# Chapter 4

# Transverse waves on a string

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In the previous three chapters, we built up the foundation for our study of waves. In the remainder of this book, we'll investigate various types of waves, such as waves on a string, sound waves, electromagnetic waves, water waves, quantum mechanical waves, and so on. In Chapters 4 through 6, we'll discuss the properties of the two basic categories of waves, namely *dispersive* waves, and *non-dispersive* waves. The rest of the book is then largely a series of applications of these results. Chapters 4 through 6 therefore form the heart of this book.

A non-dispersive system has the property that all waves travel with the same speed, independent of the wavelength and frequency. These waves are the subject of this and the following chapter (broken up into longitudinal and transverse waves, respectively). A dispersive system has the property that the speed of a wave *does* depend on the wavelength and frequency. These waves are the subject of Chapter 6. They're a bit harder to wrap your brain around, the main reason being the appearance of the so-called *group velocity.* As we'll see in Chapter 6, the difference between non-dispersive and dispersive waves boils down to the fact that for non-dispersive waves, the frequency $\omega$ and wavelength $k$ are related by a simple proportionality constant, whereas this is not the case for dispersive waves.

The outline of this chapter is as follows. In section 4.1 we derive the wave equation for transverse waves on a string. This equation will take exactly the same form as the wave equation we derived for the spring/mass system in Section 2.4, with the only difference being the change of a few letters. In Section 4.2 we discuss the reflection and transmission of a wave from a boundary. We will see that various things can happen, depending on exactly what the boundary looks like. In Section 4.3 we introduce the important concept of *impedance* and show how our previous results can be written in terms of it. In Section 4.4 we talk about the energy and power carried by a wave. In Section 4.5 we calculate the form of standing waves on a string that has boundary conditions that fall into the extremes (a fixed end or a "free" end). In Section 4.6 we introduce damping, and we see how the amplitude of a wave decreases with distance in a scenario where one end of the string is wiggled with a constant amplitude.

## 4.1 The wave equation

The most common example of a non-dispersive system is a string with transverse waves on it. We'll see below that we obtain essentially the same wave equation for *transverse* waves on a string as we obtained for the $N \to \infty$ limit of *longitudinal* waves in the mass/spring system in Section 2.4. Either of these waves could therefore be used for our discussion of the properties of non-dispersive systems. However, the reason why we've chosen to study transverse waves on a string in this chapter is that transverse waves are generally easier to visualize than longitudinal ones.

Consider a string with tension $T$ and mass density $\mu$ (per unit length). Assume that it is infinitesimally thin and completely flexible. And assume for now that it extends infinitely in both directions. We'll eventually relax this restriction. Consider *small* transverse displacements of the string (we'll be quantitative about the word "small" below). Let $x$ be the coordinate along the string, and let $\psi$ be the transverse displacement. (There's no deep reason why we're using $\psi$ for the displacement instead of the $\xi$ we used in Chapter 2.) Our goal is to find the most general form of $\psi(x, t)$.

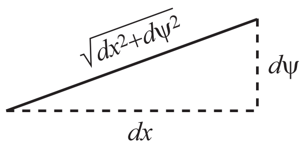



**Figure 1**

Consider two nearby points separated by a displacement $dx$ in the longitudinal direction along the string, and by a displacement $d\psi$ in the transverse direction. If $d\psi$ is small (more precisely if the slope $d\psi/dx$ is small; see below), then we can make the approximation that all points in the string move only in the transverse direction. That is, there is no longitudinal motion. This is true because in Fig. 1 the length of the hypotenuse equals

$$\sqrt{dx^2 + d\psi^2} = dx\sqrt{1 + \left(\frac{d\psi}{dx}\right)^2} \approx dx\left(1 + \frac{1}{2}\left(\frac{d\psi}{dx}\right)^2\right) = dx + d\psi\frac{1}{2}\left(\frac{d\psi}{dx}\right). \tag{1}$$

This length (which is the farthest that a given point can move to the side; it's generally less that this) differs from the length of the long leg in Fig. 1 by an amount $d\psi(d\psi/dx)/2$, which is only $(d\psi/dx)/2$ times as large as the transverse displacement $d\psi$. Since we are assuming that the slope $d\psi/dx$ is small, we can neglect the longitudinal motion in comparison with the transverse motion. Hence, all points essentially move only in the transverse direction. We can therefore consider each point to be labeled with a unique value of $x$. That is, the ambiguity between the original and present longitudinal positions is irrelevant. The string will stretch slightly, but we can always assume that the amount of mass in any given horizontal span stays essentially constant.

We see that by the phrase "small transverse displacements" we used above, we mean that the *slope* of the string is small. The slope is a dimensionless quantity, so it makes sense to label it with the word "small." It makes no sense to say that the actual transverse displacement is small, because this quantity has dimensions.

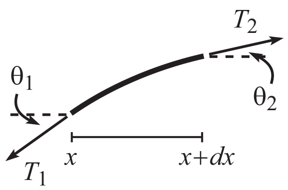



**Figure 2**

Our strategy for finding the wave equation for the string will be to write down the transverse $F = ma$ equation for a little piece of string in the span from $x$ to $x + dx$. The situation is shown in Fig. 2. (We'll ignore gravity here.) Let $T_1$ and $T_2$ be the tensions in the string at the ends of the small interval. Since the slope $d\psi/dx$ is small, the slope is essentially equal to the $\theta$ angles in the figure. (So these angles are small, even though we've drawn them with reasonable sizes for the sake of clarity.) We can use the approximation $\cos\theta \approx 1 - \theta^2/2$ to say that the longitudinal components of the tensions are equal to the tensions themselves, up to small corrections of order $\theta^2 \approx (d\psi/dx)^2$. So the longitudinal components are (essentially) equal to $T_1$ and $T_2$. Additionally, from the above reasoning concerning (essentially) no longitudinal motion, we know that there is essentially no longitudinal acceleration of the little piece in Fig. 2. So the longitudinal forces must cancel. We therefore conclude that $T_1 = T_2$. Let's call this common tension $T$.

However, although the two tensions and their longitudinal components are all equal, the same thing *cannot* be said about the transverse components. The transverse components differ by a quantity that is *first* order in $d\psi/dx$, and this difference can't be neglected. This difference is what causes the transverse acceleration of the little piece, and it can be calculated as follows.

In Fig. 2, the "upward" transverse force on the little piece at its right end is $T\sin\theta_1$, which essentially equals $T$ times the slope, because the angle is small. So the upward force at the right end is $T\psi'(x+dx)$. Likewise, the "downward" force at the left end is $-T\psi'(x)$. The net transverse force is therefore

$$F_{\text{net}} = T\Big(\psi'(x+dx) - \psi'(x)\Big) = T\,dx\frac{\psi'(x+dx)-\psi'(x)}{dx} \equiv T\,dx\frac{d^2\psi(x)}{dx^2}\,, \qquad (2)$$

where we have assumed that $dx$ is infinitesimal and used the definition of the derivative to obtain the third equality.[1] Basically, the difference in the first derivatives yields the second derivative. For the specific situation shown in Fig. 2, $d^2\psi/dx^2$ is negative, so the piece is accelerating in the downward direction. Since the mass of the little piece is $\mu\,dx$, the transverse $F = ma$ equation is

$$F_{\text{net}} = ma \quad\Longrightarrow\quad T\,dx\frac{d^2\psi}{dx^2} = (\mu\,dx)\frac{d^2\psi}{dt^2} \quad\Longrightarrow\quad \frac{d^2\psi}{dt^2} = \frac{T}{\mu}\frac{d^2\psi}{dx^2}\,. \qquad (3)$$

Since $\psi$ is a function of $x$ and $t$, let's explicitly include this dependence and write $\psi$ as $\psi(x,t)$. We then arrive at the desired wave equation (written correctly with partial derivatives now),

$$\boxed{\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2\psi(x,t)}{\partial x^2}} \qquad (\textit{wave equation}) \qquad (4)$$

This takes exactly the same form as the wave equation we found in Section 2.4 for the $N \to \infty$ limit of the spring/mass system. The only difference is the replacement of the quantity $E/\rho$ with the quantity $T/\mu$. Therefore, *all* of our previous results carry over here. In particular, the solutions take the form,

$$\boxed{\psi(x,t) = Ae^{i(\pm kx \pm \omega t)}} \qquad \text{where} \qquad \boxed{\frac{\omega}{k} = \sqrt{\frac{T}{\mu}} \equiv c} \qquad (5)$$

and where $c$ is the speed of the traveling wave (from here on, we'll generally use $c$ instead of $v$ for the speed of the wave). This form does indeed represent a traveling wave, as we saw in Section 2.4. $k$ and $\omega$ can take on any values, as long as they're related by $\omega/k = c$. The wavelength is $\lambda = 2\pi/k$, and the time period of the oscillation of any given point is $\tau = 2\pi/\omega$. So the expression $\omega/k = c$ can be written as

$$\frac{2\pi/\tau}{2\pi/\lambda} = c \quad\Longrightarrow\quad \frac{\lambda}{\tau} = c \quad\Longrightarrow\quad \lambda\nu = c, \qquad (6)$$

where $\nu = 1/\tau$ is the frequency in cycles per second (Hertz).

For a given pair of $k$ and $\omega$ values, the most general form of $\psi(x,t)$ can be written in many ways, as we saw in Section 2.4. From Eqs. (3.91) and (3.92) a couple of these ways are

$$\psi(x,t) = C_1\cos(kx+\omega t) + C_2\sin(kx+\omega t) + C_3\cos(kx-\omega t) + C_4\sin(kx-\omega t), \qquad (7)$$

and

$$\psi(x,t) = D_1\cos kx\cos\omega t + D_2\sin kx\sin\omega t + D_3\sin kx\cos\omega t + D_4\cos kx\sin\omega t. \qquad (8)$$

Of course, since the wave equation is linear, the most general solution is the sum of an arbitrary number of the expressions in, say, Eq. (8), for different pairs of $k$ and $\omega$ values (as long as each pair is related by $\omega/k = \sqrt{T/\mu} \equiv c$). For example, a possible solution is

$$\psi(x,t) = A\cos k_1x\cos ck_1t + B\cos k_1x\sin ck_1t + C\sin k_2x\sin ck_2t + \text{etc}\ldots \qquad (9)$$

[1] There is an ambiguity about whether we should write $\psi''(x)$ or $\psi''(x+dx)$ here. Or perhaps $\psi''(x+dx/2)$. But this ambiguity is irrelevant in the $dx \to 0$ limit.

**Solutions of the form $f(x-ct)$**

As we saw in Section 2.4, *any* function of the form $f(x-ct)$ satisfies the wave equation. There are two reasons why this functional form works. The first reason, as we showed in Eq. (2.97), is that if you simply plug $\psi(x,t) = f(x-ct)$ into the wave equation in Eq. (4), you will find that it works, provided that $c = \sqrt{T/\mu}$.

The second reason, as we also mentioned in Section 2.4, is due to Fourier analysis and linearity. Let's talk a little more about this reason. It is the combination of the following facts:

1. By Fourier analysis, we can write any function $f(z)$ as
$$f(z) = \int_{-\infty}^{\infty} C(k)e^{ikz}dk, \tag{10}$$
where $C(k)$ is given by Eq. (3.43). If we let $z \equiv x - ct$, then this becomes
$$f(x-ct) = \int_{-\infty}^{\infty} C(k)e^{ik(x-ct)}dk = \int_{-\infty}^{\infty} C(k)e^{i(kx-\omega t)}dk, \tag{11}$$
where $\omega \equiv ck$.

2. We showed in Section 2.4 (and you can quickly verify it again) that any exponential function of the form $e^{i(kx-\omega t)}$ satisfies the wave equation, provided that $\omega = ck$, which is indeed the case here.

3. Because the wave equation is linear, any linear combination of solutions is again a solution. Therefore, the integral in Eq. (11) (which is a continuous linear sum) satisfies the wave equation.

In this reasoning, it is critical that $\omega$ and $k$ are related linearly by $\omega = ck$, where $c$ takes on a constant value, independent of $\omega$ and $k$ (and it must be equal to $\sqrt{T/\mu}$, or whatever constant appears in the wave equation). If this relation weren't true, then the above reasoning would be invalid, for reasons we will shortly see. When we get to dispersive waves in Chapter 6, we will find that $\omega$ does *not* equal $ck$. In other words, the ratio $\omega/k$ depends on $k$ (or equivalently, on $\omega$). Dispersive waves therefore cannot be written in the form of $f(x-ct)$. It is instructive to see exactly where the above reasoning breaks down when $\omega \neq ck$. This breakdown can be seen in mathematically, and also physically.

Mathematically, it is still true that any function $f(x-ct)$ can be written in the form of $\int_{-\infty}^{\infty} C(k)e^{ik(x-ct)}dk$. However, it is *not* true that these $e^{i(kx-(ck)t)}$ exponential functions are solutions to a dispersive wave equation (we'll see in Chapter 6 what such an equation might look like), because $\omega$ doesn't take the form of $ck$ for dispersive waves. The actual solutions to a dispersive wave equation are exponentials of the form $e^{i(kx-\omega t)}$, where $\omega$ is some *nonlinear* function of $k$. That is, $\omega$ does *not* take the form of $ck$. If you want, you can write these exponential solutions as $e^{i(kx-c_k kt)}$, where $c_k \equiv \omega/k$ is the speed of the wave component with wavenumber $k$. But the point is that *a single value* of $c$ doesn't work for all values of $k$.

