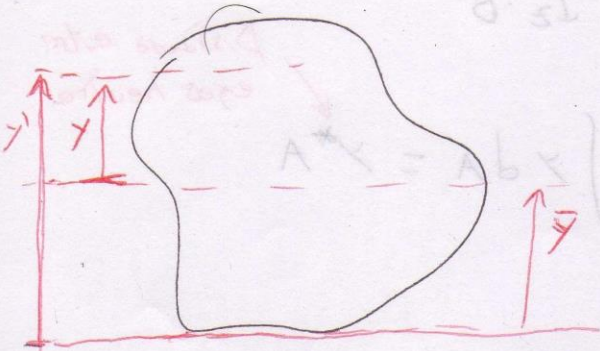


* Cálculo de propiedades de Área (I_z, Q)

↳ 2º momento de área ($\bar{I}_z$)



$$\bar{y} \text{ (Eje neutro)} = \frac{\int y' dA}{A}$$

$$\bar{I}_z = \int y'^2 dA = \int (y' - \bar{y})^2 dA$$

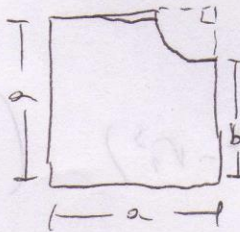
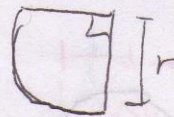
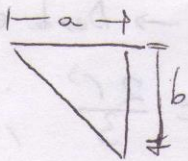
• Si hay varias fig. geométricas

$$\bar{y}_T = \frac{\sum \bar{y}_i \cdot A_i}{\sum A_i}$$

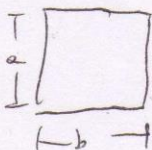
• Traslado de eje de rotación (Teorema de los ejes paralelos)

$$\bar{I}_z = I_z + d^2 A$$

* Encontrar eje neutro y ~~momento~~ I_z de:



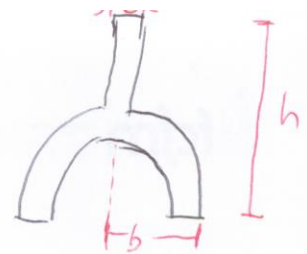
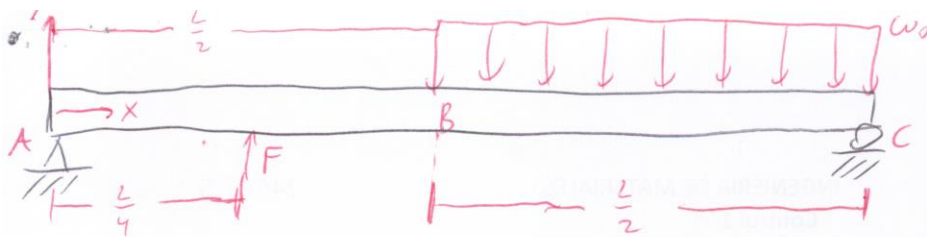
* Secciones conocidas:



$$I_z = \frac{ba^3}{12}$$



$$I_z = \frac{\pi d^4}{64}$$



(1): Semicírculo de radio b completo
(2): Semicírculo de radio $(b-e)$

$$\bar{Y}_T = \frac{\sum A_i Y_i}{\sum A_i} = \frac{(Y_1 \cdot A_1 - Y_2 \cdot A_2) + Y_3 \cdot A_3}{A_1 - A_2 + A_3}$$

$$\begin{aligned} * Y_1 &= \frac{4b}{3\pi} \\ A_1 &= \frac{\pi b^2}{2} \end{aligned}$$

$$\begin{aligned} * Y_2 &= \frac{4(b-e)}{3\pi} \\ A_2 &= \frac{\pi(b-e)^2}{2} \end{aligned}$$

$$\begin{aligned} * Y_3 &= \frac{(h-b)}{2} + b = \frac{b+h}{2} \\ A_3 &= e(h-b) \end{aligned}$$

Desde la base!

