ME3204 δ_{ij}
Mecánica de Sólidos

Guía de Ejercicios Notación Indicial

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P1.- Símbolos básicos

a) Demuestre las siguientes propiedades

$$\mathbf{A}(\mathbf{u}\otimes\mathbf{v}) = (\mathbf{A}\mathbf{u})\otimes\mathbf{v}$$
$$(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}$$
$$\mathrm{tr}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$$

b) Demuestre las siguientes propiedades del símbolo de permutacion:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$
$$\varepsilon_{jmn}\varepsilon_{imn} = 2\delta_{ij}$$
$$\varepsilon_{ijk}\varepsilon_{ijk} = 6$$

c) Usando las propiedades del símbolo de permutación, demuestre que:

$$\det(\mathbf{A}-\lambda\mathbf{I}) = -\lambda^3 + \lambda^2\mathrm{tr}\mathbf{A} - \frac{1}{2}\lambda\left[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right] + \det\mathbf{A}$$

P2.- Operadores Diferenciales

a) Evalue las derivadas de los invariantes I_1 e I_2 del tensor $\mathbf{T}$:

$$\frac{\partial(\mathrm{I}_1)}{\partial\mathbf{T}} \qquad \frac{\partial(\mathrm{I}_2)}{\partial\mathbf{T}}$$

b) Demuestre las siguientes propiedades (A_{mn} constantes)

$$\nabla(A_{mn}x_m x_n) = (A_{pn}+A_{np})x_n\mathbf{e}_p$$
$$\mathrm{div}(\Phi\mathbf{u}) = \Phi\,\mathrm{div}\,\mathbf{u} + \mathbf{u}\cdot\mathrm{grad}\,\Phi$$
$$\mathrm{grad}(\Phi\mathbf{u}) = \mathbf{u}\otimes\mathrm{grad}\,\Phi + \Phi\mathrm{grad}\,\mathbf{u}$$

c) Sea **A** un campo tensorial de 2º orden. Utilizando el teorema de la divergencia, demuestre que:

$$\int_{\partial B} \mathbf{u} \cdot \mathbf{A}\mathbf{n} = \int_{B} \mathrm{div}\mathbf{A}^T\mathbf{u}\, da$$

donde ***B*** es un volumen cerrado, ***dB*** es la superficie que encierra dicho volumen, $()^{T}$ es la traspuesta de un tensor y **n** es un vector unitario normal exterior a la superficie en ***dB***.

* Notación indicial

- Convenio de suma de Einstein: $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$A_{ij} b_j = A_{i1} b_1 + A_{i2} b_2 + A_{i3} b_3 \left(\begin{array}{l} \text{3 componentes} \\ \hookrightarrow \text{Vector} \end{array} \right)$$

- Operadores (En coord. cartesianas)
  - ↳ Producto punto: $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_j (\underset{\sim}{e_i} \cdot \underset{\sim}{e_j}) = a_i b_j \delta_{ij} = a_i b_i$
  - ↳ Producto cruz: $\underset{\sim}{a} \times \underset{\sim}{b} = \varepsilon_{ijk} a_i b_j \underset{\sim}{e_k}$
  - ↳ Producto tensorial: $\underset{\sim}{a} \otimes \underset{\sim}{b} = a_i b_j (\underset{\sim}{e_i} \otimes \underset{\sim}{e_j})$

$$* \ \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad , \quad \varepsilon_{ijk} = \begin{cases} 1 & 1,2,3 ; 3,1,2 ; 2,3,1 \\ -1 & 3,2,1 \rightarrow 1,3,2 ; 2,1,3 \\ 0 & i = j \vee j = k \vee k = i \end{cases}$$

- Operadores diferenciales (en coord. cartesianas)
  - ↳ Divergencia: $\operatorname{div} \underset{\sim}{A} = \underset{\sim}{\nabla} \cdot \underset{\sim}{A} = \dfrac{\partial}{\partial x_k} (A_{ij} \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}) \cdot \underset{\sim}{e_k}$

$$= \frac{\partial A_{ij}}{\partial x_k} \underset{\sim}{e_i} (\underset{\sim}{e_j} \cdot \underset{\sim}{e_k}) = \frac{\partial A_{ij}}{\partial x_k} e_i \delta_{jk} = \frac{\partial A_{ik}}{\partial x_k} \underset{\sim}{e_i}$$

