin{bmatrix} u = x^n \to du = nx^{n-1}dx \\ dv = \sin(x)dx \to v = -\cos(x) \end{bmatrix} \\ &= -x^n\cos(x) + n\int x^{n-1}\cos(x)dx \qquad ; \begin{bmatrix} u = x^{n-1} \to du = (n-1)x^{n-2}dx \\ dv = \cos(x)dx \to v = \sin(x) \end{bmatrix} \\ &= -x^n\cos(x) + n\left(x^{n-1}\sin(x) - (n-1)\int x^{n-2}\sin(x)dx\right) \\ &= -x^n\cos(x) + nx^{n-1}\sin(x) - n(n-1)I_{n-2}. \end{aligned}$$

b) Análogo al ejercicio a).

c)

$$\begin{aligned} I_n &= \int x^n e^x dx \qquad ; \begin{bmatrix} u = x^n \to du = nx^{n-1}dx \\ dv = e^x dx \to v = e^x \end{bmatrix} \\ &= x^n e^x - n\int x^{n-1}e^x dx = x^n e^x - nI_{n-1}. \end{aligned}$$

d)

$$\begin{aligned} I_n &= \int \sin^n(x)dx \qquad ; \begin{bmatrix} u = \sin^{n-1}(x) \to du = (n-1)\sin^{n-2}(x)\cos(x)dx \\ dv = \sin(x)dx \to v = -\cos(x) \end{bmatrix} \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)dx \\ (*) &\quad \text{Como } \sin^{n-2}(x)\cos^2(x) = \sin^{n-2}(x)(1-\sin^2(x)) = \sin^{n-2}(x) - \sin^n(x) \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int(\sin^{n-2}(x) - \sin^n(x))dx \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)(I_{n-2} - I_n) \end{aligned}$$

Despejando I_n de esta última igualdad obtenemos:

$$I_n = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}I_{n-2}.$$

e) Análogo al ejercicio d).

f)

$$\begin{aligned}
I_n &= \int x^n \sinh(2x)dx \qquad ; \left[\begin{array}{l} u = x^n \to du = nx^{n-1}dx \\ dv = \sinh(2x)dx \to v = \frac{1}{2}\cosh(2x) \end{array}\right] \\
&= \frac{1}{2}x^n \cosh(2x) - \frac{n}{2}\int x^{n-1}\cosh(2x)dx \qquad ; \left[\begin{array}{l} u = x^{n-1} \to du = (n-1)x^{n-2}dx \\ dv = \cosh(2x)dx \to v = \frac{1}{2}\sinh(2x) \end{array}\right] \\
&= \frac{1}{2}x^n \cosh(2x) - \frac{n}{2}\left(\frac{1}{2}x^{n-1}\sinh(2x) - \frac{n-1}{2}\int x^{n-2}\sinh(2x)dx\right) \\
&= \frac{1}{2}x^n \cosh(2x) - \frac{n}{4}x^{n-1}\sinh(2x) + \frac{n(n-1)}{4}I_{n-2}.
\end{aligned}$$

E5. El cambio de variable sugerido implica que:

$$u = \tan\left(\frac{x}{2}\right) \ ; \ \cos(x) = \frac{1-u^2}{1+u^2} \ ; \ \sin(x) = \frac{2u}{1+u^2} \ ; \ dx = \frac{2du}{1+u^2}$$

a)

$$\int \frac{dx}{\sin(x)} = \int \frac{2du}{2u} = \ln|u| + c = \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$$

b)

$$\begin{aligned}
\int \frac{dx}{\cos(x)} &= \int \frac{2du}{1-u^2} = \int \left(\frac{1}{1+u} + \frac{1}{1-u}\right)du = \ln|1+u| - \ln|1-u| + c \\
&= \ln\left|\frac{1+u}{1-u}\right| + c = \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + c
\end{aligned}$$

c)

$$\begin{aligned}
\int \frac{dx}{1+\sin(x)} &= \int \frac{2du}{(1+u^2)+2u} = \int \frac{2du}{(1+u)^2} \qquad ; v = 1+u \to dv = du \\
&= \int \frac{2}{v^2}dv = \frac{-2}{1+u} + c = \frac{-2}{1+\tan\left(\frac{x}{2}\right)} + c
\end{aligned}$$

