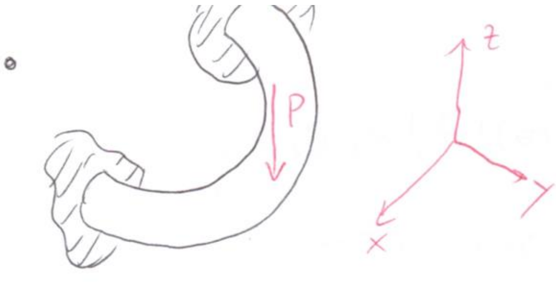
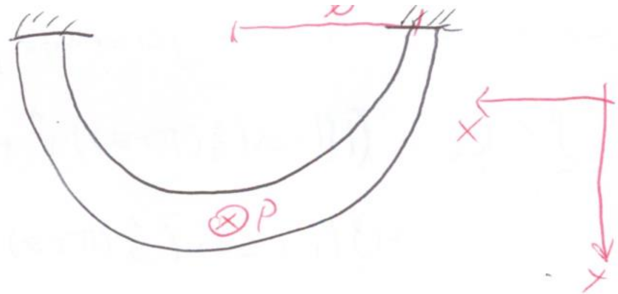




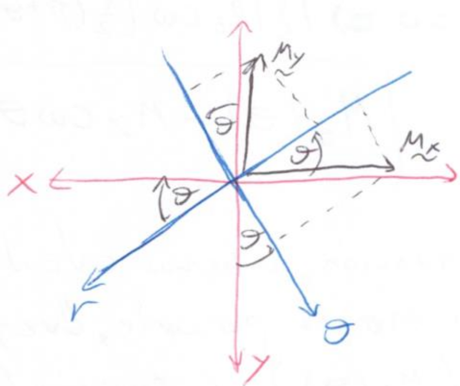
* Este problema es hiperestático. Técnicamente, debiésemos calcular las reacciones utilizando Castigliano en los puntos de apoyo. Para simplificar el problema, analizaremos la mitad de la dona

- $\sum F_z = 0 \Rightarrow \boxed{R_z = P/2}$
- $\sum_o M_x = 0 \Rightarrow -M_x - \frac{P}{2}\varnothing = 0 \Rightarrow \boxed{M_x = -\frac{P}{2}\varnothing}$
- $\sum_o M_y = 0 \Rightarrow \frac{P}{2}\varnothing - M_y = 0 \Rightarrow \boxed{M_y = \frac{P}{2}\varnothing}$

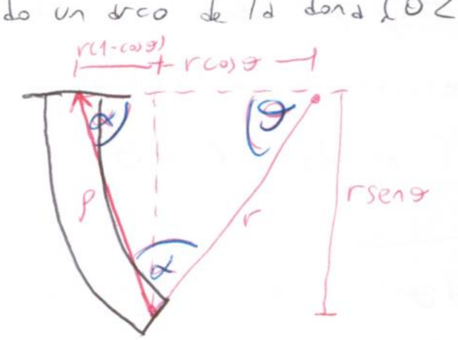
$\rightsquigarrow \underset{\sim}{R_z} = R_z \hat{z}$

$\underset{\sim}{M_x} = -M_x \cos\theta \, \hat{r} + M_x \operatorname{sen}\theta \, \hat{\theta}$

$\underset{\sim}{M_y} = -M_y \operatorname{sen}\theta \, \hat{r} + M_y \cos\theta \, \hat{\theta}$



* Analizando un arco de la dona $(0 < \theta < \pi/2)$



- $\alpha = \frac{1}{2}(\pi - \theta)$
- $|\underset{\sim}{\rho}| = \sqrt{(r(1-\cos\theta))^2 + (r\cos\theta)^2}$

  $|\underset{\sim}{\rho}| = r\sqrt{2(1-\cos\theta)}$

$\Rightarrow \underset{\sim}{\rho} = r\sqrt{2(1-\cos\theta)}\cos\alpha \, \hat{\imath} - r\sqrt{2(1-\cos\theta)}\operatorname{sen}\theta \, \hat{\jmath}$

$= |\underset{\sim}{\rho}|(\cos\alpha\cos\theta - \operatorname{sen}\alpha \operatorname{sen}\theta)\hat{r} + |\underset{\sim}{\rho}|(\operatorname{sen}\alpha\cos\theta - \cos\alpha \operatorname{sen}\theta)\hat{\theta}$