$$\therefore \bar{Y}_T = 0,07468 \text{ m} = 7,468 \text{ cm}$$

b) $\bar{I}_T = (\bar{I}_1 - \bar{I}_2 + \bar{I}_3)$ ← C/r a eje neutro de fig. completa $\bar{Y}_T$

$$\begin{aligned} \leadsto I_1 &= 0,1098 b^4 \Rightarrow \bar{I}_1 = I_1 + (Y_1 - \bar{Y}_T)^2 A_1 \\ &= 0,1098 b^4 + \left(\frac{4b}{3\pi} - \bar{Y}_T\right)^2 \cdot \frac{\pi b^2}{2} = 1643,56 \text{ cm}^4 \end{aligned}$$

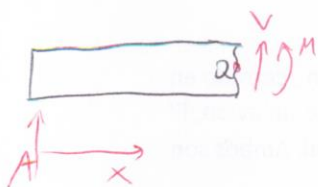
$$\begin{aligned} \leadsto I_2 &= 0,1098 (b-e)^4 \Rightarrow \bar{I}_2 = I_2 + (Y_2 - \bar{Y}_T)^2 A_2 \\ &= 0,1098 (b-e)^4 + \left(\frac{4(b-e)}{3\pi} - \bar{Y}_T\right)^2 \cdot \frac{\pi (b-e)^2}{2} \\ &= 2413,88 \text{ cm}^4 \end{aligned}$$

$$\begin{aligned} \leadsto I_3 &= \frac{e \cdot (h-b)^3}{12} \Rightarrow \bar{I}_3 = I_3 + (Y_3 - \bar{Y}_T)^2 A_3 \\ &= \frac{e(h-b)^3}{12} + \left(\frac{(h+b)}{2} - \bar{Y}_T\right)^2 \cdot e(h-b) = 283,28 \text{ cm}^4 \end{aligned}$$

$$\therefore \bar{I}_T = 4340,72 \text{ cm}^4 = 0,4341 \cdot 10^{-5} \text{ m}^4$$

c) Momento interno $M(x)$:

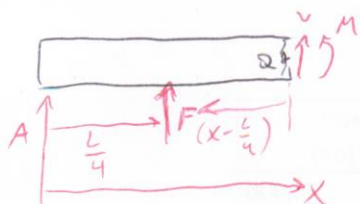
i) $0 < x < \frac{L}{4}$



$$\sum F_y = 0 \Rightarrow V(x) = A$$

$$\sum M_z = 0 \Rightarrow -Ax + M = 0 \Rightarrow \boxed{M(x) = Ax}$$

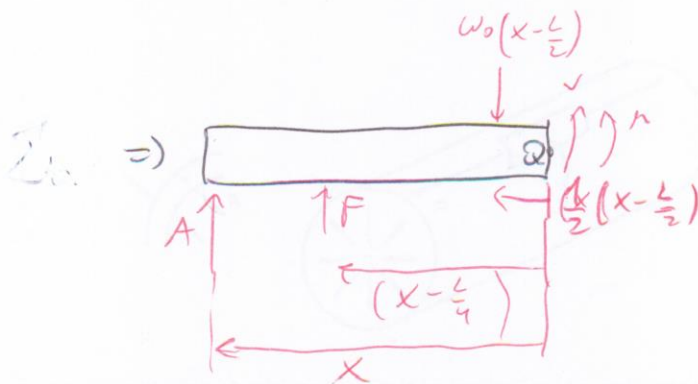
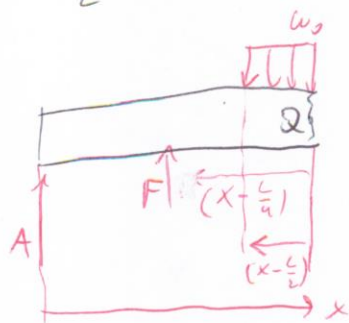
ii) $\frac{L}{4} < x < \frac{L}{2}$



$$\sum M_z = 0 \Rightarrow -Ax - F(x - \frac{L}{4}) + M = 0$$

$$\boxed{\therefore M(x) = Ax + F(x - \frac{L}{4})}$$

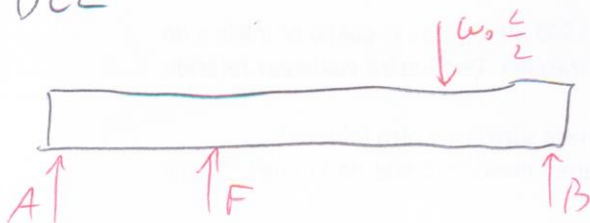
iii) $\frac{L}{2} < x < L$



$$\sum M_z = 0 \Rightarrow -Ax - F(x - \frac{L}{4}) + \frac{w_0}{2}(x - \frac{L}{2})^2 + M = 0$$

