Auxiliar #1 Mecánica Oscilaciones

$*\quad m\frac{d^2x}{dt} + kx = F_0\,\text{sen}(wt)$ se propone una solución x_p

$\quad \hookrightarrow \; w_0 = \sqrt{\frac{k}{m}}$

$x = A\,\text{sen}(wt)$ y se sustituye en la ecuación original

$\dot{x} = +A\cos(wt)w \quad ; \quad \ddot{x} = -A\,\text{sen}(wt)w^2$

$\Rightarrow \quad -mA\,\text{sen}(wt)w^2 + kA\,\text{sen}(wt) = F_0\,\text{sen}(wt) \qquad w_0 = \sqrt{\frac{k}{m}}$

$$\Rightarrow \quad A = \frac{F_0}{m(w_0^2 - w^2)}$$

$$x = x_h + x_p = \frac{F_0}{m(w_0^2-w^2)}\,\text{sen}(wt) + A\,\text{sen}(w_0t) + B\cos(w_0t)$$

con condiciones iniciales $x(0) = x_0$ y $\dot{x}(0) = V_0$

$\Rightarrow \quad x(0) = B = x_0$

$$\Rightarrow \quad \dot{x}(0) = \frac{F_0}{m(w_0^2-w^2)}w\cos(wt)\Big|_{t=0} + A\cos(w_0t)w_0\Big|_{t=0} - B\,\text{sen}(w_0t)w_0\Big|_{t=0}$$

$$= \frac{F_0 w}{m(w_0^2-w^2)} + Aw_0 = V_0 \quad \Rightarrow \quad A = \frac{V_0}{w_0} - \frac{F_0 w}{mw_0(w_0^2-w^2)}$$

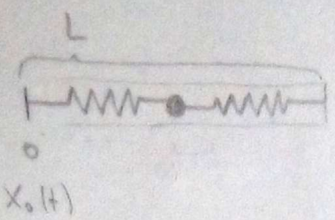
$$\Rightarrow \quad x = \frac{F_0}{m(w_0^2-w^2)}\,\text{sen}(wt) + x_0\cos(w_0t) + \frac{V_0 m(w_0^2-w^2) - F_0 w}{mw_0(w_0^2-w^2)}\,\text{sen}(w_0t)$$

$\lim_{w\to w_0}$

$$x = x_0\cos(w_0t) + \frac{V_0}{w_0}\,\text{sen}(w_0t) + \frac{F_0\,\text{sen}(w_0t)}{2mw_0^2} + \frac{F_0\cos(w_0t)\,t}{2mw_0}$$

término debido al forzaje

en $t\to\infty$ domina el término $\frac{F_0}{2mw_0}\cos(w_0t)\,t$ que diverge



$x_0(t) = C\cos(wt)$

(a) $m\ddot{x} = -k(x - l_0 - x_0(t)) + k(L - x - l_0 + x_0(t))$

$m\ddot{x} = -kx + kl_0 + kx_0(t) + kL - kx - \cancel{kl_0} + kx_0(t)$

$m\ddot{x} = -2k\left(x - \frac{L}{2} - x_0(t)\right) = -2k\left(x - \frac{L}{2}\right) + 2kC\cos(wt)$

$m\ddot{x} + 2k\left(x - \frac{L}{2}\right) = 2kC\cos(wt)$ ecuación oscilador forzado

(con $\Delta x = x - \frac{L}{2}$ y constante $2k$)

(b) $x_p = A\cos(wt)$ $\qquad \dot{x} = -A\,\mathrm{sen}(wt)\,w$ $\qquad \ddot{x} = -A\cos(wt)\,w^2$

$-mA\cos(wt)w^2 + 2kA\cos(wt) = 2kC\cos(wt)$

$\Rightarrow A = \dfrac{C\left(\frac{2k}{m}\right)}{\left(w^2 - \frac{2k}{m}\right)}$ con $w_0^2 = \dfrac{2k}{m}$

(c) Si hubiera roce viscoso $F_v = -c\dot{x}$

$m\ddot{x} = -2k\left(x - \frac{L}{2}\right) + 2kC\cos(wt) - c\dot{x}$

* Para resolver osciladores se propone solución $x = e^{pt}$

$$\dot{x} = p e^{pt}$$

$$\ddot{x} = p^2 e^{pt}$$

Oscilador amortiguado:

$m\ddot{x} = -kx - c\dot{x} \Rightarrow \ddot{x} = -w_0^2 x - \frac{c}{m}\dot{x}$, si aplicamos el método

$$\Rightarrow \quad p^2 e^{pt} = -w_0^2 e^{pt} - \frac{c}{m} p e^{pt}$$

$$\Rightarrow \quad p^2 + p\frac{c}{m} + w_0^2 = 0 \Rightarrow p = -\frac{c}{2m} \pm \underbrace{\sqrt{\left(\frac{c}{2m}\right)^2 - w_0^2}}_{\Delta = \left(\frac{c}{2m}\right)^2 - w_0^2}$$

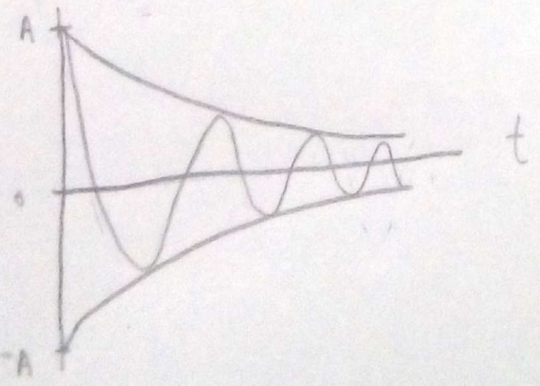
Entonces la solución general tiene la forma:

$$x(t) = \left(A_1 e^{t\sqrt{\Delta}} + A_2 e^{-t\sqrt{\Delta}}\right) e^{-\frac{c}{2m}t}$$

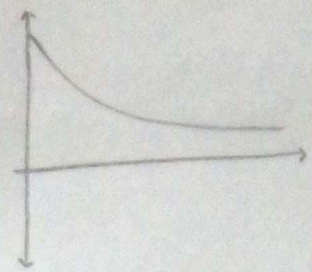
* $\Delta < 0$ osea $w_0^2 > \left(\frac{c}{2m}\right)^2$ amortiguado

$$x(t) = \left(A_1 e^{it\sqrt{\Delta}} + A_2 e^{-it\sqrt{\Delta}}\right) e^{-\frac{c}{2m}t} \; ; \; A_1 = \frac{B}{2} e^{i\theta} \quad A_2 = \frac{B}{2} e^{-i\theta}$$

$$\Rightarrow (1) \; x(t) = B e^{-\frac{c}{2m}t} \cos(\sqrt{\Delta}t + \theta) \quad \left[e^{i(\sqrt{\Delta}+\theta)} = \cos(t\sigma+\theta) + i \,\mathrm{sen}(t\sigma+\theta) \right]$$



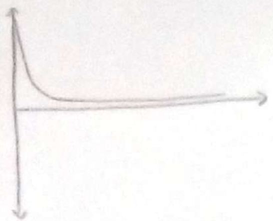
* $\Delta > 0$ osea $\omega_0^2 < \left(\frac{c}{2m}\right)^2$ sobreamortiguado



(2) $x(t) = \left(A_1 e^{\Delta t} + A_2 e^{-\Delta t}\right) e^{-\frac{c}{2m}t}$

↳ domina la solución

* $\Delta = 0$ osea $\omega_0^2 = \left(\frac{c}{2m}\right)^2$ criticamente amortiguado



(3) $x(t) = A e^{-\frac{c}{2m}t}$

Para encontrar las constantes se deben solucionar con las C.I

→ por ejemplo $x(t=0) = x_0$ $\quad \dot{x}(t=0) = 0$

(1) ⇒ $x(t=0) = B\cos(\theta) = x_0 \Rightarrow B = \frac{x_0}{\cos(\theta)}$

$\dot{x}(t=0) = -\frac{c}{2m} B\cos(\theta) - B\operatorname{sen}(\theta)\sqrt{\Delta} = 0 \Rightarrow \tan(\theta) = -\frac{c}{2m\sqrt{\Delta}}$

(2) ⇒ $x(t=0) = A_1 + A_2 = x_0$

$\dot{x}(t=0) = \left(\Delta - \frac{c}{2m}\right) A_1 - \left(\Delta + \frac{c}{2m}\right) A_2 = 0$

(3) ⇒ $x(t=0) = A = x_0$