d)

$$\begin{aligned}
\int \frac{dx}{1-\cos(x)} &= \int \frac{2du}{(1+u^2)-(1-u^2)} = \int \frac{du}{u^2} = \frac{-1}{u} + c \\
&= \frac{-1}{\tan\left(\frac{x}{2}\right)} + c
\end{aligned}$$

e)

$$\int \frac{dx}{\sin(x)+\cos(x)} = \int \frac{2du}{2u+1-u^2} = \int \frac{-2du}{(u-1+\sqrt{2})(u-1-\sqrt{2})}$$

Separemos esta última fracción:

$$\frac{-2}{(u-1+\sqrt{2})(u-1-\sqrt{2})} = \frac{A}{u-1+\sqrt{2}} + \frac{B}{u-1-\sqrt{2}} = \frac{(A+B)u-(A+B)+(B-A)\sqrt{2}}{(u-1+\sqrt{2})(u-1-\sqrt{2})}$$

Luego, imponiendo $A+B=0$ y $(B-A)\sqrt{2}=-2$, se tiene $A=\frac{1}{\sqrt{2}}$ y $B=-\frac{1}{\sqrt{2}}$. Se sigue que:

$$\begin{aligned}
\int \frac{dx}{\sin(x)+\cos(x)} &= \int \frac{-2du}{(u-1+\sqrt{2})(u-1-\sqrt{2})} \\
&= \frac{1}{\sqrt{2}} \int \left(\frac{1}{u-1+\sqrt{2}} - \frac{1}{u-1-\sqrt{2}} \right) du \\
&= \frac{1}{\sqrt{2}} \left(\ln|u-1+\sqrt{2}| - \ln|u-1-\sqrt{2}| \right) + c \\
&= \frac{1}{\sqrt{2}} \ln \left| \frac{u-1+\sqrt{2}}{u-1-\sqrt{2}} \right| + c \\
&= \frac{1}{\sqrt{2}} \ln \left| \frac{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1-\sqrt{2}} \right| + c
\end{aligned}$$

E6.

a)

$$\int \frac{dx}{\sqrt{x^2-1}} \quad ;\text{como } x^2>1, \text{ hagamos } x=\sec(v) \to dx=\sec(v)\tan(v)dv$$

$$= \int \frac{\sec(v)\tan(v)dv}{\sqrt{\sec^2(v)-1}} = \int \frac{\sec(v)\tan(v)dv}{\tan(v)}$$

$$= \int \sec(v)dv = \ln|\sec(v)+\tan(v)|+c$$

Recordando que $\tan^2(v)=\sec^2(v)-1=x^2-1$, tenemos finalmente:

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}|+c.$$

b)

$$\int \frac{dx}{\sqrt{x^2+1}} \quad ;\text{como } x\in\mathbb{R}, \text{ hagamos } x=\tan(v) \to dx=\sec^2(v)dv$$

$$= \int \frac{\sec^2(v)dv}{\sqrt{\tan^2(v)+1}} = \int \frac{\sec^2(v)dv}{\sec(v)} = \int \sec(v)dv = \ln|\sec(v)+\tan(v)|+c$$

Recordando que $\sec^2(v)=\tan^2(v)+1=x^2+1$, tenemos finalmente:

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln|x+\sqrt{x^2+1}|+c.$$

E7.

$$\begin{aligned}
&\int \frac{g(x)g'(x)}{\sqrt{1+g(x)^2}} \qquad ; u = 1+g(x)^2 \to du = 2g(x)g'(x)dx \\
= &\int \frac{du}{2\sqrt{u}} = \sqrt{u} + c = \sqrt{1+g(x)^2} + c.
\end{aligned}$$

P1.