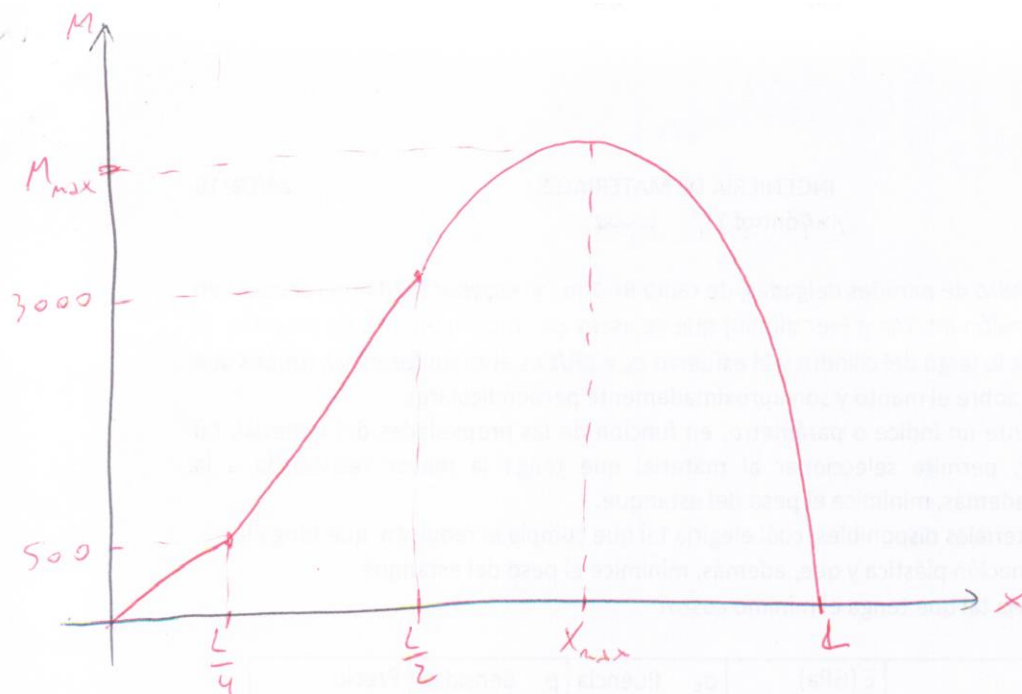
$$\boxed{\therefore M(x) = Ax + F(x - \frac{L}{4}) - \frac{w_0}{2}(x - \frac{L}{2})^2}$$

* DCL



$$\sum M_z = 0 \Rightarrow A \cdot L - F \cdot \frac{3L}{4} + \frac{w_0 L^2}{8} = 0$$

$$\boxed{\therefore A = \frac{w_0 L}{8} - \frac{3}{4} F = 500 \text{ N}}$$



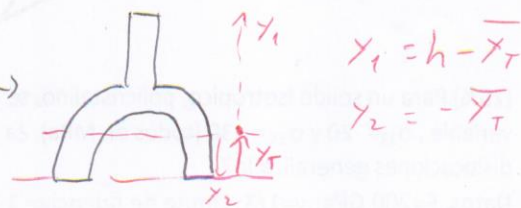
• Si el máximo se encuentra en el 3º tramo:

$$\left. \frac{dM(x)}{dx} \right|_{M_{max}} = 0 \Rightarrow A + F - w_0 \left(x - \frac{L}{2} \right) = 0$$

$$x_{max} = \frac{A+F}{w_0} + \frac{L}{2} = 2,625 \text{ m}$$

$$\Rightarrow M_{max} = 3781,25 \text{ N}\cdot\text{m}$$

• Es el máximo $\Rightarrow \sigma = - \frac{M_{max}}{I_z} \cdot y_{max} \rightarrow$



$$\Rightarrow y_1 = 9,532 \text{ cm} \text{ y } y_2 = -7,468 \text{ cm}$$

$$\therefore \sigma = \frac{-3781,25 \text{ N}\cdot\text{m}}{4,341 \cdot 10^{-5} \text{ m}^4} \cdot 9,532 \cdot 10^{-2} \text{ m} \Rightarrow \sigma_{max} = -8,303 \text{ MPa}$$