(1º orden)

  - ↳ Curl: $\operatorname{curl} \underset{\sim}{A} = \underset{\sim}{\nabla} \times \underset{\sim}{v} = \varepsilon_{kij} \dfrac{\partial v_i}{\partial x_k} \underset{\sim}{e_j}$

  - ↳ Gradiente: • $\operatorname{Grad} \underset{\sim}{A} = \underset{\sim}{\nabla} \otimes \underset{\sim}{A} = \dfrac{\partial}{\partial x_k} (A_{ij} \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}) \otimes \underset{\sim}{e_k}$

$$= \frac{\partial A_{ij}}{\partial x_k} \underset{\sim}{e_i} \otimes \underset{\sim}{e_j} \otimes \underset{\sim}{e_k} \ \text{(3º orden!)}$$

• $\operatorname{Grad} \underset{\sim}{v} = \dfrac{\partial}{\partial x_j} (v_i \underset{\sim}{e_i}) \otimes \underset{\sim}{e_j} = \dfrac{\partial v_i}{\partial x_j} \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$ (2º orden!)

$\boxed{\text{ETC!!}}$

P1)

a) $\cdot\ \underset{\sim}{A}(\underset{\sim}{u} \otimes \underset{\sim}{v}) = A_{ij}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j} (u_k \underset{\sim}{e_k} \otimes v_l \underset{\sim}{e_l})$

$= A_{ij} u_k v_l\, (\underset{\sim}{e_i} \otimes \underset{\sim}{e_j})(\underset{\sim}{e_k} \otimes \underset{\sim}{e_l})$

$= A_{ij} u_k v_l\, (\underset{\sim}{e_i} \otimes \underset{\sim}{e_l})\, \delta_{jk}$

$= A_{ij} u_j v_l (\underset{\sim}{e_i} \otimes \underset{\sim}{e_l})$

$= (A_{ij} u_j \underset{\sim}{e_i}) \otimes v_l \underset{\sim}{e_l} = (\underset{\sim}{A}\underset{\sim}{u}) \otimes \underset{\sim}{v}$ //

$\cdot\ (\underset{\sim}{a} \otimes \underset{\sim}{b})^T = [(a_i \underset{\sim}{e_i}) \otimes (b_j \underset{\sim}{e_j})]^T = (a_i b_j\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j})^T$

$= (a_i b_j\, \underset{\sim}{e_j} \otimes \underset{\sim}{e_i}) = [(b_j \underset{\sim}{e_j}) \otimes (a_i \underset{\sim}{e_i})] = (\underset{\sim}{b} \otimes \underset{\sim}{a})$ //

$\cdot\ tr(\underset{\sim}{a} \otimes \underset{\sim}{b}) = tr(a_i b_j\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}) = a_i b_j\, tr(\underset{\sim}{e_i} \otimes \underset{\sim}{e_j})$

$= a_i b_j \delta_{ij} = a_i b_i = \underset{\sim}{a} \cdot \underset{\sim}{b}$ //

b) $\cdot\ \varepsilon_{ijk}\varepsilon_{lmn}$ ~~$= \varepsilon_{ijk}\varepsilon_{lmn}$ → Separemos por casos:~~

↳ La def. del símbolo de permutación en función del delta de Kronecker es:

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

⤳ En nuestro caso:

$$\varepsilon_{ijk}\varepsilon_{imn} = \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} = \begin{vmatrix} 1 & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

$$= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn})$$
$$+ \delta_{in}(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki})$$

* Del 2° término

$$\delta_{im}(\delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn}) = \delta_{im}\delta_{ki}\delta_{jn} - \delta_{im}\delta_{ji}\delta_{kn})$$
$$= \delta_{mk}\delta_{jn} - \delta_{mj}\delta_{kn}$$

* Del 3° término

$$\delta_{in}(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}) = \delta_{in}\delta_{ji}\delta_{km} - \delta_{in}\delta_{ki}\delta_{jm}$$
$$= \delta_{jn}\delta_{mk} - \delta_{mj}\delta_{kn}$$

$$\Rightarrow \varepsilon_{ijk}\varepsilon_{imn} = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \cancel{\delta_{mk}\delta_{jn}} + \cancel{\delta_{mj}\delta_{kn}}$$
$$+ \cancel{\delta_{jn}\delta_{mk}} - \cancel{\delta_{mj}\delta_{kn}}$$

$$\boxed{\therefore \varepsilon_{ijk}\varepsilon_{imn} = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})}$$