$$\begin{aligned}
\int \frac{\sin(x)}{1+\sin(x)}dx &= \int \left(1 - \frac{1}{1+\sin(x)}\right) dx \\
&= x - \int \frac{dx}{1+\sin(x)} \qquad ; \text{usamos el cambio } u = \tan\left(\frac{x}{2}\right) \\
&= x - \int \frac{2du}{(1+u^2)+2u} = x - \int \frac{2du}{(1+u)^2} \qquad ; v = 1+u \to dv = du \\
&= x - \int \frac{2du}{v^2} = x + \frac{2}{v} + c \\
&= x + \frac{2}{1+u} + c = x + \frac{2}{1+\tan\left(\frac{x}{2}\right)} + c
\end{aligned}$$

P2.a)

1.

$$\begin{aligned}
I_1 &= \int \frac{\cos(x)}{\cos(x)}dx = \int dx = x + c \\
I_2 &= \int \frac{\cos(2x)}{\cos^2(x)}dx = \int \frac{2\cos^2(x)-1}{\cos^2(x)}dx \\
&= \int (2 - \sec^2(x))dx = 2x - \tan(x) + c
\end{aligned}$$

2.

$$\begin{aligned}
J_n &= \int \frac{\sin(x)}{\cos^{n+1}(x)}dx \qquad ; \left[\begin{array}{l} u = \frac{1}{\cos^{n+1}(x)} \to du = (n+1)\frac{\sin(x)}{\cos^{n+2}(x)}dx \\ dv = \sin(x)dx \to v = -\cos(x) \end{array}\right] \\
&= \frac{-1}{\cos^n(x)} + (n+1)\int \frac{\sin(x)}{\cos^{n+1}(x)}dx = \frac{-1}{\cos^n(x)} + (n+1)J_n
\end{aligned}$$

Luego, despejando J_n de la última igualdad se obtiene:

$$J_n = \frac{1}{n\cos^n(x)}$$

3.

$$\begin{aligned} I_{n+1} &= \int \frac{\cos(nx+x)}{\cos^{n+1}(x)}dx = \int \frac{\cos(nx)\cos(x)-\sin(nx)\sin(x)}{\cos^{n+1}(x)}dx \\ &= \int \frac{\cos(nx)}{\cos^{n}(x)}dx - \int \frac{\sin(nx)\sin(x)}{\cos^{n+1}(x)}dx = I_n - \int \frac{\sin(nx)\sin(x)}{\cos^{n+1}(x)}dx \end{aligned}$$

Concentrémonos en esta última integral. Notemos que podemos ocupar la parte a.2 apropiadamente para integrar por partes:

$$\int \frac{\sin(nx)\sin(x)}{\cos^{n+1}(x)}dx \qquad ; \left[\begin{array}{l} u = \sin(nx) \to du = n\cos(nx) \\ dv = \frac{\sin(x)}{\cos^{n+1}(x)}dx \to v = \frac{1}{n\cos^n(x)} \end{array} \right]$$

$$= \frac{\sin(nx)}{n\cos^n(x)} - \int \frac{\cos(nx)}{\cos^n(x)}dx = \frac{\sin(nx)}{n\cos^n(x)} - I_n$$

Luego tenemos finalmente:

$$I_{n+1} = I_n - \left(\frac{\sin(nx)}{n\cos^n(x)} - I_n\right) = 2I_n - \frac{\sin(nx)}{n\cos^n(x)}.$$

P2.b)

$$\int \frac{\sqrt{a^2-x^2}}{x^2}dx \qquad ;\text{ya que } x^2 \le a^2\text{, hacemos } x = a\cos(v) \to dx = -a\sin(v)dv$$

$$\begin{aligned} &= -\int \frac{\sqrt{a^2-a^2\cos^2(v)}}{a^2\cos^2(v)}a\sin(v)dv = -\int \tan^2(v)dv \\ &= \int (1-\sec^2(v))dv = v - \tan(v) + c \end{aligned}$$

Recordemos que $\cos(v) = \frac{x}{a}$, por lo que:

$$\tan^2(v) = \frac{1}{\cos^2(v)} - 1 = \frac{a^2}{x^2} - 1 = \frac{a^2-x^2}{x^2}$$

Finalmente tenemos:

$$\int \frac{\sqrt{a^2-x^2}}{x^2}dx = \arccos\left(\frac{x}{a}\right) - \frac{\sqrt{a^2-x^2}}{x} + c.$$