(Compresión)

- Cálculo de deflexión por integración:

$$\frac{d^4 y}{dx^4} = -\frac{w(x)}{EI} \quad \leftarrow \text{Fza distribuida}$$

* Modelación de Fzas mediante función escalón ($v(x-x_0)$) para Fzas distribuidas, y delta de Dirac ($\delta(x-x_0)$) para Fzas puntuales

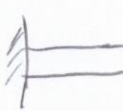
* Lo que ocurre en los extremos de la viga se reemplaza por condiciones de borde (Se necesitan 4 cond. de borde)

* Las cond. de borde son condiciones en la deflexión en los extremos, el ángulo de deflexión ($\frac{dy}{dx}$), el momento ($\frac{d^2 y}{dx^2}$) o de Fza de corte ($\frac{d^3 y}{dx^3}$)

$$\hookrightarrow \boxed{M = EI \frac{d^2 y}{dx^2}} \quad \wedge \quad \boxed{V = -EI \frac{d^3 y}{dx^3}}$$

* Algunos bordes:

 $\Rightarrow$ No hay deflexión $\rightarrow y(0) = 0$
No hay momento $\rightarrow \frac{d^2 y}{dx^2} = 0$

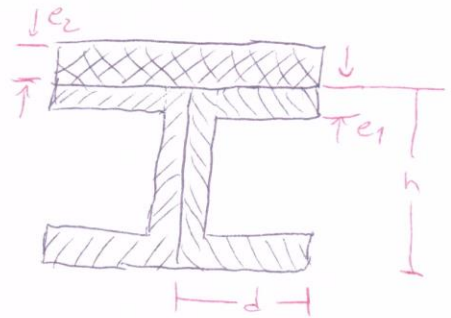
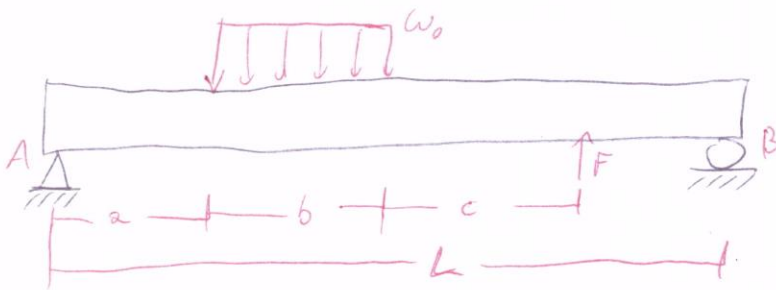
 $\Rightarrow$ No hay deflexión $\rightarrow y(0) = 0$
No hay ángulo $\rightarrow \frac{dy}{dx}(0) = 0$

 $\Rightarrow$ No hay momento $\rightarrow \frac{d^2 y}{dx^2}(0) = 0$

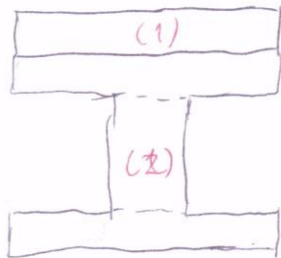
Balance de fzas $\rightarrow V(0) = -F = -k y(0)$

$\therefore -EI \frac{d^3 y}{dx^3}(0) = -k y(0)$

P1



a)



$$\bar{y}_T = \frac{\sum \bar{y}_i \cdot A_i}{\sum A_i} \quad / \quad \begin{aligned} \bar{y}_1 &= h + \frac{e_2}{2} ; A_1 = 2de_2 \\ \bar{y}_2 &= \frac{h}{2} ; A_2 = 4de_1 + 2e_1(h-2e_1) \end{aligned}$$

$$\Rightarrow \bar{y}_T = \frac{(h + \frac{e_2}{2}) \cdot 2de_2 + \frac{h}{2} (4de_1 + 2e_1(h-2e_1))}{2de_2 + 4de_1 + 2e_1(h-2e_1)}$$

$$\boxed{\bar{y}_T = 0,177 \text{ m}}$$

$$\sim \bar{I}_2 = \bar{I}_1 + \bar{I}_2$$

$$\hookrightarrow \bar{I}_1 = I_1 + (\bar{y}_1 - \bar{y}_T)^2 \cdot A_1 = \frac{2de_2^3}{12} + (\bar{y}_1 - \bar{y}_T)^2 \cdot 2de_2$$