In short, if by the $\omega$ in Eq. (11) we mean $\omega_k$ (which equals $ck$ for nondispersive waves, but not for dispersive waves), then for dispersive waves, the above reasoning breaks down in the second equality in Eq. (11), because the coefficient of $t$ in the first integral is $ck$ (times $-i$), which isn't equal to the $\omega$ coefficient in the second integral. If on the other hand we want to keep the $\omega$ in Eq. (11) defined as $ck$, then for dispersive waves, the above reasoning breaks down in step 2. The exponential function $e^{i(kx-\omega t)}$ with $\omega = ck$ is simply not a solution to a dispersive wave equation.

The physical reason why the $f(x-ct)$ functional form doesn't work for dispersive waves is the following. Since the speed $c_k$ of the Fourier wave components depends on $k$ in a dispersive wave, the wave components move with different speeds. The shape of the wave at some initial time will therefore not be maintained as $t$ increases (assuming that the wave contains more than a single Fourier component). This implies that the wave cannot be written as $f(x-ct)$, because this wave *does* keep the same shape (because the argument $x-ct$ doesn't change if $t$ increases by $\Delta t$ and $x$ increases by $c\,\Delta t$).

The distortion of the shape of the wave can readily be seen in the case where there are just two wave components, which is much easier to visualize than the continuous infinity of components involved in a standard Fourier integral. If the two waves have $(k,\omega)$ values of $(1,1)$ and $(2,1)$, then since the speed is $\omega/k$, the second wave moves with half the speed of the first. Fig. 3 shows the sum of these two waves at two different times. The total wave clearly doesn't keep the same shape, so it therefore can't be written in the form of $f(x-ct)$.

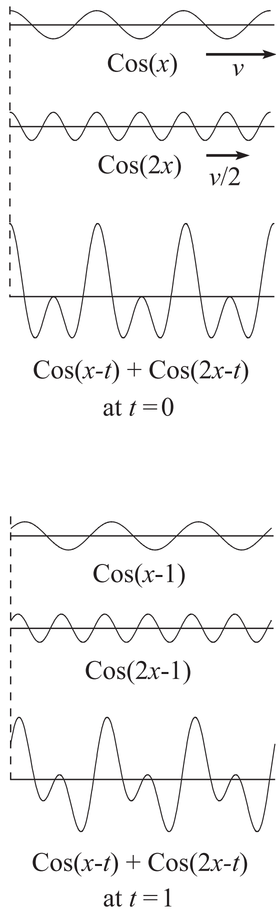



**Figure 3**

**Fourier transform in 2-D**

Having learned about Fourier transforms in Chapter 3, we can give another derivation of the fact that any solution to the wave equation in Eq. (4) can be written in terms of $e^{i(kx-\omega t)}$ exponentials, where $\omega = ck$. We originally derived these solutions in Section 2.4 by guessing exponentials, with the reasoning that since all (well enough behaved) functions can be built up from exponentials due to Fourier analysis, it suffices to consider exponentials. However, you might still feel uneasy about this "guessing" strategy, so let's be a little more systematic with the following derivation. This derivation involves looking at the Fourier transform of a function of two variables. In Chapter 3, we considered functions of only one variable, but the extension to two variables in quite straightforward.

Consider a wave $\psi(x,t)$ on a string, and take a snapshot at a given time. If $f(x)$ describes the wave at this instant, then from 1-D Fourier analysis we can write

$$f(x) = \int_{-\infty}^{\infty} C(k) e^{ikx} dk, \tag{12}$$

where $C(k)$ is given by Eq. (3.43). If we take a snapshot at a slightly later time, we can again write $\psi(x,t)$ in terms of its Fourier components, but the coefficients $C(k)$ will be slightly different. In other words, the $C(k)$'s are functions of time. So let's write them as $C(k,t)$. In general, we therefore have

$$\psi(x,t) = \int_{-\infty}^{\infty} C(k,t) e^{ikx} dk. \tag{13}$$

This equation says that at any instant we can decompose the snapshot of the string into its $e^{ikx}$ Fourier components.

We can now do the same thing with the $C(k,t)$ function that we just did with the $\psi(x,t)$ function. But we'll now consider "slices" with constant $k$ value instead of constant $t$ value. If $g(t)$ describes the function $C(k,t)$ for a particular value of $k$, then from 1-D Fourier analysis we can write

$$g(t) = \int_{-\infty}^{\infty} \beta(\omega) e^{i\omega t} d\omega. \tag{14}$$

If we consider a slightly different value of $k$, we can again write $C(k,t)$ in terms of its Fourier components, but the coefficients $\beta(\omega)$ will be slightly different. That is, they are functions of $k$, so let's write them as $\beta(k,\omega)$. In general, we have

$$C(k,t) = \int_{-\infty}^{\infty} \beta(k,\omega) e^{i\omega t} d\omega. \tag{15}$$

This equation says that for a given value of $k$, we can decompose the function $C(k,t)$ into its $e^{i\omega t}$ Fourier components. Plugging this expression for $C(k,t)$ into Eq. (13) gives

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\beta(k,\omega)e^{i\omega t}d\omega\right)e^{ikx}dk \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta(k,\omega)e^{i(kx+\omega t)}dk\,d\omega. \qquad (16)\end{aligned}$$

This is a general result for *any* function of two variables; it has nothing to do with the wave equation in Eq. (4). This result basically says that we can just take the Fourier transform in each dimension separately.

Let's now apply this general result to the problem at hand. That is, let's plug the above expression for $\psi(x,t)$ into Eq. (4) and see what it tells us. We will find that $\omega$ must be equal to $ck$. The function $\beta(k,\omega)$ is a constant as far as the $t$ and $x$ derivatives in Eq. (4) are concerned, so we obtain

$$\begin{aligned}0 &= \frac{\partial^2\psi(x,t)}{\partial t^2} - c^2\frac{\partial^2\psi(x,t)}{\partial x^2} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta(k,\omega)\left(\frac{\partial^2 e^{i(kx+\omega t)}}{\partial t^2} - c^2\frac{\partial^2 e^{i(kx+\omega t)}}{\partial x^2}\right)dk\,d\omega \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta(k,\omega)e^{i(kx+\omega t)}\Big(-\omega^2 - c^2(-k^2)\Big)\,dk\,d\omega. \qquad (17)\end{aligned}$$

Since the $e^{i(kx+\omega t)}$ exponentials here are linearly independent functions, the only way that this sum can be zero for all $x$ and $t$ is if the coefficient of each separate $e^{i(kx+\omega t)}$ term is zero. That is,

$$\beta(k,\omega)(\omega^2 - c^2k^2) = 0 \qquad (18)$$

for all values of $k$ and $\omega$.[2] There are two ways for this product to be zero. First, we can have $\beta(k,\omega) = 0$ for particular values of $k$ and $\omega$. This certainly works, but since $\beta(k,\omega)$ indicates how much of $\psi(x,t)$ is made up of $e^{i(kx+\omega t)}$ for these particular values of $k$ and $\omega$, the $\beta(k,\omega) = 0$ statement tells us that this particular $e^{i(kx+\omega t)}$ exponential doesn't appear in $\psi(x,t)$. So we don't care about how $\omega$ and $k$ are related.

The other way for Eq. (18) to be zero is if $\omega^2 - c^2k^2 = 0$. That is, $\omega = \pm ck$, as we wanted to show. We therefore see that if $\beta(k,\omega)$ is nonzero for particular values of $k$ and $\omega$ (that is, if $e^{i(kx+\omega t)}$ appears in $\psi(x,t)$), then $\omega$ must be equal to $\pm ck$, if we want the wave equation to be satisfied.

## 4.2 Reflection and transmission

### 4.2.1 Applying the boundary conditions

Instead of an infinite uniform string, let's now consider an infinite string with density $\mu_1$ for $-\infty < x < 0$ and $\mu_2$ for $0 < x < \infty$. Although the density isn't uniform, the tension is still uniform throughout the entire string, because otherwise there would be a nonzero horizontal acceleration somewhere.

Assume that a wave of the form $\psi_{\rm i}(x,t) = f_{\rm i}(x - v_1 t)$ (the "i" here is for "incident") starts off far to the left and heads rightward toward $x = 0$. It turns out that it will be much

[2] Alternatively, Eq. (17) says that $\beta(k,\omega)(\omega^2 - c^2k^2)$ is the 2-D Fourier transform of zero. So it must be zero, because it can be found from the 2-D inverse-transform relations analogous to Eq. (3.43), with a zero appearing in the integrand.

more convenient to instead write the wave as

$$\psi_{\mathrm{i}}(x,t) = f_{\mathrm{i}}\left(t - \frac{x}{v_1}\right) \tag{19}$$

for a redefined function $f_{\mathrm{i}}$, so we'll use this form. Note that $\psi$ is a function of two variables, whereas $f$ is a function of only one. From Eq. (5), the speed $v_1$ equals $\sqrt{T/\mu_1}$.

What happens when the wave encounters the boundary at $x = 0$ between the different densities? The most general thing that can happen is that there is some reflected wave,

$$\psi_{\mathrm{r}}(x,t) = f_{\mathrm{r}}\left(t + \frac{x}{v_1}\right), \tag{20}$$

moving leftward from $x = 0$, and also a transmitted wave,

$$\psi_{\mathrm{t}}(x,t) = f_{\mathrm{t}}\left(t - \frac{x}{v_2}\right), \tag{21}$$

moving rightward from $x = 0$ (where $v_2 = \sqrt{T/\mu_2}$). Note the "+" sign in the argument of $f_{\mathrm{r}}$, since the reflected wave is moving leftward.

In terms of the above functions, the complete expressions for the waves on the left and right side of $x = 0$ are, respectively,

$$\begin{aligned} \psi_{\mathrm{L}}(x,t) &= \psi_{\mathrm{i}}(x,t) + \psi_{\mathrm{r}}(x,t) = f_{\mathrm{i}}(t - x/v_1) + f_{\mathrm{r}}(t + x/v_1), \\ \psi_{\mathrm{R}}(x,t) &= \psi_{\mathrm{t}}(x,t) = f_{\mathrm{t}}(t - x/v_2). \end{aligned} \tag{22}$$

If we can find the reflected and transmitted waves in terms of the incident wave, then we will know what the complete wave looks like everywhere. Our goal is therefore to find $\psi_{\mathrm{r}}(x,t)$ and $\psi_{\mathrm{t}}(x,t)$ in terms of $\psi_{\mathrm{i}}(x,t)$. To do this, we will use the two boundary conditions at $x = 0$. Using Eq. (22) to write the waves in terms of the various $f$ functions, the two boundary conditions are:

- The string is continuous. So we must have (for all $t$)

$$\psi_{\mathrm{L}}(0,t) = \psi_{\mathrm{R}}(0,t) \quad \Longrightarrow \quad \boxed{f_{\mathrm{i}}(t) + f_{\mathrm{r}}(t) = f_{\mathrm{t}}(t)} \tag{23}$$

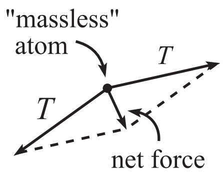


**Figure 4**

- The slope is continuous. This is true for the following reason. If the slope were different on either side of $x = 0$, then there would be a net (non-infinitesimal) force in some direction on the atom located at $x = 0$, as shown in Fig. 4. This (nearly massless) atom would then experience an essentially infinite acceleration, which isn't physically possible. (Equivalently, a nonzero force would have the effect of instantaneously readjusting the string to a position where the slope was continuous.) Continuity of the slope gives (for all $t$)

$$\left.\frac{\partial \psi_{\mathrm{L}}(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_{\mathrm{R}}(x,t)}{\partial x}\right|_{x=0} \quad \Longrightarrow \quad -\frac{1}{v_1} f_{\mathrm{i}}'(t) + \frac{1}{v_1} f_{\mathrm{r}}'(t) = -\frac{1}{v_2} f_{\mathrm{t}}'(t). \tag{24}$$

Integrating this and getting the $v$'s out of the denominators gives

$$\boxed{v_2 f_{\mathrm{i}}(t) - v_2 f_{\mathrm{r}}(t) = v_1 f_{\mathrm{t}}(t)} \tag{25}$$

We have set the constant of integration equal to zero because we are assuming that the string has no displacement before the wave passes by.

Solving Eqs. (23) and (25) for $f_{\rm r}(t)$ and $f_{\rm t}(t)$ in terms of $f_{\rm i}(t)$ gives

$$f_{\rm r}(s) = \frac{v_2 - v_1}{v_2 + v_1} f_{\rm i}(s), \qquad \text{and} \qquad f_{\rm t}(s) = \frac{2v_2}{v_2 + v_1} f_{\rm i}(s), \tag{26}$$

where we have written the argument as $s$ instead of $t$ to emphasize that these relations hold for *any* arbitrary argument of the $f$ functions. The argument need not have anything to do with the time $t$. The $f$'s are functions of one variable, and we've chosen to call that variable $s$ here.