P3.a)

$$\int \frac{dx}{x(\ln(x)+\ln^2(x))} \qquad ; u = \ln(x) \to du = \frac{dx}{x}$$

$$\begin{aligned} &= \int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right)du = \ln|u| - \ln|u+1| + c \\ &= \ln\left|\frac{u}{u+1}\right| + c = \ln\left|\frac{\ln(x)}{\ln(x)+1}\right| + c \end{aligned}$$

P3.b)

$$
\begin{aligned}
& \int \frac{\cos(x)}{1+\cos(x)}dx = \int \left(1 - \frac{1}{1+\cos(x)}\right)dx \\
= & \quad x - \int \frac{dx}{1+\cos(x)} \qquad ; u = \tan\left(\frac{x}{2}\right) \\
= & \quad x - \int \frac{2du}{(1+u^2)-(1-u^2)} = x - \int du = x - u + c = x - \tan\left(\frac{x}{2}\right) + c
\end{aligned}
$$

P3.c)

$$
\begin{aligned}
I &= \int \cos(\ln(x))dx \qquad ; \left[\begin{array}{l} u = \cos(\ln(x)) \to du = \frac{-\sin(\ln(x))}{x}dx \\ dv = dx \to v = x \end{array}\right] \\
&= x\cos(\ln(x)) + \int \sin(\ln(x))dx \\
&= x\cos(\ln(x)) + J \\
J &= \int \sin(\ln(x))dx \qquad ; \left[\begin{array}{l} u = \sin(\ln(x)) \to du = \frac{\cos(\ln(x))}{x}dx \\ dv = dx \to v = x \end{array}\right] \\
&= x\sin(\ln(x)) - \int \cos(\ln(x))dx \\
&= x\sin(\ln(x)) - I
\end{aligned}
$$

Juntando ambos resultados obtenemos el sistema:

$$
\begin{aligned}
I &= x\cos(\ln(x)) + J \\
J &= x\sin(\ln(x)) - I
\end{aligned}
$$

De donde se despeja:

$$
\begin{aligned}
I &= \frac{x}{2}(\sin(\ln(x)) + \cos(\ln(x))) + c \\
J &= \frac{x}{2}(\sin(\ln(x)) - \cos(\ln(x))) + c.
\end{aligned}
$$

P4.a)

$$\begin{aligned}\frac{5x^2+12x+1}{x^3+3x^2-4} &= \frac{5x^2+12x+1}{(x-1)(x+2)^2} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\\ &= \frac{(A+B)x^2+(4A+B+C)x+(4A-2B-C)}{(x-1)(x+2)^2}\end{aligned}$$

Luego resolvemos el sistema:

$$\left[\begin{array}{rcl} A+B &=& 5\\ 4A+B+C &=& 12\\ 4A-2B-C &=& 1\end{array}\right] \Rightarrow \begin{array}{l} A=2\\ B=3\\ C=1\end{array}$$

Usando lo anterior en la integral obtenemos:

$$\begin{aligned}\int \frac{5x^2+12x+1}{x^3+3x^2-4}dx &= 2\int\frac{dx}{x-1}+3\int\frac{dx}{x+2}+\underbrace{\int\frac{dx}{(x+2)^2}}_{w=x+2\to dw=dx}\\ &= 2\ln|x-1|+3\ln|x+2|+\int w^{-2}dw\\ &= 2\ln|x-1|+3\ln|x+2|-w^{-1}+c\\ &= \ln\left|(x-1)^2(x+2)^3\right|-\frac{1}{x+2}+c\end{aligned}$$

P4.b)

$$\begin{aligned} I_{m,n} &= \int x^m\ln^n(x)dx \qquad ;\left[\begin{array}{l} u=\ln^n(x)\to du=\frac{n\ln^{n-1}(x)}{x}dx\\ dv=x^m dx\to v=\frac{x^{m+1}}{m+1}\end{array}\right]\\ &= \frac{x^{m+1}\ln^n(x)}{m+1}-\frac{n}{m+1}\int x^m\ln^{n-1}(x)dx\\ &= \frac{x^{m+1}\ln^n(x)}{m+1}-\frac{n}{m+1}I_{m,n-1}\end{aligned}$$