- $\varepsilon_{jmn}\varepsilon_{imn} = 2\delta_{ij}$. A partir de la ec. anterior:

$$\boxed{\begin{aligned}\varepsilon_{jmn}\varepsilon_{imn} &= \cancel{(\ldots)} = \varepsilon_{mnj}\varepsilon_{mni} \\ &= (\delta_{nn}\delta_{ji} - \delta_{ni}\delta_{jn}) = (3\delta_{ji} - \delta_{ij}) = 2\delta_{ij}\end{aligned}}$$

- $\varepsilon_{ijk}\varepsilon_{ijk} = 6$. A partir de la ec. anterior

$$\varepsilon_{ijk}\varepsilon_{ijk} = 2\cdot\delta_{ii} = 2\cdot 3 \rightsquigarrow \boxed{\varepsilon_{ijk}\varepsilon_{ijk} = 6}$$

c) $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = -\lambda^3 + \lambda^2 tr(\underset{\sim}{A}) - \frac{1}{2}\lambda\left[(tr(\underset{\sim}{A}))^2 - tr(\underset{\sim}{A}^2)\right] + \det(\underset{\sim}{A}$

$$\boxed{\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = -\lambda^3 + I_1 \lambda^2 - I_2 \cdot \frac{\lambda}{2} + I_3}$$

- El determinante en notación indicial se escribe como:

$$\det(\underset{\sim}{S}) = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr} S_{ip} S_{jq} S_{kr}$$

- Aplicándolo a $\det(\underset{\sim}{A} - \lambda\underset{\sim}{I})$: $(\underset{\sim}{I} = \delta_{ij})$

$$\det(\underset{\sim}{A} - \lambda\underset{\sim}{I}) = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}(A_{ip} - \lambda\delta_{ip})(A_{jq} - \lambda\delta_{jq})(A_{kr} - \lambda\delta_{kr})$$

- Expandiendo y dejando ~~factorizado~~ factorizado en potencias de λ, se tiene:

$$\det(\underset{\sim}{A} - \lambda\underset{\sim}{I}) = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\Big[\overbrace{A_{ip}A_{jq}A_{kr}}^{(1)} - \underbrace{\lambda(A_{ip}A_{jq}\delta_{kr} + A_{ip}\delta_{jq}A_{kr} + \delta_{ip}A_{jq}A_{kr})}_{(2)} + \underbrace{\lambda^2(A_{ip}\delta_{jq}\delta_{kr} + \delta_{ip}A_{jq}\delta_{kr} + \delta_{ip}\delta_{jq}A_{kr})}_{(3)} - \underbrace{\lambda^3\delta_{ip}\delta_{jq}\delta_{kr}}_{(4)}\Big]$$

- De (1) $\rightsquigarrow \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}A_{ip}A_{jq}A_{kr} = \det(\underset{\sim}{A})$ (por definición)

- De (2) $\rightsquigarrow -\frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\lambda(A_{ip}A_{jq}\delta_{kr} + A_{ip}\delta_{jq}A_{kr} + \delta_{ip}A_{jq}A_{kr})$

  * Tomemos el 1º término

  $\rightsquigarrow A_{ip}A_{jq}\delta_{kr}\varepsilon_{ijk}\varepsilon_{pqr} = A_{ip}A_{jq}\varepsilon_{ijk}\varepsilon_{pqk} = A_{ip}A_{jq}(\varepsilon_{kij}\varepsilon_{kpq})$

  (por (b)) $= A_{ip}A_{jq}(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})$

  $= A_{ip}A_{jq}\delta_{ip}\delta_{jq} - A_{ip}A_{jq}\delta_{iq}\delta_{jp}$

  $= A_{ii}A_{jj} - A_{ij}A_{ji} = tr(A)\cdot tr(A) - tr(A^2)$

  $= (tr(A))^2 - tr(A^2)$ //

* Esto sucede análogamente con los otros 2 términos, luego

$\therefore (2) = \frac{1}{6}\lambda\left(3(trA)^2 + 3tr(A^2)\right) = \frac{1}{2}\lambda\left((trA)^2 + tr(A^2)\right)$ //