### 4.2.2 Reflection

We derived the relations in Eq. (26) by considering how the $\psi(x,t)$'s relate at $x = 0$. But how do we relate them at other $x$ values? We can do this in the following way. Let's look at the reflected wave first. If we replace the $s$ in Eq. (26) by $t + x/v_1$ (which we are free to do, because the argument of the $f$'s can be whatever we want it to be), then we can write $f_{\rm r}$ as

$$f_{\rm r}\left(t + \frac{x}{v_1}\right) = \frac{v_2 - v_1}{v_2 + v_1} f_{\rm i}\left(t - \frac{-x}{v_1}\right). \tag{27}$$

If we recall the definition of the $f$'s in Eqs. (19-21), we can write this result in terms of the $\psi$'s as

$$\boxed{\psi_{\rm r}(x,t) = \frac{v_2 - v_1}{v_2 + v_1}\psi_{\rm i}(-x,t)} \tag{28}$$

This is the desired relation between $\psi_{\rm r}$ and $\psi_{\rm i}$, and its interpretation is the following. It says that at a given time $t$, the value of $\psi_{\rm r}$ at position $x$ equals $(v_2 - v_1)/(v_2 + v_1)$ times the value of $\psi_{\rm i}$ at position *negative* $x$. This implies that the speed of the $\psi_{\rm r}$ wave equals the speed of the $\psi_{\rm i}$ wave (but with opposite velocity), and it also implies that the width of the $\psi_{\rm r}$ wave equals the width of the $\psi_{\rm i}$ wave. But the height is decreased by a factor of $(v_2 - v_1)/(v_2 + v_1)$.

Only negative values of $x$ are relevant here, because we are dealing with the reflected wave which exists only to the left of $x = 0$. Therefore, since the expression $\psi_{\rm i}(-x,t)$ appears in Eq. (28), the $-x$ here means that only *positive* position coordinates are relevant for the $\psi_{\rm i}$ wave. You might find this somewhat disconcerting, because the $\psi_{\rm i}$ function isn't applicable to the right of $x = 0$. However, we can mathematically imagine $\psi_{\rm i}$ still proceeding to the right. So we have the pictures shown in Fig. 5. For simplicity, let's say that $v_2 = 3v_1$, which means that the $(v_2 - v_1)/(v_2 + v_1)$ factor equals 1/2. Note that in any case, this factor lies between $-1$ and 1. We'll talk about the various possibilities below.

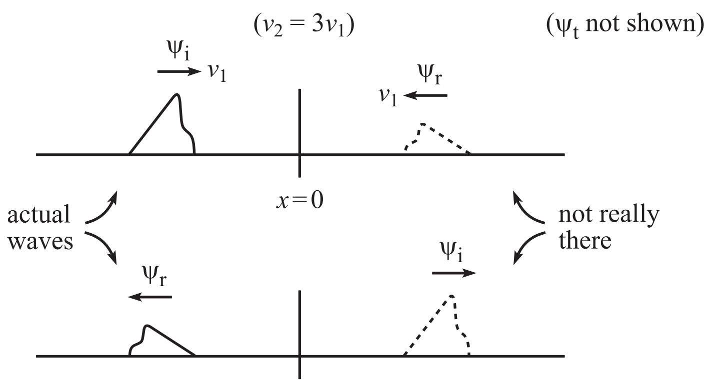


**Figure 5**

In the first picture in Fig. 5, the incident wave is moving in from the left, and the reflected wave is moving in from the right. The reflected wave doesn't actually exist to the right of $x = 0$, of course, but it's convenient to imagine it coming in as shown. Except for the scale factor of $(v_2 - v_1)/(v_2 + v_1)$ in the vertical direction (only), $\psi_\text{r}$ is simply the mirror image of $\psi_\text{i}$.

In the second picture in Fig. 5, the incident wave has passed the origin and continues moving to the right, where it doesn't actually exist. But the reflected wave is now located on the left side of the origin and moves to the left. This is the real piece of the wave. For simplicity, we haven't shown the transmitted $\psi_\text{t}$ wave in these pictures (we'll deal with it below), but it's technically also there.

In between the two times shown in Fig. 5, things aren't quite as clean, because there are $x$ values near the origin (to the left of it) where both $\psi_\text{i}$ and $\psi_\text{r}$ are nonzero, and we need to add them to obtain the complete wave, $\psi_\text{L}$, in Eq. (22). But the procedure is straightforward in principle. The two $\psi_\text{i}$ and $\psi_\text{r}$ waves simply pass through each other, and the value of $\psi_\text{L}$ at any point to the left of $x = 0$ is obtained by adding the values of $\psi_\text{i}$ and $\psi_\text{r}$ at that point. Remember that only the region to the left of $x = 0$ is real, as far as the reflected wave is concerned. The wave to the right of $x = 0$ that is generated from $\psi_\text{i}$ and $\psi_\text{r}$ is just a convenient mathematical construct.

Fig. 6 shows some successive snapshots that result from an easy-to-visualize incident square wave. The bold-line wave indicates the actual wave that exists to the left of $x = 0$. We haven't drawn the transmitted wave to the right of $x = 0$. You should stare at this figure until it makes sense. This wave actually isn't physical; its derivative isn't continuous, so it violates the second of the above boundary conditions (although we can imagine rounding off the corners to eliminate this issue). Also, its derivative isn't small, which violates our assumption at the beginning of this section. However, we're drawing this wave so that the important features of reflection can be seen. Throughout this book, if we actually drew realistic waves, the slopes would be so small that it would be nearly impossible to tell what was going on.

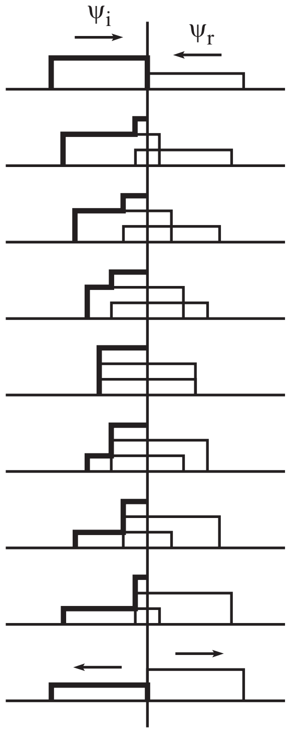



**Figure 6**

### 4.2.3 Transmission

Let's now look at the transmitted wave. If we replace the $s$ by $t - x/v_2$ in Eq. (26), we can write $f_\text{t}$ as

$$f_\text{t}\left(t - \frac{x}{v_2}\right) = \frac{2v_2}{v_2 + v_1} f_\text{i}\left(t - \frac{(v_1/v_2)x}{v_1}\right). \tag{29}$$

Using the definition of the $f$'s in Eqs. (19-21), we can write this in terms of the $\psi$'s as

$$\boxed{\psi_\text{t}(x, t) = \frac{2v_2}{v_2 + v_1}\psi_\text{i}\Big((v_1/v_2)x, t\Big)} \tag{30}$$

This is the desired relation between $\psi_\text{t}$ and $\psi_\text{i}$, and its interpretation is the following. It says that at a given time $t$, the value of $\psi_\text{t}$ at position $x$ equals $2v_2/(v_2 + v_1)$ times the value of $\psi_\text{i}$ at position $(v_1/v_2)x$. This implies that the speed of the $\psi_\text{t}$ wave is $v_2/v_1$ times the speed of the $\psi_\text{i}$ wave, and it also implies that the width of the $\psi_\text{t}$ wave equals $v_2/v_1$ times the width of the $\psi_\text{i}$ wave. These facts are perhaps a little more obvious if we write Eq. (30) as $\psi_\text{t}\big((v_2/v_1)x, t\big) = 2v_2/(v_2 + v_1) \cdot \psi_\text{i}(x, t)$.

Only positive values of $x$ are relevant here, because we are dealing with the transmitted wave which exists only to the right of $x = 0$. Therefore, since the expression $\psi_\text{i}\big((v_1/v_2)x, t\big)$ appears in Eq. (30), only positive position coordinates are relevant for the $\psi_\text{i}$ wave. As in the case of reflection above, although the $\psi_i$ function isn't applicable for positive $x$, we can mathematically imagine $\psi_i$ still proceeding to the right. The situation is shown in Fig. 7. As above, let's say that $v_2 = 3v_1$, which means that the $2v_2/(v_2 + v_1)$ factor equals 3/2. Note that in any case, this factor lies between 0 and 2. We'll talk about the various possibilities below.

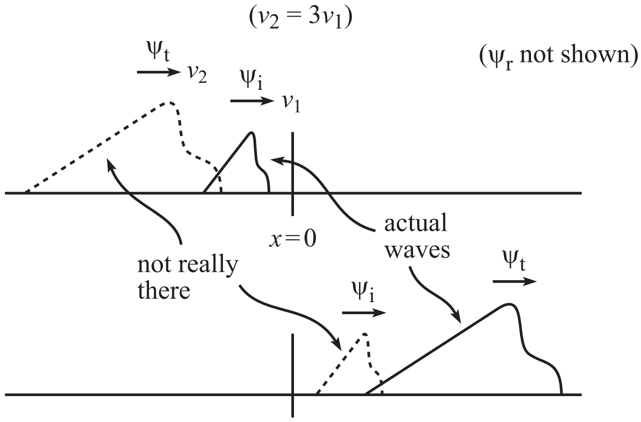



**Figure 7**

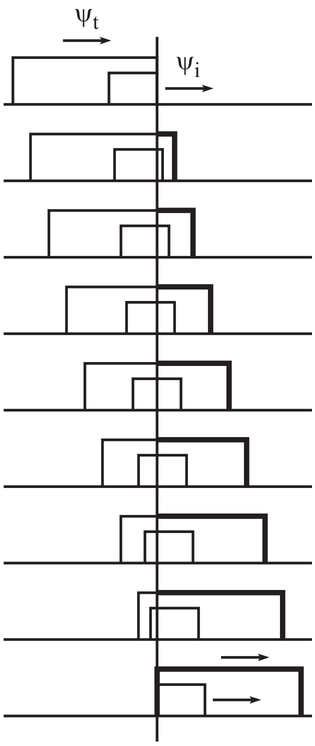



**Figure 8**

In the first picture in Fig. 7, the incident wave is moving in from the left, and the transmitted wave is also moving in from the left. The transmitted wave doesn't actually exist to the left of $x = 0$, of course, but it's convenient to imagine it coming in as shown. With $v_2 = 3v_1$, the transmitted wave is 3/2 as tall and 3 times as wide as the incident wave.

In the second picture in Fig. 7, the incident wave has passed the origin and continues moving to the right, where it doesn't actually exist. But the transmitted wave is now located on the right side of the origin and moves to the right. This is the real piece of the wave. For simplicity, we haven't shown the reflected $\psi_r$ wave in these pictures, but it's technically also there.

In between the two times shown in Fig. 7, things are easier to deal with than in the reflected case, because we don't need to worry about taking the sum of two waves. The transmitted wave consists only of $\psi_t$. We don't have to add on $\psi_i$ as we did in the reflected case. In short, $\psi_L$ equals $\psi_i + \psi_r$, whereas $\psi_R$ simply equals $\psi_t$. Equivalently, $\psi_i$ and $\psi_r$ have physical meaning only to the left of $x = 0$, whereas $\psi_t$ has physical meaning only to the right of $x = 0$.

Fig. 8 shows some successive snapshots that result from the same square wave we considered in Fig. 6. The bold-line wave indicates the actual wave that exists to the right of $x = 0$. We haven't drawn the reflected wave to the left of $x = 0$. We've squashed the $x$ axis relative to Fig. 6, to make a larger region viewable. These snapshots are a bit boring compared with those in Fig. 6, because there is no need to add any waves. As far as $\psi_t$ is concerned on the right side of $x = 0$, what you see is what you get. The entire wave (on both sides of $x = 0$) is obtained by juxtaposing the bold waves in Figs. 6 and 8, after expanding Fig. 8 in the horizontal direction to make the unit sizes the same (so that the $\psi_i$ waves have the same width).

### 4.2.4 The various possible cases

For convenience, let's define the reflection and transmission coefficients as

$$\boxed{R \equiv \frac{v_2 - v_1}{v_2 + v_1} \qquad \text{and} \qquad T \equiv \frac{2v_2}{v_2 + v_1}} \tag{31}$$

With these definitions, we can write the reflected and transmitted waves in Eqs. (28) and (30) as

$$\begin{aligned} \psi_{\mathrm{r}}(x,t) &= R\psi_{\mathrm{i}}(-x,t), \\ \psi_{\mathrm{t}}(x,t) &= T\psi_{\mathrm{i}}\big((v_1/v_2)x,t\big). \end{aligned} \tag{32}$$

$R$ and $T$ are the amplitudes of $\psi_{\mathrm{r}}$ and $\psi_{\mathrm{t}}$ relative to $\psi_{\mathrm{i}}$. Note that $1 + R = T$ always. This is just the statement of continuity of the wave at $x = 0$.