Para calcular lo pedido debemos usar $m=2$ y $n=1$, con lo que se tiene:

$$I_{2,1}=\frac{x^{2+1}\ln^1(x)}{2+1}-\frac{1}{2+1}I_{2,0}=\frac{x^3\ln(x)}{3}-\frac{1}{3}\int x^2dx=\frac{x^3\ln(x)}{3}-\frac{x^3}{9}+c$$

P5.a)

$$\frac{x}{(1+x^2)(1+x)} = \frac{Ax+B}{1+x^2} + \frac{C}{1+x} = \frac{(A+C)x^2+(A+B)x+(B+C)}{(1+x^2)(1+x)}$$

$$\begin{bmatrix} A+C & = & 0 \\ A+B & = & 1 \\ B+C & = & 0 \end{bmatrix} \Rightarrow \begin{array}{l} A = \frac{1}{2} \\ B = \frac{1}{2} \\ C = \frac{-1}{2} \end{array}$$

Usando el resultado en la integral obtenemos:

$$\begin{aligned}
\int \frac{x}{(1+x^2)(1+x)}dx &= \frac{1}{2}\left(\int \frac{1+x}{1+x^2}dx - \int \frac{dx}{1+x}\right) = \frac{1}{2}\left(\int \frac{x}{1+x^2}dx + \int \frac{dx}{1+x^2} - \int \frac{dx}{1+x}\right) \\
&= \frac{1}{2}\left(\int \frac{x}{1+x^2}dx + \arctan(x) - \ln|1+x|\right) \qquad ; u = 1+x^2 \to du = 2xdx \\
&= \frac{1}{2}\left(\int \frac{du}{2u}dx + \arctan(x) - \ln|1+x|\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\ln|u| + \arctan(x) - \ln|1+x|\right) + c \\
&= \frac{1}{4}\left(\ln\left(\frac{1+x^2}{(1+x)^2}\right) + 2\arctan x\right) + c
\end{aligned}$$

P5.b)

$$\int \frac{\sin(x)}{1+\sin(x)+\cos(x)}dx \qquad ; \text{usamos el cambio de variable } u = \tan\left(\frac{x}{2}\right)$$

$$\begin{aligned}
&= \int \frac{2u}{(1+u^2)+2u+(1-u^2)} \cdot \frac{2du}{1+u^2} = 2\int \frac{udu}{(1+u)(1+u^2)} \\
&= 2\cdot\frac{1}{4}\left(\ln\left(\frac{1+u^2}{(1+u)^2}\right) + 2\arctan u\right) + c \qquad \text{[Por parte 5.a]} \\
&= \frac{1}{2}\left(\ln\left(\frac{1+\tan\left(\frac{x}{2}\right)^2}{(1+\tan\left(\frac{x}{2}\right))^2}\right) + x\right) + c
\end{aligned}$$

P5.c)

$$\int \operatorname{arc\,sen}\left(\sqrt{\frac{x}{1+x}}\right)dx \qquad ; \begin{bmatrix} w = \operatorname{arc\,sen}\left(\sqrt{\frac{x}{1+x}}\right) \to dw = \frac{dx}{2\sqrt{x}(x+1)} \\ dv = dx \to v = x \end{bmatrix}$$

$$= x\operatorname{arc\,sen}\left(\sqrt{\frac{x}{1+x}}\right) - \int \frac{xdx}{2\sqrt{x}(x+1)}$$

Calculemos esta última integral:

$$\begin{aligned}
&\int \frac{xdx}{2\sqrt{x}(x+1)} \qquad ; u = \sqrt{x} \to du = \frac{dx}{2\sqrt{x}} \\
&= \int \frac{u^2}{1+u^2}du = \int \left(1 - \frac{1}{1+u^2}\right)du \\
&= u - \arctan(u) + c = \sqrt{x} - \arctan(\sqrt{x}) + c
\end{aligned}$$

Juntando todo obtenemos el resultado final:

$$\int \operatorname{arc\,sen}\left(\sqrt{\frac{x}{1+x}}\right)dx = x\operatorname{arc\,sen}\left(\sqrt{\frac{x}{1+x}}\right) - \sqrt{x} + \arctan(\sqrt{x}) + c$$