• De (3) $\sim \frac{\lambda^2}{6}\left(A_{1p}\delta_{jq}\delta_{kr} + \delta_{1p}A_{jq}\delta_{kr} + \delta_{1p}\delta_{jq}A_{kr}\right)\varepsilon_{1jk}\varepsilon_{pqr}$

* Tomemos el 1º término:

$\rightarrow \varepsilon_{1jk}\varepsilon_{pqr}A_{1p}\delta_{jq}\delta_{kr} = \varepsilon_{1jk}\varepsilon_{pjk}A_{1p} = \varepsilon_{jk1}\varepsilon_{jkp}A_{1p}$

(Por (b)) $= 2\delta_{1p}A_{1p} = 2A_{11} = 2tr(\underline{A})$ //

* Esto sucede análogamente con los otros 2 términos, luego:

$\therefore (3) = \frac{\lambda^2}{6}\left(6\,tr(\underline{A})\right) = \lambda^2 tr(\underline{A})$

• De (4) $\sim -\frac{\lambda^3}{6}\delta_{1p}\delta_{jq}\delta_{kr}\varepsilon_{1jk}\varepsilon_{pqr} = -\frac{\lambda^3}{6}\varepsilon_{1jk}\varepsilon_{1jk}$

$= -\frac{\lambda^3}{\cancel{6}}\cdot\cancel{6}$ (Por (b))

$\therefore (4) = -\lambda^3$

$$\boxed{\begin{aligned} &\therefore \det(\underline{A} - \lambda\underline{I}) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 \\ &I_1 = tr(\underline{A}) \\ &I_2 = \frac{1}{2}\left[(tr\underline{A})^2 - tr(\underline{A}^2)\right] \\ &I_3 = \det(\underline{A}) \end{aligned}}$$ //

[2]

$$* \quad \frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad ; \quad \boxed{\frac{\partial T_{ij}}{\partial T_{kl}} = \delta_{ik}\,\delta_{jl}}$$

$$\bullet \quad \frac{\partial I_1}{\partial \underset{\sim}{T}} = \frac{\partial\, tr(\underset{\sim}{T})}{\partial \underset{\sim}{T}} = \frac{\partial T_{ii}}{\partial T_{jk}} = \delta_{ij}\,\delta_{ik} = \delta_{jk} = \underset{\approx}{I}$$

$$\bullet \quad \frac{\partial I_2}{\partial \underset{\sim}{T}} = \frac{1}{2}\,\frac{\partial}{\partial \underset{\sim}{T}}\left((tr\,\underset{\sim}{T})^2 - tr(\underset{\sim}{T})^2\right) = \frac{1}{2}\left(\frac{\partial (tr\,\underset{\sim}{T})^2}{\partial \underset{\sim}{T}} - \frac{\partial (tr(T^2))}{\partial \underset{\sim}{T}}\right)$$

$$= \frac{1}{2}\left(\frac{\partial (T_{ii}T_{jj})}{\partial T_{kl}} - \frac{\partial (T_{ij}T_{ji})}{\partial T_{kl}}\right)$$

$$= \frac{1}{2}\left(\delta_{ik}\delta_{il}T_{jj} + T_{ii}\delta_{jk}\delta_{kl} - \delta_{ik}\delta_{jl}T_{ji} - T_{ij}\delta_{jk}\delta_{il}\right)$$

$$= \frac{1}{2}\left(\delta_{kl}T_{jj} + \delta_{kl}T_{ii} - 2\,T_{lk}\right) \quad (\text{original } T_{kl}!)$$

$$= \frac{1}{2}\left(\underset{\sim}{I}\cdot tr(\underset{\sim}{T}) + \underset{\sim}{I}\cdot Tr(\underset{\sim}{T}) - 2\,T^T\right)$$

$$\boxed{\therefore \quad \frac{\partial I_2}{\partial \underset{\sim}{T}} = tr(\underset{\sim}{T})\cdot \underset{\sim}{I} - \underset{\sim}{T}^T}$$

b) $\cdot\ \nabla(A_{mn} x_n x_m) = \frac{\partial}{\partial x_p}(A_{mn} x_n x_m)\, \underset{\sim}{e_p}$