Since $v = \sqrt{T/\mu}$, and since the tension $T$ is uniform throughout the string, we have $v_1 \propto 1/\sqrt{\mu_1}$ and $v_2 \propto 1/\sqrt{\mu_2}$. So we can alternatively write $R$ and $T$ in the terms of the densities on either side of $x = 0$:

$$\boxed{R \equiv \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \qquad \text{and} \qquad T \equiv \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}} \tag{33}$$

There are various cases to consider:

- BRICK WALL ON RIGHT: $\mu_2 = \infty$ $(v_2 = 0)$ $\Longrightarrow$ $R = -1,\ T = 0$. Nothing is transmitted, since $T = 0$. And the reflected wave has the same size as the incident wave, but is inverted due to the $R = -1$ value. This is shown in Fig. 9.

  The inverted nature of the wave isn't intuitively obvious, but it's believable for the following reason. When the wave encounters the wall, the wall pulls down on the string in Fig. 9. This downward force causes the string to overshoot the equilibrium position and end up in the inverted orientation. Of course, you may wonder why the downward force causes the string to overshoot the equilibrium position instead of, say, simply returning to the equilibrium position. But by the same token, you should wonder why a ball that collides elastically with a wall bounces back with the same speed, as opposed to ending up at rest.

  We can deal with both of these situations by invoking conservation of energy. Energy wouldn't be conserved if in the former case the string ended up straight, and if in the latter case the ball ended up at rest, because there would be zero energy in the final state. (The wall is "infinitely" massive, so it can't pick up any energy.)

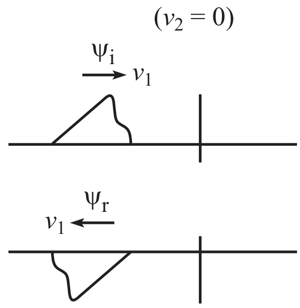



**Figure 9**

- LIGHT STRING ON LEFT, HEAVY STRING ON RIGHT: $\mu_1 < \mu_2 < \infty$ $(v_2 < v_1)$ $\Longrightarrow$ $-1 < R < 0,\ 0 < T < 1$. This case is in between the previous and following cases. There is partial (inverted) reflection and partial transmission. See Fig. 10 for the particular case where $\mu_2 = 4\mu_1 \Longrightarrow v_2 = v_1/2$. The reflection and transmission coefficients in this case are $R = -1/3$ and $T = 2/3$.

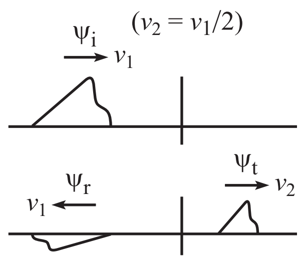



**Figure 10**

- UNIFORM STRING: $\mu_2 = \mu_1$ $(v_2 = v_1)$ $\Longrightarrow$ $R = 0,\ T = 1$. Nothing is reflected. The string is uniform, so the "boundary" point is no different from any other point. The wave passes right through, as shown in Fig. 11.

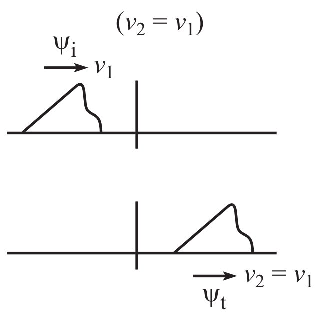



**Figure 11**

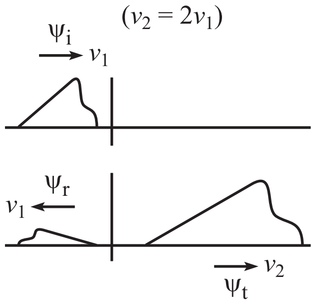



**Figure 12**

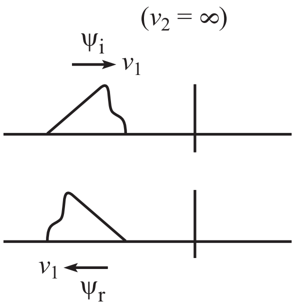



**Figure 13**

- Heavy string on left, light string on right: $0 < \mu_2 < \mu_1 \ (v_2 > v_1) \implies 0 < R < 1, \ 1 < T < 2$. This case is in between the previous and following cases. There is partial reflection and partial transmission. See Fig. 12 for the particular case where $\mu_2 = \mu_1/4 \Longrightarrow v_2 = 2v_1$. The reflection and transmission coefficients in this case are $R = 1/3$ and $T = 4/3$.

- Zero-mass string on right: $\mu_2 = 0 \ (v_2 = \infty) \implies R = 1, \ T = 2$. There is complete (rightside up) reflection in this case, as shown in Fig. 13. Although the string on the right side technically moves, it has zero mass so it can't carry any energy. All of the energy is therefore contained in the reflected wave (we'll talk about energy in Section 4.4). So in this sense there is total reflection. In addition to carrying no energy, the movement of the right part of the string isn't even wave motion. The whole thing always remains in a straight horizontal line and just rises and falls (technically it's a wave with infinite wavelength).[3]

  As with the brick-wall case above, the right-side-up nature of the wave isn't intuitively obvious, but it's believable for the following reason. When the wave encounters the boundary, the zero-mass string on the right side is always horizontal, so it can't apply any transverse force on the string on the left side. Since there is nothing pulling the string down, it can't end up on the other side of the equilibrium position as it did in the brick-wall case. The fact that it actually ends up with the same shape is a consequence of energy conservation, because the massless string on the right can't carry any energy.

## 4.3 Impedance

### 4.3.1 Definition of impedance

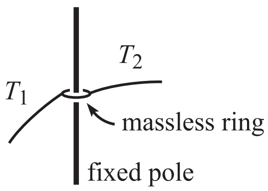



**Figure 14**

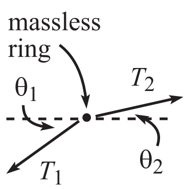



**Figure 15**

In the previous section, we allowed the density to change at $x = 0$, but we assumed that the tension was uniform throughout the string. Let's now relax this condition and allow the tension to also change at $x = 0$. The previous treatment is easily modified, and we will find that a new quantity, called the *impedance*, arises.

You may be wondering how we can arrange for the tension to change at $x = 0$, given that any difference in the tension should cause the atom at $x = 0$ to have "infinite" acceleration. But we can eliminate this issue by using the setup shown in Fig. 14. The boundary between the two halves of the string is a massless ring, and this ring encircles a fixed frictionless pole. The pole balances the difference in the longitudinal components of the two tensions, so the net longitudinal force on the ring is zero, consistent with the fact that it is constrained to remain on the pole and move only in the transverse direction.

The net *transverse* force on the massless ring must also be zero, because otherwise it would have infinite transverse acceleration. This zero-transverse-force condition is given by $T_1 \sin\theta_1 = T_2 \sin\theta_2$, where the angles are defined in Fig. 15. In terms of the derivatives on either side of $x = 0$, this relation can be written as (assuming, as we always do, that the slope of the string is small)

$$T_1 \frac{\partial \psi_{\mathrm{L}}(x,t)}{\partial x}\bigg|_{x=0} = T_2 \frac{\partial \psi_{\mathrm{R}}(x,t)}{\partial x}\bigg|_{x=0}. \tag{34}$$

[3]The right part of the string must be straight (and hence horizontal, so that it doesn't head off to $\pm\infty$) because if it were curved, then the nonzero second derivative would imply a nonzero force on a given piece of the string, resulting in infinite acceleration, because the piece is massless. Alternatively, the transmitted wave is stretched horizontally by a factor $v_2/v_1 = \infty$ compared with the incident wave. This implies that it is essentially horizontal.

In the case of uniform tension discussed in the previous section, we had $T_1 = T_2$, so this equation reduced to the first equality in Eq. (24). With the tensions now distinct, the only modification to the second equality in Eq. (24) is the extra factors of $T_1$ and $T_2$. So in terms of the $f$ functions, Eq. (34) becomes

$$-\frac{T_1}{v_1} f_{\rm i}'(t) + \frac{T_1}{v_1} f_{\rm r}'(t) = -\frac{T_2}{v_2} f_{\rm t}'(t). \tag{35}$$

The other boundary condition (the continuity of the string) is unchanged, so all of the results in the previous section can be carried over, with the only modification being that wherever we had a $v_1$, we now have $v_1/T_1$. And likewise for $v_2$. The quantity $v/T$ can be written as

$$\frac{v}{T} = \frac{\sqrt{T/\mu}}{T} = \frac{1}{\sqrt{T\mu}} \equiv \frac{1}{Z}\,, \tag{36}$$

where

$$\boxed{Z \equiv \frac{T}{v} = \sqrt{T\mu}} \tag{37}$$

is called the *impedance*. We'll discuss $Z$ in depth below, but for now we'll simply note that the results in the previous section are modified by replacing $v_1$ with $1/\sqrt{T_1\mu_1} \equiv 1/Z_1$, and likewise for $v_2$. The reflection and transmission coefficients in Eq. (31) therefore become

$$R = \frac{\frac{1}{Z_2} - \frac{1}{Z_1}}{\frac{1}{Z_2} + \frac{1}{Z_1}} = \boxed{\frac{Z_1 - Z_2}{Z_1 + Z_2}} \qquad \text{and} \qquad T = \frac{\frac{2}{Z_2}}{\frac{1}{Z_2} + \frac{1}{Z_1}} = \boxed{\frac{2Z_1}{Z_1 + Z_2}} \tag{38}$$

Note that $Z$ grows with both $T$ and $\mu$.

### Physical meaning of impedance

What is the meaning of the impedance? It makes our formulas look nice, but does it have any actual physical significance? Indeed it does. Consider the transverse force that the ring *applies* to the string on its *left*. Since there is zero net force on the ring, this force also equals the transverse force that the *right* string applies *to* the ring, which is

$$F_y = T_2 \frac{\partial \psi_{\rm R}(x,t)}{\partial x}\bigg|_{x=0} = T_2 \frac{\partial f_{\rm t}(t - x/v_2)}{\partial x}\bigg|_{x=0}, \tag{39}$$

where we have labeled the transverse direction as the $y$ direction. But the chain rule tells us that the $x$ and $t$ partial derivatives of $f_t$ are related by

$$\frac{\partial f_{\rm t}(t - x/v_2)}{\partial x} = -\frac{1}{v_2} \cdot \frac{\partial f_{\rm t}(t - x/v_2)}{\partial t}\,. \tag{40}$$

Substituting this into Eq. (39) and switching back to the $\psi_{\rm R}(x,t)$ notation gives

$$F_y = -\frac{T_2}{v_2} \cdot \frac{\partial \psi_{\rm R}(x,t)}{\partial t}\bigg|_{x=0} = -\frac{T_2}{v_2} \cdot v_y \equiv -b v_y, \tag{41}$$

where $v_y = \partial\psi_{\rm R}(x,t)/\partial t$ is the transverse velocity of the ring (at $x = 0$), and where $b$ is defined to be $T_2/v_2$.

This force $F_y$ (which again, is the force that the ring *applies* to the string on its *left*) has the interesting property that it is proportional to the (negative of the) transverse velocity. It therefore acts exactly like a *damping* force. If we removed the right string and replaced the ring with a massless plate immersed in a fluid (in other words, a piston), and if we arranged things (the thickness of the fluid and the cross-sectional area of the plate) so that the damping coefficient was $b$, then the left string wouldn't have any clue that the right string was removed and replaced with the damping mechanism. As far as the left string is concerned, the right string acts exactly like a resistance that is being dragged against.

Said in another way, if the left string is replaced by your hand, and if you move your hand so that the right string moves just as it was moving when the left string was there, then you can't tell whether you're driving the right string or driving a piston with an appropriately-chosen damping coefficient. This is true because by Newton's third law (and the fact that the ring is massless), the force that the ring applies to the string on the right is $+bv_y$. The plus sign signifies a driving force (that is, the force is doing positive work) instead of a damping force (where the force does negative work).

From Eq. (41), we have $F_y/v_y = -T_2/v_2$.[4] At different points in time, the ring has different velocities $v_y$ and exerts different forces $F_y$ on the left string. But the ratio $F_y/v_y$ is always equal to $-T_2/v_2$, which is constant, given $T_2$ and $\mu_2$ (since $v_2 = \sqrt{T_2/\mu_2}$). So, since $F_y/v_y = -T_2/v_2$ is constant in time, it makes sense to give it a name, and we call it the *impedance*, $Z$. This is consistent with the $Z \equiv T/v$ definition in Eq. (37). From Eq. (41), the impedance $Z$ is simply the damping coefficient $b$. Large damping therefore means large impedance, so the "impedance" name makes colloquial sense.