$$\boxed{\bar{I}_1 = 5,538 \cdot 10^{-5} \text{ [m}^4\text{]}}$$

$$\hookrightarrow \bar{I}_2 = \left(\frac{2de_1^3}{12} + \left(\frac{e_1}{2} - \frac{h}{2} \right)^2 \cdot 2de_1 \right) + \left(\frac{2de_1^3}{12} + \left(h + \frac{e_1}{2} - \frac{h}{2} \right)^2 \cdot 2de_1 \right)$$

$$+ (\bar{y}_T - \bar{y}_2)^2 \cdot A_2$$

$$\hookrightarrow \bar{I}_2 = 2 \left(\frac{dh^3}{12} - \frac{(d-e_1)(h-2e_1)^3}{12} + (\bar{y}_T - \bar{y}_2)^2 \frac{A_2}{2} \right) = 1,701 \cdot 10^{-4} \text{ m}^4$$

$$\therefore \bar{I}_z = \cancel{7.239} = 2.255 \cdot 10^{-4} \text{ [m}^4\text{]}$$

$$b) \quad \frac{d^4 y}{dx^4} = - \frac{w(x)}{EI_z} \quad \begin{aligned} w(x) &= w_0(r(x-a) - r(x-a-b)) - \\ w(x) &= w_0(r(x-a) - r(x-a-b)) \\ &\quad + F\delta(x-a-b-c) \end{aligned}$$

$$\Rightarrow \frac{d^4 y}{dx^4} = - \frac{1}{EI_z} \left[w_0 [r(x-a) - r(x-\cancel{a}-b)] + F\delta(x-\cancel{a}-b-c) \right]$$

$$\frac{d^3 y}{dx^3} = - \frac{1}{EI_z} \left[w_0 ((x-a)r(x-a) - (x-k_1)r(x-k_1)) - F r(x-k_2) \right] + \alpha$$

$$\frac{d^2 y}{dx^2} = - \frac{1}{EI_z} \left[w_0 \left(\frac{(x-a)^2}{2} r(x-a) - \frac{(x-k_1)^2}{2} r(x-k_1) \right) - F(x-k_2) \cdot r(x-k_2) \right] + \alpha x + \beta$$

$$\beta \frac{dy}{dx} = - \frac{1}{EI_z} \left[w_0 \left(\frac{(x-a)^3}{6} r(x-a) - \frac{(x-k_1)^3}{6} r(x-k_1) \right) - F \frac{(x-k_2)^2}{2} r(x-k_2) \right] + \frac{\alpha x^2}{2} + \beta x + \gamma$$

$$y(x) = - \frac{1}{EI_z} \left[w_0 \left(\frac{(x-a)^4}{24} r(x-a) - \frac{(x-k_1)^4}{24} r(x-k_1) \right) - F \frac{(x-k_2)^3}{6} r(x-k_2) \right] + \frac{\alpha x^3}{6} + \frac{\beta x^2}{2} + \gamma x + \delta$$

• Condiciones de Borde:

$$1-) \frac{d^2 y}{dx^2}(0) = 0 \Rightarrow \beta = 0; \quad 2-) y(0) = 0 \Rightarrow \delta = 0$$

$$3-) \frac{d^2 y}{dx^2}(L) = 0 \Rightarrow \left[- \frac{1}{EI_z} \left[w_0 \left(\frac{(L-a)^2}{2} - \frac{(L-k_1)^2}{2} \right) - \frac{F(L-k_2)^2}{6} \right] + \alpha L + \beta L \right] = 0 \quad (1)$$

$$4-) y'(0) = 0 \Rightarrow \left[- \frac{1}{EI_z} \left[w_0 \left(\frac{(L-a)^3}{24} - \frac{(L-k_1)^3}{24} \right) - \frac{F(L-k_2)^3}{6} \right] + \frac{\alpha L^2}{6} + \gamma L \right] = 0 \quad (2)$$

$$* \delta_c(1) \rightarrow \alpha = \frac{1}{LEI_z} \left[\omega_0 \left(\frac{(L-a)^2}{2} - \frac{(L-k_1)^2}{2} \right) - F \frac{(L-k_2)^3}{6} \right]$$