$$= \left(A_{mn}\frac{\partial x_n}{\partial x_p} x_m + A_{mn}\frac{\partial x_m}{\partial x_p} x_n\right) \underset{\sim}{e_p}$$

$$= \left(A_{mn}\delta_{np} x_m + A_{mn}\delta_{mp} x_n\right) \underset{\sim}{e_p}$$

$$= \left(A_{mp} x_m + A_{pn} x_n\right) \underset{\sim}{e_p} \quad \left(\begin{matrix} m,n \text{ indices} \\ \text{muda} \end{matrix}\right)$$

$$= \left(A_{np} x_n + A_{pn} x_n\right) \underset{\sim}{e_p}$$

$$= \left(A_{np} + A_{pn}\right) \underset{\sim}{x_n}\, \underset{\sim}{e_p} \; //$$

$\cdot\ \mathrm{div}(\phi \underset{\sim}{u}) = \frac{\partial}{\partial x_j}(\phi\, u_i \underset{\sim}{e_i}) = u_i \underset{\sim}{e_i} \cdot \frac{\partial \phi}{\partial x_j} \cdot \underset{\sim}{e_j} + \phi \frac{\partial u_i}{\partial x_j}(e_i \cdot e_j)$

$$= u_i \frac{\partial \phi}{\partial x_j}(\underset{\sim}{e_i}\cdot\underset{\sim}{e_j}) + \phi \frac{\partial u_i}{\partial x_j}(\underset{\sim}{e_i}\cdot\underset{\sim}{e_j})$$

$$= u_i \frac{\partial \phi}{\partial x_j}\delta_{ij} + \phi \frac{\partial u_i}{\partial x_j}\delta_{ij}$$

$$= u_i \frac{\partial \phi}{\partial x_i} + \phi \frac{\partial u_i}{\partial x_i}$$

$$= \underset{\sim}{u} \cdot \mathrm{grad}\,\phi + \phi\, \mathrm{div}\, \underset{\sim}{u} \; //$$

$\cdot\ \mathrm{grad}(\phi \underset{\sim}{u}) = \underset{\sim}{\nabla} \otimes (\phi \underset{\sim}{u}) = \frac{\partial}{\partial x_j}(\phi\, \cancel{u_i}\, \underset{\sim}{e_i}) \otimes \underset{\sim}{e_j}$

$$= u_i \frac{\partial \phi}{\partial x_j}(\underset{\sim}{e_i} \otimes \underset{\sim}{e_j}) + \phi \frac{\partial u_i}{\partial x_j}(\underset{\sim}{e_i} \otimes \underset{\sim}{e_j})$$

$$= \underset{\sim}{u} \otimes \mathrm{grad}\,\phi + \phi\, \mathrm{grad}\, \underset{\sim}{u} \; //$$

c) Analicemos $\underset{\sim}{u} \cdot \underset{\sim}{A} \underset{\sim}{n}$

$$\underset{\sim}{u} \cdot \underset{\sim}{A} \underset{\sim}{n} = u_i \underset{\sim}{e_i} \cdot (A_{jk} \underset{\sim}{e_j} \otimes \underset{\sim}{e_k} \, n_l \underset{\sim}{e_l})$$

$$= u_i \underset{\sim}{e_i} \cdot (A_{jk} n_l (\underset{\sim}{e_j} \otimes \underset{\sim}{e_k}) \underset{\sim}{e_l})$$

$$= u_i \underset{\sim}{e_i} \cdot (A_{jk} n_l \underset{\sim}{e_j} \delta_{kl})$$

$$= u_i \underset{\sim}{e_i} \cdot (A_{jk} n_k \underset{\sim}{e_j})$$

$$= u_i A_{jk} n_k \, e_i \cdot e_j = u_i A_{jk} n_k \delta_{ij}$$

$$= u_i A_{ik} n_k = A_{ik} u_i n_k = \underset{\sim}{A}^T \underset{\sim}{u} \underset{\sim}{n}$$

$$\Rightarrow \int_{\partial B} \underset{\sim}{u} \cdot \underset{\sim}{A} \underset{\sim}{n} \, da = \int_{\partial B} (A^T \underset{\sim}{u}) \underset{\sim}{n} \, da = \int_B \operatorname{div} (\underset{\sim}{A}^T \underset{\sim}{u}) \, dV \quad //$$

Teorema de la divergencia