Since the impedance $Z \equiv T/v$ equals $\sqrt{T\mu}$ from Eq. (37), it is a property of the string itself (given $T$ and $\mu$), and not of a particular wave motion on the string. From Eq. (38) we see that if $Z_1 = Z_2$, then $R = 0$ and $T = 1$. In other words, there is total transmission. In this case we say that the strings are "impedance matched." We'll talk much more about this below, but for now we'll just note that there are many ways to make the impedances match. One way is to simply have the left and right strings be identical, that is, $T_1 = T_2$ and $\mu_1 = \mu_2$. In this case we have a uniform string, so the wave just moves merrily along and nothing is reflected. But $T_2 = 3T_1$ and $\mu_2 = \mu_1/3$ also yields matching impedances, as do an infinite number of other scenarios. All we need is for the product $T\mu$ to be the same in the two pieces, and then the impedances match and everything is transmitted. However, in these cases, it isn't obvious that there is no reflected wave, as it was for the uniform string. The reason for zero reflection is that the left string can't tell the difference between an identical string on the right, or a piston with a damping coefficient of $\sqrt{T_1\mu_1}$, or a string with $T_2 = 3T_1$ and $\mu_2 = \mu_1/3$. They all feel exactly the same, as far as the left string is concerned; they all generate a transverse force of the form, $F_y = -\sqrt{T_1\mu_1} \cdot v_y$. So if there is no reflection in one case (and there certainly isn't in the case of an identical string), then there is no reflection in any other case. As far as reflection and transmission go, a string is completely characterized by one quantity: the impedance $Z \equiv \sqrt{T\mu}$. Nothing else matters. Other quantities such as $T$, $\mu$, and $v = \sqrt{T/\mu}$ are relevant for various other considerations, but only the combination $Z \equiv \sqrt{T\mu}$ appears in the $R$ and $T$ coefficients.

Although the word "impedance" makes colloquial sense, there is one connotation that might be misleading. You might think that a small impedance allows a wave to transmit easily and reflect essentially nothing back. But this isn't the case. Maximal transmission occurs when the impedances *match*, not when $Z_2$ is small. (If $Z_2$ is small, say $Z_2 = 0$, then Eq. (38) tells us that we actually have *total* reflection with $R = 1$.) When we discuss energy in Section 4.4, we'll see that impedance matching yields maximal energy transfer, consistent with the fact that no energy remains on the left side, because there is no reflected wave.

---

[4] Remember that $F_y$ and $v_y$ are the *transverse* force and velocity, which are generally very small, given our usual assumption of a small slope of the string. But $T_2$ and $v_2$ are the tension and wave speed on the right side, which are "everyday-sized" quantities. What we showed in Eq. (41) was that the two ratios, $F_y/v_y$ and $-T_2/v_2$, are always equal.

### Why $F_y$ is proportional to $\sqrt{T\mu}$

We saw above that the transverse force that the left string (or technically the ring at the boundary) applies to the right string is $F_y = +bv_y \equiv Zv_y$. So if you replace the left string with your hand, then $F_y = Zv_y$ is the transverse force than you must apply to the right string to give it the same motion that it had when the left string was there. The impedance $Z$ gives a measure of how hard it is to wiggle the end of the string back and forth. It is therefore reasonable that $Z = \sqrt{T_2\mu_2}$ grows with both $T_2$ and $\mu_2$. In particular, if $\mu_2$ is large, then more force should be required in order to wiggle the string in a given manner.

However, although this general dependence on $\mu_2$ seems quite intuitive, you have to be careful, because there is a common incorrect way of thinking about things. The reason why the force grows with $\mu_2$ is *not* the same as the reason why the force grows with $m$ in the simple case of a single point mass (with no string or anything else attached to it). In that case, if you wiggle a point mass back and forth, then the larger the mass, the larger the necessary force, due to $F = ma$.

But in the case of a string, if you grab onto the leftmost atom of the right part of the string, then this atom is essentially massless, so your force isn't doing any of the "$F = ma$" sort of acceleration. All your force is doing is simply balancing the transverse component of the $T_2$ tension that the right string applies to its leftmost atom. This transverse component is nonzero due to the (in general) nonzero slope. So as far as your force is concerned, all that matters are the values of $T_2$ and the slope. And the *slope* is where the dependence on $\mu_2$ comes in. If $\mu_2$ is large, then $v_2 = \sqrt{T_2/\mu_2}$ is small, which means that the wave in the right string is squashed by a factor of $v_2/v_1$ compared with the wave on the left string. This then means that the slope of the right part is larger by a factor that is proportional to $1/v_2 = \sqrt{\mu_2/T_2}$, which in turn means that the transverse force is larger. Since the transverse force is proportional to the product of the tension and the slope, we see that it is proportional to $T_2\sqrt{\mu_2/T_2} = \sqrt{T_2\mu_2}$. To sum up: $\mu_2$ affects the impedance not because of an $F = ma$ effect, but rather because $\mu$ affects the wave's speed, and hence slope, which then affects the transverse component of the force.

A byproduct of this reasoning is that the dependence of the transverse force on $T_2$ takes the form of $\sqrt{T_2}$. This comes from the expected factor of $T_2$ which arises from the fact that the transverse force is proportional to the tension. But there is an additional factor of $1/\sqrt{T_2}$, because the transverse force is also proportional to the slope, which behaves like $1/\sqrt{T_2}$ from the argument in the previous paragraph.

### Why $F_y$ is proportional to $v_y$

We saw above that if the right string is removed and if the ring is attached to a piston with a damping coefficient of $b = \sqrt{T_2\mu_2}$, then the left string can't tell the difference. Either way, the force on the left string takes the form of $-bv_y \equiv -b\dot{y}$. If instead of a piston we attach the ring to a transverse *spring*, then the force that the ring applies to the left string (which equals the force that the spring applies to the ring, since the ring is massless) is $-ky$. And if the ring is instead simply a *mass* that isn't attached to anything, then the force it applies to the left string is $-m\ddot{y}$ (equal and opposite to the $F_y = ma_y$ force that the string applies to the mass). Neither of these scenarios mimics the correct $-b\dot{y}$ force that the right string actually applies.

This $-b\dot{y}$ force from the right string is a consequence of the "wavy-ness" of waves, for the following reason. The transverse force $F_y$ that the right string applies to the ring is proportional to the slope of the wave; it equals the tension times the slope, assuming the slope is small:

$$F_y = T_2 \frac{\partial \psi_{\mathrm{R}}}{\partial x} \,. \tag{42}$$

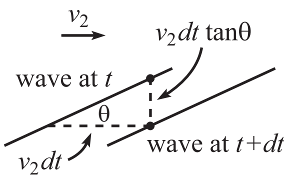



**Figure 16**

And the transverse velocity is also proportional to the (negative of the) slope, due to the properties of Fig. 16. The left tilted segment is a small piece of the wave at a given time $t$, and the right tilted segment is the piece at a later time $t + dt$ (the wave is moving to the right). The top dot is the location of a given atom at time $t$, and the bottom dot is the location of the same atom at time $t + dt$. The wave moves a distance $v_2\,dt$ to the right, so from the triangle shown, the dot moves a distance $(v_2\,dt)\tan\theta$ *downward*. The dot's velocity is therefore $-v_2\tan\theta$, which is $-v_2$ times the slope. That is,

$$v_y = -v_2 \frac{\partial \psi_{\mathrm{R}}}{\partial x}\,. \tag{43}$$

We see that both the transverse force in Eq. (42) and the transverse velocity in Eq. (43) are proportional to the slope, with the constants of proportionality being $T_2$ and $-v_2$, respectively. The ratio $F_y/v_y$ is therefore independent of the slope. It equals $-T_2/v_2$, which is constant in time.

Fig. 16 is the geometric explanation of the mathematical relation in Eq. (40). Written in terms of $\psi_{\mathrm{R}}$ instead of $f_{\mathrm{t}}$, Eq. (40) says that

$$\frac{\partial \psi_{\mathrm{R}}}{\partial t} = -v_2 \cdot \frac{\partial \psi_{\mathrm{R}}}{\partial x}\,. \tag{44}$$

That is, the transverse velocity is $-v_2$ times the slope, which is simply the geometrically-derived result in Eq. (43). Note that this relation holds only for a *single* traveling wave. If we have a wave that consists of, say, two different traveling waves, $\psi(x,t) = f_{\mathrm{a}}(t - x/v_{\mathrm{a}}) + f_{\mathrm{b}}(t - x/v_{\mathrm{b}})$, then

$$\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{\partial f_{\mathrm{a}}}{\partial t} + \frac{\partial f_{\mathrm{b}}}{\partial t}\,, \\
\frac{\partial \psi}{\partial x} &= -\frac{1}{v_{\mathrm{a}}} \cdot \frac{\partial f_{\mathrm{a}}}{\partial t} - \frac{1}{v_{\mathrm{b}}} \cdot \frac{\partial f_{\mathrm{a}}}{\partial t}\,.
\end{aligned} \tag{45}$$

Looking at the righthand sides of these equations, we see that it is *not* the case that $\partial\psi/\partial t = -v \cdot \partial\psi/\partial x$ for a particular value of $v$. It doesn't work for $v_1$ or $v_2$ or anything else. This observation is relevant to the following question.

In the discussion leading up to Eq. (41), we considered the transverse force that the ring applies to the string on its *left*, and the result was $-(T_2/v_2)v_y$. From Newton's third law, the transverse force that the ring applies to the string on its *right* is therefore $+(T_2/v_2)v_y$. However, shouldn't we be able to proceed through the above derivation with "left" and "right" reversed, and thereby conclude that the force that the ring applies to the string on its *right* is equal to $+(T_1/v_1)v_y$? The answer had better be a "no," because this result isn't consistent with the $+(T_2/v_2)v_y$ result unless $T_1/v_1 = T_2/v_2$, which certainly doesn't hold for arbitrary choices of strings. Where exactly does the derivation break down? The task of Problem [to be added] is to find out.

### 4.3.2 Examples of impedance matching

In this section we'll present (without proof) a number of examples of impedance matching. The general point of matching impedances is to yield the maximal energy transfer from one thing to another. As we saw above in the case of the string with different densities, if the impedances are equal, then nothing is reflected. So 100% of the energy is transferred, and you can't do any better than that, due to conservation of energy.

If someone gives you two impedance-matched strings (so the product $T\mu$ is the same in both), then all of the wave is transmitted. If the above conservation-of-energy argument slips your mind, you might (erroneously) think that you could increase the amount of transmitted energy by, say, decreasing $\mu_2$. This has the effect of decreasing $Z_2$ and thus increasing the transmission coefficient $T$ in Eq. (38). And you might think that the larger amplitude of the transmitted wave implies a larger energy. However, this isn't the case, because less mass is moving now in the right string, due to the smaller value of $\mu_2$. There are competing effects, and it isn't obvious from this reasoning which effect wins. But it *is* obvious from the conservation-of-energy argument. If the $Z$'s aren't equal, then there is nonzero reflection, by Eq. (38), and therefore less than 100% of the energy is transmitted. This can also be demonstrated by explicitly calculating the energy. We'll talk about energy in Section 4.4 below.

The two basic ways to match two impedances are to (1) simply make one of them equal to the other, or (2) keep them as they are, but insert a large number of things (whatever type of things the two original ones are) between them with impedances that gradually change from one to the other. It isn't obvious that this causes essentially all of the energy to be transferred, but this can be shown without too much difficulty. The task of Problem [to be added] is to demonstrate this for the case of the "Gradually changing string density" mentioned below. Without going into too much detail, here are a number of examples of impedance matching:

Electrical circuits: The general expression for impedance is $Z = F/v$. In a purely resistive circuit, the analogs of $Z$, $F$, and $v$ are, respectively, the resistance $R$, the voltage $V$, and the current $I$. So $Z = F/v$ becomes $R = V/I$, which is simply Ohm's law. If a source has a given impedance (resistance), and if a load has a variable impedance, then we need to make the load's impedance equal to the source's impedance, if the goal is to have the maximum power delivered to the load. In general, impedances can be complex if the circuit isn't purely resistive, but that's just a fancy way of incorporating a phase difference when $V$ and $I$ (or $F$ and $v$ in general) aren't in phase.

Gradually changing string density: If we have two strings with densities $\mu_1$ and $\mu_2$, and if we insert between them a long string (much longer than the wavelength of the wave being used) whose density gradually changes from $\mu_1$ to $\mu_2$, then essentially all of the wave is transmitted. See Problem [to be added].

Megaphone: A tapered megaphone works by the same principle. If you yell into a simple cylinder, then it turns out that the abrupt change in cross section (from a radius of $r$ to a radius of essentially infinity) causes reflection. The impedance of a cavity depends on the cross section. However, in a megaphone the cross section varies gradually, so not much sound is reflected back toward your mouth. The same effect is relevant for a horn.

Ultrasound: The gel that is put on your skin has the effect of impedance matching the waves in the device to the waves in your body.

Ball collisions: Consider a marble that collides elastically with a bowling ball. The marble will simply bounce off and give essentially none of its energy to the bowling ball. All of the energy will remain in the marble. (And similarly, if a bowling ball collides with a marble, then the bowling ball will simply plow through and keep essentially all of the energy.) But if a series of many balls, each with slightly increasing size, is placed between them (see Fig. 17), then it turns out that essentially all of the marble's energy will end up in the bowling ball. Not obvious, but true. And conversely, if the bowling ball is the one that is initially moving (to the left), then essentially all of its energy will end up in the marble, which will therefore be moving very fast.