$$\boxed{\alpha = \cancel{3,167 \cdot 10^{-4}} = -7,849 \cdot 10^{-5}}$$

$$* \delta_c(2)$$

$$\gamma = \frac{1}{LEI_z} \left[\omega_0 \left(\frac{(L-a)^4}{24} - \frac{(L-k_1)^4}{24} \right) - \frac{F(L-k_2)^5}{6} \right] - \frac{\alpha L^2}{6}$$

$$\boxed{\gamma = 1,402 \cdot 10^{-3}}$$

* Una vez calculadas las ctes., calculamos la deflexión en $x = L/2$

$$\gamma(L/2) = -\frac{1}{EI_z} \left[\omega_0 \cdot \frac{(L/2 - a)^4}{24} \right] + \frac{\alpha (L/2)^5}{6} + \gamma \cdot \frac{L}{2}$$

$$\boxed{\gamma(L/2) = 5,563 \cdot 10^{-3} \text{ [m]}}$$

* El est. normal ~~máximo~~ se calcula como:

$$\cancel{\sigma_{max} = -M(x) \cdot \gamma}$$

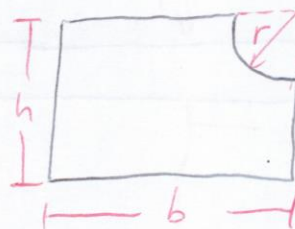
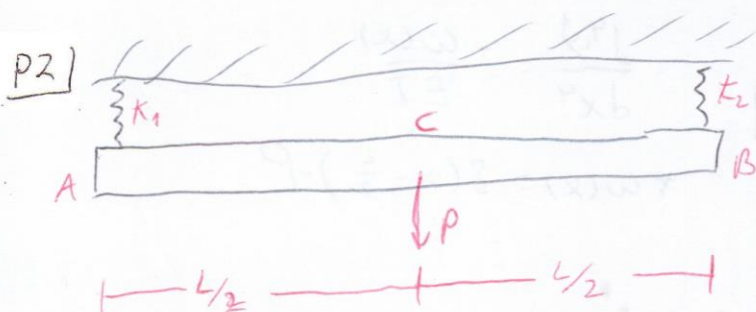
$$\sigma(x) = -\frac{M(x) \cdot \gamma}{I_z} \quad ; \quad M(x) = EI \frac{d^2 \gamma}{dx^2}$$

$$\boxed{\therefore \sigma(x) = -\frac{d^2 \gamma}{dx^2} EI \gamma}$$

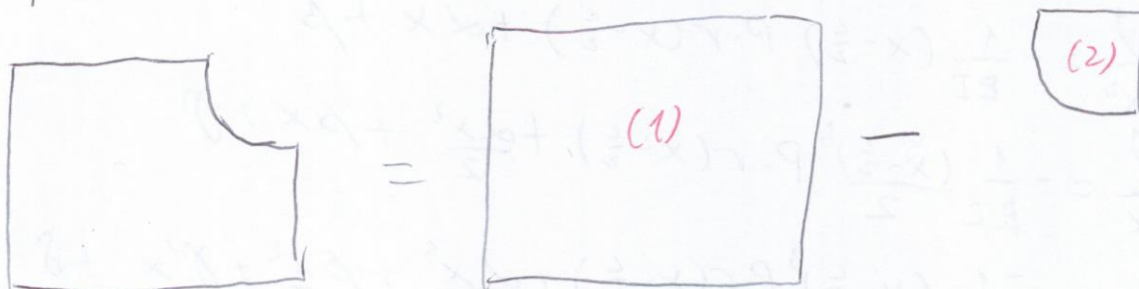
$$\hookrightarrow \frac{d^2 \gamma}{dx^2} \left(\frac{L}{2} \right) = -\frac{\omega_0}{2EI_z} \left(\frac{L}{2} - a \right)^2 + \alpha \cdot \frac{L}{2} = -4,279 \cdot 10^{-4}$$

$$\hookrightarrow \gamma_{max} = -\bar{\gamma}_1 = -0,177$$

$$\boxed{\therefore \sigma(L/2) = \cancel{18147466} \text{ [Pa]} \quad 1,515 \cdot 10^7 \text{ [Pa]}}$$



e) Propriedades de área:



$$(1): \bar{Y}_1 = \frac{h}{2} \quad \text{e} \quad I_1 = \frac{b \cdot h^3}{12}$$

$$(2): \left. \begin{array}{l} \bar{Y}_2 = \frac{4r}{3\pi} \quad \text{e} \quad I_2 = \frac{\pi r^4}{16} \end{array} \right\} \begin{array}{l} \text{Desde a base} \\ \bar{Y}_2 = h - \frac{4r}{3\pi} \end{array}$$

$$\Rightarrow \bar{Y}_T = \frac{\bar{Y}_1 \cdot A_1 - \bar{Y}_2 \cdot A_2}{A_1 - A_2} = \frac{\frac{h}{2} \cdot hb - \left(h - \frac{4r}{3\pi}\right) \cdot \frac{1}{4} \pi r^2}{hb - \frac{1}{4} \pi r^2} = 4,567 \text{ cm}$$

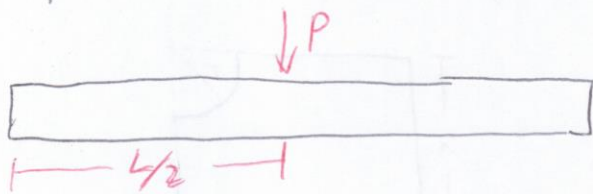
$$\boxed{\bar{Y}_T = 4,567 \text{ cm}}$$

$$\Rightarrow \bar{I}_1 = I_1 + S^2 A = \frac{bh^3}{12} + \left(\frac{h}{2} - \bar{Y}_T\right)^2 bh \Rightarrow \boxed{\bar{I}_1 = 1,278 \cdot 10^{-5} \text{ m}^4}$$

$$\bar{I}_2 = I_2 + S^2 A = \frac{\pi r^4}{16} + \left(\bar{Y}_T - \left(h - \frac{4r}{3\pi}\right)\right)^2 \cdot \frac{1}{4} \pi r^2 \Rightarrow \boxed{\bar{I}_2 = 3,38 \cdot 10^{-6} \text{ m}^4}$$

$$\therefore \bar{I}_T = \bar{I}_1 - \bar{I}_2 \Rightarrow \boxed{\bar{I}_T = 9,401 \cdot 10^{-6} \text{ m}^4}$$

b) Aplicando la ec. de la curva elástica



$$\frac{d^4 y}{dx^4} = -\frac{w(x)}{EI}$$

$$* w(x) = \delta(x - \frac{L}{2}) \cdot P$$

• Integrando:

$$\leadsto \frac{d^3 y}{dx^3} = -\frac{1}{EI} \cdot P \cdot r(x - \frac{L}{2}) + \alpha$$

$$\leadsto \frac{d^2 y}{dx^2} = -\frac{1}{EI} (x - \frac{L}{2}) \cdot P \cdot r(x - \frac{L}{2}) + \alpha x + \beta$$

$$\leadsto \frac{dy}{dx} = -\frac{1}{EI} \frac{(x - \frac{L}{2})^2}{2} P \cdot r(x - \frac{L}{2}) + \frac{\alpha x^2}{2} + \beta x + \gamma$$

$$\Rightarrow \hat{y}(x) = \frac{-1}{6EI} (x - \frac{L}{2})^3 P \cdot r(x - \frac{L}{2}) + \frac{\alpha x^3}{6} + \beta \frac{x^2}{2} + \gamma x + \delta$$

• Condiciones de Borde:

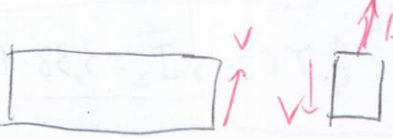
$$\hookrightarrow \boxed{\frac{d^2 y}{dx^2}(0) = 0 \text{ (1)} \quad \frac{d^2 y}{dx^2}(L) = 0 \text{ (2)}}$$