**Figure 17**

It is nebulous what impedance means for one-time evens like collisions between balls, because we defined impedance for waves. But the above string of balls is certainly similar to a longitudinal series of masses (with increasing size) and springs. The longitudinal waves that travel along this spring/mass system consist of many "collisions" between the masses. In the original setup with just the string of balls and no springs, when two balls collide they smush a little and basically act like springs. Well, sort of; they can only repel and not attract. At any rate, if you abruptly increased the size of the masses in the spring/mass system by a large factor, then not much of the wave would make it through. But gradually increasing the masses would be just like gradually increasing the density $\mu$ in the "Gradually changing string density" example above.

Lever: If you try to lift a refrigerator that is placed too far out on a lever, you're not going to be able to do it. If you jumped on your end, you would just bounce off like on a springboard. You'd keep all of the energy, and none of it would be transmitted. But if you move the refrigerator inward enough, you'll be able to lift it. However, if you move it in too far (let's assume it's a point mass), then you're back to essentially not being able to lift it, because you'd have to move your end of the lever, say, a mile to lift the refrigerator up by a foot. So there is an optimal placement.

Bicycle: The gears on a bike act basically the same way as a lever. If you're in too high a gear, you'll find that it's too hard on your muscles; you can't get going fast. And likewise, if you're in too low a gear, your legs will just spin wildly, and you'll be able to go only so fast. There is an optimal gear ratio that allows you to transfer the maximum amount of energy from chemical potential energy (from your previous meal) to kinetic energy.

Rolling a ball up a ramp: This is basically just like a lever or a bike. If the ramp is too shallow, then the ball doesn't gain much potential energy. And if it's too steep, then you might not be able to move the ball at all.

## 4.4 Energy

**Energy**

What is the energy of a wave? Or more precisely, what is the energy density per unit length? Consider a little piece of the string between $x$ and $x + dx$. In general, this piece has both kinetic and potential energy. The kinetic energy comes from the transverse motion (we showed in the paragraph following Eq. (1) that the longitudinal motion is negligible), so it equals

$$K_{dx} = \frac{1}{2}(dm)v_y^2 = \frac{1}{2}(\mu\, dx)\left(\frac{\partial\psi}{\partial t}\right)^2. \tag{46}$$

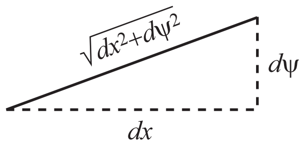


**Figure 18**

We have used the fact that since there is essentially no longitudinal motion, the mass within the span from $x$ to $x + dx$ is always essentially equal to $\mu\, dx$.

The potential energy depends on the stretch of the string. In general, a given piece of the string is tilted and looks like the piece shown in Fig. 18. As we saw in Eq. (1), the Taylor series $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$ gives the length of the piece as

$$dx\sqrt{1+\left(\frac{\partial\psi}{\partial x}\right)^2} \approx dx + \frac{dx}{2}\left(\frac{\partial\psi}{\partial x}\right)^2. \tag{47}$$

The piece is therefore stretched by an amount, $d\ell \approx (dx/2)(\partial\psi/\partial x)^2$.[5] This stretch is caused by the external tension forces at the two ends. These forces do an amount $T d\ell$ of

[5]Actually, I think this result is rather suspect, although it doesn't matter in the end. See the remark below.

work, and this work shows up as potential energy in the piece, exactly in the same way that a normal spring acquires potential energy if you grab an end and stretch it. So the potential energy of the piece is

$$U_{dx} = \frac{1}{2} T dx \left(\frac{\partial \psi}{\partial x}\right)^2. \tag{48}$$

The total energy per unit length (call it $\mathcal{E}$) is therefore

$$\begin{aligned}\mathcal{E}(x,t) = \frac{K_{dx}+U_{dx}}{dx} &= \frac{\mu}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 \\ &= \frac{\mu}{2}\left[\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{T}{\mu}\left(\frac{\partial \psi}{\partial x}\right)^2\right] \\ &= \frac{\mu}{2}\left[\left(\frac{\partial \psi}{\partial t}\right)^2 + v^2\left(\frac{\partial \psi}{\partial x}\right)^2\right], \end{aligned} \tag{49}$$

where we have used $v = \sqrt{T/\mu}$. This expression for $\mathcal{E}(x,t)$ is valid for an arbitrary wave. But let's now look at the special case of a single traveling wave, which takes the form of $\psi(x,t) = f(x \pm vt)$. For such a wave, the energy density can be further simplified. As we have seen many times, the partial derivatives of a single traveling wave are related by $\partial\psi/\partial t = \pm v\, \partial\psi/\partial x$. So the two terms in the expression for $\mathcal{E}(x,t)$ are equal at a given point and at a given time. We can therefore write

$$\boxed{\mathcal{E}(x,t) = \mu\left(\frac{\partial \psi}{\partial t}\right)^2} \quad \text{or} \quad \boxed{\mathcal{E}(x,t) = \mu v^2\left(\frac{\partial \psi}{\partial x}\right)^2} \quad \text{(for traveling waves)} \tag{50}$$

Or equivalently, we can use $Z \equiv \sqrt{T\mu}$ and $v = \sqrt{T/\mu}$ to write

$$\mathcal{E}(x,t) = \frac{Z}{v}\left(\frac{\partial \psi}{\partial t}\right)^2, \qquad \text{or} \qquad \mathcal{E}(x,t) = Zv\left(\frac{\partial \psi}{\partial x}\right)^2. \tag{51}$$

For sinusoidal traveling waves, the energy density is shown in Fig. 19 (with arbitrary units on the axes). The energy-density curve moves right along with the wave.

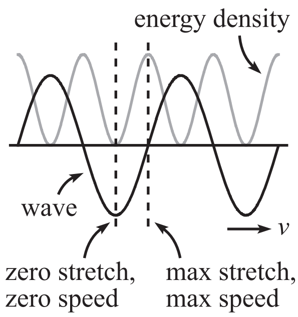



**Figure 19**

Remark : As mentioned in Footnote 5, the length of the string given in Eq. (47) and the resulting expression for the potential energy given in Eq. (48) are highly suspect. The reason is the following. In writing down Eq. (47), we made the assumption that all points on the string move in the transverse direction; we assumed that the longitudinal motion is negligible. This is certainly the case (in an exact sense) if the string consists of little masses that are hypothetically constrained to ride along rails pointing in the transverse direction. If these masses are connected by little stretchable pieces of massless string, then Eq. (48) correctly gives the potential energy.

However, all that we were able to show in the reasoning following Eq. (1) was that the points on the string move in the transverse direction, *up to errors of order* $dx(\partial\psi/\partial x)^2$. We therefore have no right to trust the result in Eq. (48), because it is of the *same* order. But even if this result is wrong and if the stretching of the string is distributed differently from Eq. (47), the *total* amount of stretching is the same. Therefore, because the work done on the string, $T d\ell$, is linear in $d\ell$, the total potential energy is independent of the particular stretching details. The $(T/2)(\partial\psi/\partial x)^2$ result therefore correctly yields the *average* potential energy density, even though it may be incorrect at individual points. And since we will rarely be concerned with more than the average, we can use Eq. (48), and everything is fine. ♣

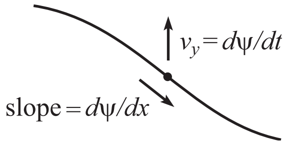



**Figure 20**

**Power**

What is the power transmitted across a given point on the string? Equivalently, what is the rate of energy flow past a given point? Equivalently again, at what rate does the string to the left of a point do work on the string to the right of the point? In Fig. 20, the left part of the string pulls on the dot with a transverse force of $F_y = -T\,\partial\psi/\partial x$. The power flow across the dot (with rightward taken to be positive) is therefore

$$P(x,t) = \frac{dW}{dt} = \frac{F_y\,d\psi}{dt} = F_y\frac{\partial\psi}{\partial t} = Fv_y = \left(-T\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\psi}{\partial t}\right). \tag{52}$$

This expression for $P(x,t)$ is valid for an arbitrary wave. But as with the energy density above, we can simplify the expression if we consider the special case of single traveling wave. If $\psi(x,t) = f(x \pm vt)$, then $\partial\psi/\partial x = \pm(1/v)\partial\psi/\partial t$. So we can write the power as (using $T/v = T/\sqrt{T/\mu} = \sqrt{T\mu} \equiv Z$)

$$P(x,t) = \mp\frac{T}{v}\left(\frac{\partial\psi}{\partial t}\right)^2 \quad\Longrightarrow\quad \boxed{P(x,t) = \mp Z\left(\frac{\partial\psi}{\partial t}\right)^2 = \mp v\mathcal{E}(x,t)} \tag{53}$$

where we have used Eq. (51). We see that the magnitude of the power is simply the wave speed times the energy density. It is positive for a rightward traveling wave $f(x-vt)$, and negative for a leftward traveling wave $f(x+vt)$ (we're assuming that $v$ is a positive quantity here). This makes sense, because the energy plot in Fig. 19 just travels along with the wave at speed $v$.

**Momentum**

A wave on a string carries energy due to the transverse motion. Does such a wave carry momentum? Well, there is certainly nonzero momentum in the transverse direction, but it averages out to zero because half of the string is moving one way, and half is moving the other way.

What about the longitudinal direction? We saw above that the points on the string move only negligibly in the longitudinal direction, so there is no momentum in that direction. Even though a traveling wave makes it look like things are moving in the longitudinal direction, there is in fact no such motion. Every point in the string is moving only in the transverse direction. Even if the points did move non-negligible distances, the momentum would still average out to zero, consistent with the fact that there is no overall longitudinal motion of the string. The general kinematic relation $p = mv_{\text{CM}}$ holds, so if the CM of the string doesn't move, then the string has no momentum.

There are a few real-world examples that might make you think that standard traveling waves can carry momentum. One example is where you try (successfully) to move the other end of a rope (or hose, etc.) you're holding, which lies straight on the ground, by flicking the rope. This causes a wave to travel down the rope, which in turn causes the other end to move farther away from you. Everyone has probably done this at one time or another, and it works. However, you can bet that you moved your hand forward during the flick, and this is what gave the rope some longitudinal momentum. You certainly must have moved your hand forward, of course, because otherwise the far end couldn't have gotten farther away from you (assuming that the rope can't stretch significantly). If you produce a wave on a rope by moving your hand only up and down (that is, transversely), then the rope will not have any longitudinal momentum.

Another example that might make you think that waves carry momentum is the case of sound waves. Sound waves are longitudinal waves, and we'll talk about these in the following chapter. But for now we'll just note that if you're standing in front of a very large speaker (large enough so that you feel the sound vibrations), then it *seems* like the sound is applying a net force on you. But it isn't. As we'll see in the next chapter, the pressure on you alternates sign, so half the time the sound wave is pushing you away from the speaker, and half the time it's drawing you closer. So it averages out to zero.[6]

An exception to this is the case of a pulse or an explosion. In this case, something at the source of the pulse must have actually moved forward, so there is some net momentum. If we have only half (say, the positive half) of a cycle of a sinusoidal pressure wave, then this part can push you away without the (missing) other half drawing you closer. But this isn't how normal waves (with both positive and negative parts) work.

# 4.5 Standing waves

## 4.5.1 Semi-infinite string

### Fixed end

Consider a leftward-moving sinusoidal wave that is incident on a brick wall at its left end, located at $x = 0$. (We've having the wave move leftward instead of the usual rightward for later convenience.) The most general form of a leftward-moving sinusoidal wave is

$$\psi_{\mathrm{i}}(x,t) = A\cos(\omega t + kx + \phi), \tag{54}$$

where $\omega$ and $k$ satisfy $\omega/k = \sqrt{T/\mu} = v$. $A$ is the amplitude, and $\phi$ depends on the arbitrary choice of the $t = 0$ time. Since the brick wall has "infinite" impedance, Eq. (38) gives $R = -1$. Eq. (32) then gives the reflected rightward-moving wave as

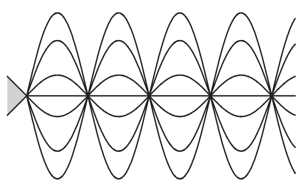
$$\psi_{\mathrm{r}}(x,t) = R\psi_{\mathrm{i}}(-x,t) = -A\cos(\omega t - kx + \phi). \tag{55}$$

The total wave is therefore

$$\begin{aligned} \psi(x,t) = \psi_{\mathrm{i}}(x,t) + \psi_{\mathrm{r}}(x,t) &= A\cos(\omega t + \phi + kx) - A\cos(\omega t + \phi - kx) \\ &= \boxed{-2A\sin(\omega t + \phi)\sin kx} \end{aligned} \tag{56}$$

As a double check, this satisfies the boundary condition $\psi(0,t) = 0$ for all $t$. The sine function of $x$ is critical here. A cosine function wouldn't satisfy the boundary condition at $x = 0$. In contrast with this, it doesn't matter whether we have a sine or cosine function of $t$, because a phase shift in $\phi$ can turn one into the other.

For a given value of $t$, a snapshot of this wave is a sinusoidal function of $x$. The wavelength of this function is $\lambda = 2\pi/k$, and the amplitude is $|2A\sin(\omega t + \phi)|$. For a given value of $x$, each point oscillates as a sinusoidal function of $t$. The period of this function is $\tau = 2\pi/\omega$, and the amplitude is $|2A\sin kx|$. Points for which $kx$ equals $n\pi$ always have $\psi(x,t) = 0$, so they never move. These points are called nodes.



(fixed left end)

**Figure 21**

Fig. 21 shows the wave at a number of different times. A wave such as the one in Eq. (56) is called a "standing wave." All points on the string have the same phase (or differ by $\pi$), as far as the oscillations in time go. That is, all of the points come to rest at the same time (at the maximal displacement from equilibrium), and they all pass through the origin at the same time, etc. This is *not* true for traveling waves. In particular, in a traveling

[6] The opening scene from the first *Back to the Future* movie involves Marty McFly standing in front of a humongous speaker with the power set a bit too high. He then gets blown backwards when he plays a chord. This isn't realistic, but ingenious movies are allowed a few poetic licenses.

wave, the points with $\psi = 0$ are moving with maximal speed, and the points with maximum $\psi$ are instantaneously at rest.

If you don't want to have to invoke the $R = -1$ coefficient, as we did above, another way of deriving Eq. (56) is to apply the boundary condition at $x = 0$ to the most general form of the wave given in Eq. (8). Since $\psi(0, t) = 0$ for all $t$, we can have only the $\sin kx$ terms in Eq. (8). Therefore,

$$\begin{aligned}\psi(x,t) &= D_2 \sin kx \sin \omega t + D_3 \sin kx \cos \omega t \\ &= (D_2 \sin \omega t + D_3 \cos \omega t) \sin kx \\ &\equiv B \sin(\omega t + \phi) \sin kx, \end{aligned} \tag{57}$$

where $B$ and $\phi$ are determined by $B \cos \phi = D_2$ and $B \sin \phi = D_3$.

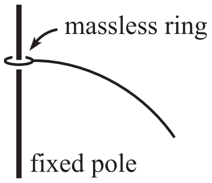



**Figure 22**

### Free end

Consider now a leftward-moving sinusoidal wave that has its left end free (located at $x = 0$). By "free" here, we mean that a massless ring at the end of the string is looped around a fixed frictionless pole pointing in the transverse direction; see Fig. 22. So the end is free to move transversely but not longitudinally. The pole makes it possible to maintain the tension $T$ in the string. Equivalently, you can consider the string to be infinite, but with a density of $\mu = 0$ to the left of $x = 0$.

As above, the most general form of a leftward-moving sinusoidal wave is

$$\psi_{\mathrm{i}}(x,t) = A \cos(\omega t + kx + \phi), \tag{58}$$

Since the massless ring (or equivalently the $\mu = 0$ string) has zero impedance, Eq. (38) gives $R = +1$. Eq. (32) then gives the reflected rightward-moving wave as

$$\psi_{\mathrm{r}}(x,t) = R\psi_{\mathrm{i}}(-x,t) = +A \cos(\omega t - kx + \phi). \tag{59}$$

The total wave is therefore

$$\begin{aligned}\psi(x,t) = \psi_{\mathrm{i}}(x,t) + \psi_{\mathrm{r}}(x,t) &= A \cos(\omega t + \phi + kx) + A \cos(\omega t + \phi - kx) \\ &= \boxed{2A \cos(\omega t + \phi) \cos kx} \end{aligned} \tag{60}$$

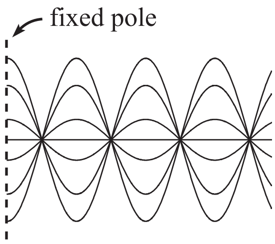



**Figure 23**

As a double check, this satisfies the boundary condition, $\partial\psi/\partial x|_{x=0} = 0$ for all $t$. The slope must always be zero at $x = 0$, because otherwise there would be a net transverse force on the massless ring, and hence infinite acceleration. If we choose to construct this setup with a $\mu = 0$ string to the left of $x = 0$, then this string will simply rise and fall, always remaining horizontal. You can assume that the other end is attached to something very far to the left of $x = 0$.

Fig. 23 shows the wave at a number of different times. This wave is similar to the one in Fig. 21 (it has the same amplitude, wavelength, and period), but it is shifted a quarter cycle in both time and space. The time shift isn't of too much importance, but the space shift is critical. The boundary at $x = 0$ now corresponds to an "antinode," that is, a point with the maximum oscillation amplitude.

As in the fixed-end case, if you don't want to have to invoke the $R = 1$ coefficient, you can apply the boundary condition, $\partial\psi/\partial x|_{x=0} = 0$, to the most general form of the wave given in Eq. (8). This allows only the $\cos kx$ terms.

In both this case and the fixed-end case, $\omega$ and $k$ can take on a continuous set of values, as long as they are related by $\omega/k = \sqrt{T/\mu} = v$.[7] In the finite-string cases below, we will find that they can take on only discrete values.

[7]Even though the above standing waves don't travel anywhere and thus don't have a speed, it still makes sense to label the quantity $\sqrt{T/\mu}$ as $v$, because standing waves can be decomposed into two oppositely-moving traveling waves, as shown in Eqs. (56) and (60).

### 4.5.2 Finite string

We'll now consider the three possible (extreme) cases for the boundary conditions at the two ends of a finite string. We can have both ends fixed, or one fixed and one free, or both free. Let the ends be located at $x = 0$ and $x = L$. In general, the boundary conditions (for all $t$) are $\psi = 0$ at a fixed end (because the end never moves), and $\partial\psi/\partial x = 0$ at a free end (because the slope must be zero so that there is no transverse force on the massless endpoint).

#### Two fixed ends

If both ends of the string are fixed, then the two boundary conditions are $\psi(0,t) = 0$ and $\psi(L,t) = 0$ for all $t$. Eq. (56) gives the most general form of a wave (with particular $\omega$ and $k$ values) satisfying the first of these conditions. So we just need to demand that the second one is also true. If we plug $x = L$ into Eq. (56), the only way to have $\psi(L,t) = 0$ for all $t$ is to have $\sin kL = 0$. This implies that $kL$ must equal $n\pi$ for some integer $n$. So

$$k_n = \frac{n\pi}{L}\,, \tag{61}$$

where we have added the subscript to indicate that $k$ can take on a discrete set of values associated with the integers $n$. $n$ signifies which "mode" the string is in. The wavelength is $\lambda_n = 2\pi/k_n = 2L/n$. So the possible wavelengths are all integral divisors of $2L$ (which is twice the length of the string). Snapshots of the first few modes are shown below in the first set of waves in Fig. 24. The $n$ values technically start at $n = 0$, but in this case $\psi$ is identically equal to zero since $\sin(0) = 0$. This is certainly a physically possible location of the string, but it is the trivial scenario. So the $n$ values effectively start at 1.

The angular frequency $\omega$ is still related to $k$ by $\omega/k = \sqrt{T/\mu} = v$, so we have $\omega_n = vk_n$. Remember that $v$ depends only on $T$ and $\mu$, and not on $n$ (although this won't be true when we get to dispersion in Chapter 6). The frequency in Hertz is then

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{vk_n}{2\pi} = \frac{v(n\pi/L)}{2\pi} = \frac{nv}{2L}\,. \tag{62}$$

The possible frequencies are therefore integer multiples of the "fundamental" frequency, $\nu_1 = v/2L$. In short, the additional boundary condition at the right end constrains the system so that only discrete values of $\omega$ and $k$ are allowed. Physically, the wave must undergo an integral number of half oscillations in space, in order to return to the required value of zero at the right end. This makes it clear that $\lambda$ (and hence $k$) can take on only discrete values. And since the ratio $\omega/k$ is fixed to be $\sqrt{T/\mu}$, this means that $\omega$ (and $\nu$) can take on only discrete values too. To summarize:

$$\boxed{\lambda_n = \frac{2L}{n} \qquad \text{and} \qquad \nu_n = \frac{nv}{2L}} \tag{63}$$

The product of these quantities is $\lambda_n\nu_n = v$, as it should be.

Since the wave equation in Eq. (4) is linear, the most general motion of a string with two fixed ends is an arbitrary linear combination of the solutions in Eq. (56), with the restriction that $k$ takes the form $k_n = n\pi/L$ (and $\omega/k$ must equal $v$). So the most general expression for $\psi(x,t)$ is (we'll start the sum at $n = 0$, even though this term doesn't contribute anything)

$$\boxed{\psi(x,t) = \sum_{n=0}^{\infty} B_n \sin(\omega_n t + \phi_n)\sin k_n x} \qquad \text{where } k_n = \frac{n\pi}{L}\,, \quad \text{and} \quad \omega_n = vk_n. \tag{64}$$

The $B$ here equals the $-2A$ from Eq. (56). Note that the amplitudes and phases of the various modes can in general be different.

### One fixed end, one free end

Now consider the case where one end is fixed and the other is free. Let's take the fixed end to be the one at $x = 0$. The case where the fixed end is located at $x = L$ gives the same general result; it's just the mirror image of the result we will obtain here.

The two boundary conditions are $\psi(0, t) = 0$ and $\partial\psi/\partial x|_{x=L} = 0$ for all $t$. Eq. (56) again gives the most general form of a wave satisfying the first of these conditions. So we just need to demand that the second one is also true. From Eq. (56), the slope $\partial\psi/\partial x$ is proportional to $\cos kx$. The only way for this to be zero at $x = L$ is for $kL$ to equal $(n + 1/2)\pi$ for some integer $n$. So

$$k_n = \frac{(n + 1/2)\pi}{L}\,. \tag{65}$$

$n$ starts at zero here. Unlike in the two-fixed-ends case, the $n = 0$ value now gives a nontrivial wave.

The wavelength is $\lambda_n = 2\pi/k_n = 2L/(n + 1/2)$. These wavelengths are most easily seen in pictures, and snapshots of the first few modes are shown below in the second set of waves in Fig. 24. The easiest way to describe the wavelengths in words is to note that the number of oscillations that fit on the string is $L/\lambda_n = n/2 + 1/4$. So for the lowest mode (the $n = 0$ one), a quarter of a wavelength fits on the string. The higher modes are then obtained by successively adding on half an oscillation (which ensures that $x = L$ is always located at an antinode, with slope zero). The frequency $\nu_n$ can be found via Eq. (62), and we can summarize the results:

$$\boxed{\lambda_n = \frac{2L}{n + 1/2} \qquad \text{and} \qquad \nu_n = \frac{(n + 1/2)v}{2L}} \tag{66}$$

Similar to Eq. (64), the most general motion of a string with one fixed end and one free end is a linear combination of the solutions in Eq. (56):

$$\boxed{\psi(x, t) = \sum_{n=0}^{\infty} B_n \sin(\omega_n t + \phi_n) \sin k_n x} \qquad \text{where} \;\; k_n = \frac{(n + 1/2)\pi}{L}\,, \quad \text{and} \quad \omega_n = v k_n. \tag{67}$$

If we instead had the left end as the free one, then Eq. (60) would be the relevant equation, and the $\sin kx$ here would instead be a $\cos kx$. As far as the dependence on time goes, it doesn't matter whether it's a sine or cosine function of $t$, because a redefinition of the $t = 0$ point yields a phase shift that can turn sines into cosines, and vice versa. We aren't free to redefine the $x = 0$ point, because we have a physical wall there.

### Two free ends

Now consider the case with two free ends. The two boundary conditions are $\partial\psi/\partial x|_{x=0} = 0$ and $\partial\psi/\partial x|_{x=L} = 0$ for all $t$. Eq. (60) gives the most general form of a wave satisfying the first of these conditions. So we just need to demand that the second one is also true. From Eq. (60), the slope $\partial\psi/\partial x$ is proportional to $\sin kx$. The only way for this to be zero at $x = L$ is for $kL$ to equal $n\pi$ for some integer $n$. So

$$k_n = \frac{n\pi}{L}\,, \tag{68}$$

which is the same as in the case of two fixed ends. The wavelength is $\lambda_n = 2\pi/k_n = 2L/n$. So the possible wavelengths are all integral divisors of $2L$, again the same as in the case of two fixed ends. Snapshots of the first few modes are shown below in the third set of waves in Fig. 24.

The $n$ values technically start at $n = 0$. In this case $\psi$ has no dependence on $x$, so we simply have a flat line (not necessarily at $\psi = 0$). The line just sits there at rest, because the frequency is again given by Eq. (62) and is therefore zero (the $k_n$ values, and hence $\omega_n$ values, are the same as in the two-fixed-ends case, so $\omega_0 = 0$). This case isn't as trivial as the $n = 0$ case for two fixed ends; the resulting constant value of $\psi$ might be necessary to satisfy the initial conditions of the string. This constant value is analogous to the $a_0$ term in the Fourier-series expression in Eq. (3.1).

As with two fixed ends, an integral number of half oscillations must fit into $L$, but they now start and end at antinodes instead of nodes. The frequency $\nu_n$ is the same as in the two-fixed-ends case, so as in that case we have:

$$\boxed{\lambda_n = \frac{2L}{n} \qquad \text{and} \qquad \nu_n = \frac{nv}{2L}} \tag{69}$$

Similar to Eqs. (64) and (67), the most general motion of a string with two free ends is a linear combination of the solutions in Eq. (60):

$$\boxed{\psi(x,t) = \sum_{n=0}^{\infty} B_n \cos(\omega_n t + \phi_n) \cos k_n x} \qquad \text{where} \quad k_n = \frac{n\pi}{L}, \quad \text{and} \quad \omega_n = vk_n. \tag{70}$$

Fig. 24 summarizes the above results.