$\hookrightarrow$ Corte dif. en $x=0 \leadsto$  $\left. \begin{array}{l} V(0) + A = 0 \\ \Rightarrow V(0) = -A \end{array} \right\}$

$$* V(x) = -EI \frac{d^3 y}{dx^3}(x) \quad \Rightarrow \quad A = -k_1 \cdot \hat{y}(0)$$

$$\boxed{\therefore -EI \frac{d^3 y}{dx^3}(0) = k_1 \hat{y}(0) \text{ (3)}}$$

$\hookrightarrow$ Corte dif. en $x=L$

 $\left. \begin{array}{l} V(L) - B = 0 \Rightarrow V(L) = B \\ * B = -k_2 \hat{y}(L) \end{array} \right\}$

$$\boxed{\therefore EI \frac{d^3 y}{dx^3}(L) = k_2 \hat{y}(L) \text{ (4)}}$$

• Aplicando las cond. de Borde

$$(1) \frac{d^2 y}{dx^2}(0) = 0 \Rightarrow \boxed{\beta = 0}$$

$$(2) \frac{d^2 y}{dx^2}(L) = 0 \leadsto -\frac{1}{EI} \left(L - \frac{L}{2}\right) \cdot P + 2L = 0$$

$$\Rightarrow \boxed{\alpha = \frac{P}{2EI}}$$

$$(3) -EI \frac{d^3 y}{dx^3}(0) = K_1 y(0)$$

$$-2EI = K_1 \delta \leadsto \delta = -\frac{2EI}{K_1} = -\frac{P}{2EI} \frac{EI}{K_1} \Rightarrow \boxed{\delta = -\frac{P}{2K_1}}$$

$$(4) EI \frac{d^3 y}{dx^3}(L) = K_2 y(L)$$

$$-P + 2EI = K_2 \left(-\frac{1}{6EI} \left(L - \frac{L}{2}\right)^3 P + \frac{2L^3}{6} + \gamma L + \delta \right)$$

~~* Despejando~~

$$-P + \frac{P}{2} = K_2 \left(-\frac{L^3 P}{48EI} + \frac{2L^3}{6} + \frac{P}{2K_1} \right) + K_2 L \gamma$$

$$\therefore \gamma = \frac{1}{K_2 L} \left(-\frac{P}{2} + K_2 P \left(-\frac{L^3}{48EI} + \frac{L^3}{12EI} - \frac{1}{2K_1} \right) \right)$$

• Una vez calculadas todas las constantes, evaluemos $y\left(\frac{L}{2}\right)$ para encontrar el desplazamiento en el punto C.

$$y\left(\frac{L}{2}\right) = \alpha \frac{L^3}{6} + \gamma \frac{L}{2} + \delta = \alpha \frac{\left(\frac{L}{2}\right)^3}{6} + \gamma \frac{L}{2} + \delta$$

• Calculando los coef:

$$\alpha = 1,4 \cdot 10^{-3}$$

$$\gamma = -7,4 \cdot 10^{-5}$$

$$\delta = -8,3 \cdot 10^{-3}$$

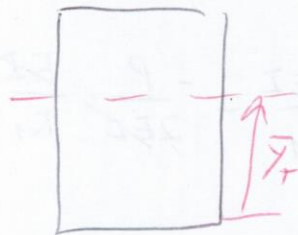
$$\therefore y\left(\frac{L}{2}\right) = -0,0241 \text{ m} = -2,41 \text{ cm}$$

a) $M = EI \frac{d^2 y}{dx^2} \Rightarrow M_{max} \sim \text{derivar } M(x) \text{ obteniendo } x \text{ para luego evaluar en } M(x)$

$\Rightarrow \frac{d^3 y}{dx^3} = 0 \sim \frac{1}{EI} P \cdot r(x - \frac{L}{2}) + \frac{P}{2EI}$

c) $M_{max} \sim \left(\frac{d^2 y}{dx^2} \right)_{max} \sim x = \frac{L}{2}$

$y_{max} \sim$



* $y = -\bar{y}_+ = -4,967 \text{ cm}$

* $y = h - \bar{y}_+ = 5,433 \text{ cm} \rightarrow M_{max} \text{ (Ext. sup)}$

$\therefore \sigma_{max} = -\frac{M}{I} \wedge M = EI \frac{d^2 y}{dx^2}$

$\Rightarrow \sigma_{max} = -EI \left(\frac{d^2 y}{dx^2} \right)_{max} \left. \vphantom{\frac{d^2 y}{dx^2}} \right\} \frac{d^2 y}{dx^2} \left(\frac{L}{2} \right) = \frac{PL}{4EI}$

$\therefore \sigma_{max} = -43,342 \text{ MPa}$