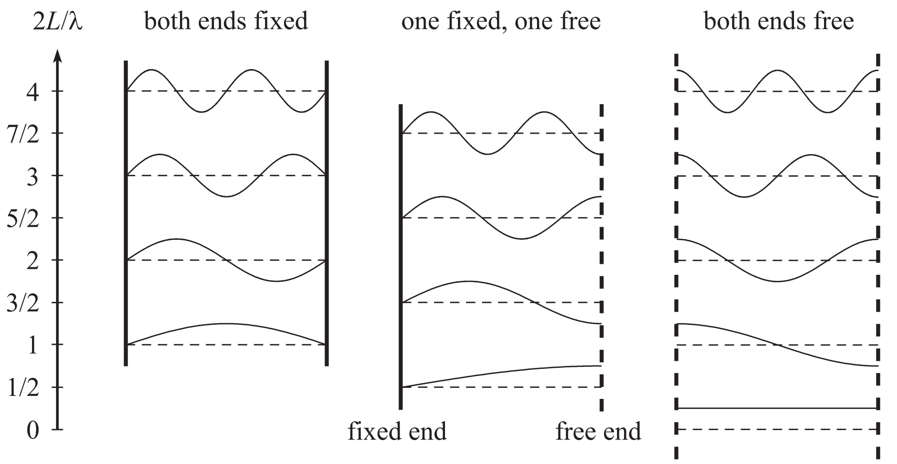



**Figure 24**

### Power in a standing wave

We saw above in Section 4.4 that not only do traveling waves contain energy, they also contain an energy flow along the string. That is, they transmit power. A given point on the string does work (which may be positive or negative, depending on the direction of the wave's velocity) on the part of the string to its right. And it does the opposite amount of work on the string to its left.

A reasonable question to ask now is: Is there energy flow in standing waves? There is certainly an energy *density*, because in general the string both moves and stretches. But is there any energy transfer along the string?

Intuitively, a standing wave is the superposition of two oppositely-moving traveling waves with equal amplitudes. These traveling waves have equal and opposite energy flow, on average, so we expect the net energy flow in a standing wave to be zero, on average. (We'll see below where this "on average" qualification arises.) Alternatively, we can note that there can't be any net energy flow in either direction in a standing wave, due to the left-right symmetry of the system. If you flip the paper over, so that right and left are reversed, then a standing wave looks exactly the same, whereas a traveling wave doesn't, because it's now moving in the opposite direction.

Mathematically, we can calculate the energy flow (that is, the power) as follows. The expression for the power in Eq. (52) is valid for an arbitrary wave. This is the power flow across a given point, with rightward taken to be positive. Let's see what Eq. (52) reduces to for a standing wave. We'll take our standing wave to be $A\sin\omega t\sin kx$. (We could have any combination of sines and cosines here; they all give the same general result.) Eq. (52) becomes

$$\begin{aligned} P(x,t) &= \left(-T\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\psi}{\partial t}\right) \\ &= -T\big(kA\sin\omega t\cos kx\big)\big(\omega A\cos\omega t\sin kx\big) \\ &= -TA^2\omega k\big(\sin kx\cos kx\big)\big(\sin\omega t\cos\omega t\big). \end{aligned} \tag{71}$$

In general, this is nonzero, so there *is* energy flow across a given point. However, at a given value of $x$, the average (over one period) of $\sin\omega t\cos\omega t$ (which equals $(1/2)\sin 2\omega t$) is zero. So the *average* power is zero, as we wanted to show.

The difference between a traveling wave and a standing wave is the following. Mathematically: for a traveling wave of the form $A\cos(kx-\omega t)$, the two derivatives in Eq. (71) produce the same $\sin(kx-\omega t)$ function, so we end up with the *square* of a function, which is always positive. There can therefore be no cancelation. But for a standing wave, Eq. (71) yields a bunch of different functions and no squares, and they average out to zero.

Physically: in a traveling wave, the transverse force that a given dot on the string applies to the string on its right is always in phase (or 180° out of phase, depending on the direction of the wave's motion) with the velocity of the dot. This is due to the fact that $\partial\psi/\partial x$ is proportional to $\partial\psi/\partial t$ for a traveling wave. So the power, which is the product of the transverse force and the velocity, always has the same sign. There is therefore never any cancelation between positive and negative amounts of work being done.

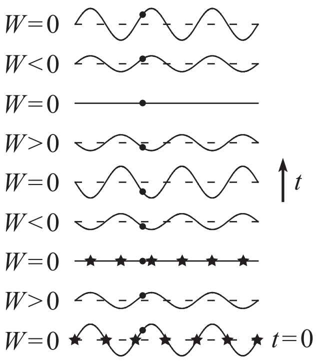



**Figure 25**

However, for the standing wave $A\sin\omega t\sin kx$, the transverse force is proportional to $-\partial\psi/\partial x = -kA\sin\omega t\cos kx$, while the velocity is proportional to $\partial\psi/\partial t = \omega A\cos\omega t\sin kx$. For a given value of $x$, the $\sin kx$ and $\cos kx$ functions are constant, so the $t$ dependence tells us that the transverse force is 90° ahead of the velocity. So half the time the force is in the same direction as the velocity, and half the time it is in the opposite direction. The product integrates to zero, as we saw in Eq. (71).

The situation is summarized in Fig. 25, which shows a series of nine snapshots throughout a full cycle of a standing wave. $W$ is the work that the dot on the string does on the string to its *right*. Half the time $W$ is positive, and half the time it is negative. The stars show the points with maximum energy density. When the string is instantaneously at rest at maximal curvature, the nodes have the greatest energy density, in the form of potential energy. The nodes are stretched maximally, and in contrast there is never any stretching at the antinodes. There is no kinetic energy anywhere in the string. A quarter cycle later, when the string is straight and moving quickest, the antinodes have the greatest energy density, in the form of kinetic energy. The antinodes are moving fastest, and in contrast there is never any motion at the nodes. There is no potential energy anywhere in the string.

We see that the energy continually flows back and forth between the nodes and antinodes. Energy never flows across a node (because a node never moves and therefore can do no work), nor across an antinode (because an antinode never applies a transverse force and therefore can do no work). The energy in each "half bump" (a quarter of a wavelength) of the sinusoidal curve is therefore constant. It flows back and forth between one end (a node/antinode) and the other end (an antinode/node). In other words, it flows back and forth across a point such as the dot we have chosen in the figure. This is consistent with the fact that the dot is doing work (positive or negative), except at the quarter-cycle points where $W = 0$.

## 4.6 Attenuation

What happens if we add some damping to a transverse wave on a string? This damping could arise, for example, by immersing the string in a fluid. As with the drag force in the spring/mass system we discussed in Chapter 1, we'll assume that this drag force is proportional to the (transverse) velocity of the string. Now, we usually idealize a string as having negligible thickness, but such a string wouldn't experience any damping. So for the purposes of the drag force, we'll imagine that the string has some thickness that produces a drag force of $-(\beta\, dx)v_y$ on a length $dx$ of string, where $\beta$ is the drag coefficient per unit length. The longer the piece, the larger the drag force.

The transverse forces on a little piece of string are the drag force along with the force arising from the tension and the curvature of the string, which we derived in Section 4.1. So the transverse $F = ma$ equation for a little piece is obtained by taking the $F = ma$ equation in Eq. (3) and tacking on the drag force. Since $v_y = \partial\psi/\partial t$, the desired $F = ma$ (or rather, $ma = F$) equation is

$$(\mu\, dx)\frac{\partial^2\psi}{\partial t^2} = T\, dx\frac{\partial^2\psi}{\partial x^2} - (\beta\, dx)\frac{\partial\psi}{\partial t} \quad\Longrightarrow\quad \boxed{\frac{\partial^2\psi}{\partial t^2} + \Gamma\frac{\partial\psi}{\partial t} = v^2\frac{\partial^2\psi}{\partial x^2}} \tag{72}$$

where $\Gamma \equiv \beta/\mu$ and $v^2 = T/\mu$. To solve this equation, we'll use our trusty method of guessing an exponential solution. If we guess

$$\psi(x,t) = De^{i(\omega t - kx)} \tag{73}$$

and plug this into Eq. (72), we obtain, after canceling the factor of $De^{i(\omega t-kx)}$,

$$-\omega^2 + \Gamma(i\omega) = -v^2k^2. \tag{74}$$

This equation tells us how $\omega$ and $k$ are related, but it doesn't tell us what the motion looks like. The motion can take various forms, depending on what the given boundary conditions are. To study a concrete example, let's look at the following system.

Consider a setup where the left end of the string is located at $x = 0$ (and it extends rightward to $x = \infty$), and we arrange for that end to be driven up and down sinusoidally with a constant amplitude $A$. In this scenario, $\omega$ must be real, because if it had a complex value of $\omega = a + bi$, then the $e^{i\omega t}$ factor in $\psi(x,t)$ would involve a factor of $e^{-bt}$, which decays with time. But we're assuming a steady-state solution with constant amplitude $A$ at $x = 0$, so there can be no decay in time. Therefore, $\omega$ must be real. The $i$ in Eq. (74) then implies that $k$ must have an imaginary part. Define $K$ and $-i\kappa$ be the real and imaginary parts of $k$, that is,

$$k = \frac{1}{v}\sqrt{\omega^2 - i\Gamma\omega} \equiv K - i\kappa. \tag{75}$$

If you want to solve for $K$ and $\kappa$ in terms of $\omega$, $\Gamma$, and $v$, you can square both sides of this equation and solve a quadratic equation in $K^2$ or $\kappa^2$. But we won't need the actual values. You can show however, that if $K$ and $\omega$ have the same sign (which they do, since we're looking at a wave that travels rightward from $x = 0$), then $\kappa$ is positive. Plugging $k \equiv K - i\kappa$ into Eq. (73) gives

$$\psi(x,t) = De^{-\kappa x}e^{i(\omega t - Kx)}. \tag{76}$$

A similar solution exists with the opposite sign in the imaginary exponent (but the $e^{-\kappa x}$ stays the same). The sum of these two solutions gives the actual physical (real) solution,

$$\boxed{\psi(x,t) = Ae^{-\kappa x}\cos(\omega t - Kx + \phi)} \tag{77}$$

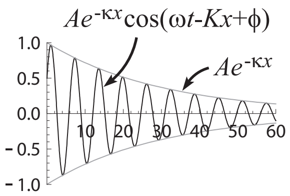



**Figure 26**

where the phase $\phi$ comes from the possible phase of $D$, which may be complex. (Equivalently, you can just take the real part of the solution in Eq. (76).) The coefficient $A$ has been determined by the boundary condition that the amplitude equals $A$ at $x = 0$. Due to the $e^{-\kappa x}$ factor, we see that $\psi(x,t)$ decays with *distance*, and not time. $\psi(x,t)$ is a rightward-traveling wave with the function $Ae^{-\kappa x}$ as its envelope. A snapshot in time is shown in Fig. 26, where we have arbitrarily chosen $A = 1$, $\kappa = 1/30$, $K = 1$, and $\phi = \pi/3$. The snapshot corresponds to $t = 0$. Such a wave is called an *attenuated* wave, because it tapers off as $x$ grows.

Let's consider the case of small damping. If $\Gamma$ is small (more precisely, if $\Gamma/\omega$ is small), then we can use a Taylor series to write the $k$ in Eq. (75) as

$$k = \frac{\omega}{v}\sqrt{1 - \frac{i\Gamma}{\omega}} \approx \frac{\omega}{v}\left(1 - \frac{i\Gamma}{2\omega}\right) = \frac{\omega}{v} - \frac{i\Gamma}{2v} \equiv K - i\kappa. \tag{78}$$

Therefore, $\kappa = \Gamma/2v$. The envelope therefore takes the form, $Ae^{-\Gamma x/2v}$. So after each distance of $2v/\Gamma$, the amplitude decreases by a factor $1/e$. If $\Gamma$ is very small, then this distance is very large, which makes sense. Note that this distance $2v/\Gamma$ doesn't depend on $\omega$. No matter how fast or slow the end of the string is wiggled, the envelope dies off on the same distance scale of $2v/\Gamma$ (unless $\omega$ is slow enough so that we can't work in the approximation where $\Gamma/\omega$ is small). Note also that $K \approx \omega/v$ in the $\Gamma \to 0$ limit. This must be the case, of course, because we must obtain the undamped result of $k = \omega/v$ when $\Gamma = 0$. The opposite case of large damping (more precisely, large $\Gamma/\omega$) is the subject of Problem [to be added].

If we instead have a setup with a uniform wave (standing or traveling) on a string, and if we then immerse the whole thing at once in a fluid, then we will have decay in *time*, instead of distance. The relation in Eq. (74) will still be true, but we will now be able to say that $k$ must be real, because all points on the string are immersed at once, so there is no preferred value of $x$, and hence no decay as a function of $x$. The $i$ in Eq. (74) then implies that $\omega$ must have an imaginary part, which leads to a time-decaying $e^{-\alpha t}$ exponential factor in $\psi(x,t)$.