$= |\underset{\sim}{\rho}|\cos(\alpha+\theta)\,\hat{r} + |\underset{\sim}{\rho}|\operatorname{sen}(\alpha-\theta)\,\hat{\theta}$

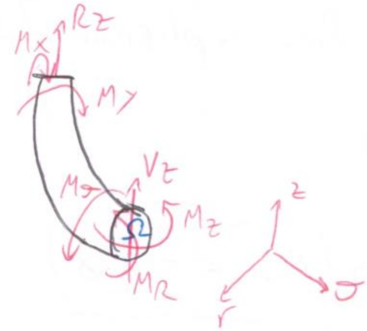
$\therefore \boxed{\underset{\sim}{\rho} = |\underset{\sim}{\rho}|\cos\left(\tfrac{1}{2}(\pi+\theta)\right)\hat{r} + |\underset{\sim}{\rho}|\operatorname{sen}\left(\tfrac{1}{2}(\pi+\theta)\right)\hat{\theta}}$

• Calculando el momento generado por R_z:

$$\underset{\sim}{\rho} \times R_z = \left(|\underset{\sim}{\rho}|\cos\left(\tfrac{1}{2}(\pi+\theta)\right)\hat{r} + |\underset{\sim}{\rho}|\operatorname{sen}\left(\tfrac{1}{2}(\pi+\theta)\right)\hat{\theta}\right)\times\left(R_z\hat{z}\right)$$

$$= |\underset{\sim}{\rho}| R_z \cos\left(\tfrac{1}{2}(\pi+\theta)\right)\hat{\theta} + |\underset{\sim}{\rho}| R_z \operatorname{sen}\left(\tfrac{1}{2}(\pi+\theta)\right)\hat{r}$$

• Cargas internas



$$\cdot \sum F_z = 0 \Rightarrow \boxed{V_z = -R_z = \frac{-P}{2}}$$

$$\cdot \sum_\Omega M_z = 0 \Rightarrow \boxed{M_z = 0}$$

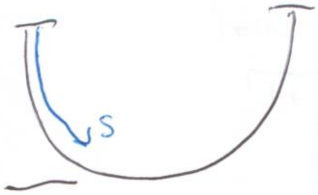
$$\cdot \sum_\Omega M_R = 0 \Rightarrow |\underset{\sim}{\rho}| R_z \operatorname{sen}\left(\tfrac{1}{2}(\pi+\theta)\right) - M_x\cos\theta - M_y \operatorname{sen}\theta + M_R(\theta) = 0$$

$$\boxed{M_R(\theta) = M_y \operatorname{sen}\theta + M_x\cos\theta - |\underset{\sim}{\rho}|\frac{P}{2}\operatorname{sen}\left(\tfrac{1}{2}(\pi+\theta)\right)}$$

$$\cdot \sum_\Omega M_\theta = 0 \Rightarrow |\underset{\sim}{\rho}| R_z \cos\left(\tfrac{1}{2}(\pi+\theta)\right) + M_x \operatorname{sen}\theta + M_y\cos\theta + M_\theta(\theta) = 0$$

$$\boxed{M_\theta(\theta) = M_y\cos\theta - M_x \operatorname{sen}\theta - |\underset{\sim}{\rho}|\frac{P}{2}\cos\left(\tfrac{1}{2}(\pi+\theta)\right)}$$

• A continuación, debemos calcular las energías de deformación. Despreciaremos energía por corte. Luego, existe energía de deformación por flexión ($M_R(\theta)$) y torsión ($M_\theta(\theta)$). Hay que tener en cuenta que se debe integrar a través de la línea de la sección de la figura:



$$U_{fl} = \int_S \frac{M(s)^2}{EI}\,ds \quad \wedge \quad ds = r\cdot d\theta$$

$$\boxed{\begin{aligned} \therefore U_{fl} &= 2\cdot\int_0^{\pi/2}\frac{M_R(\theta)^2}{2EI}\,r\,d\theta \\ U_{tor} &= 2\cdot\int_0^{\pi/2}\frac{M_\theta(\theta)^2}{2GJ}\,r\,d\theta \end{aligned}}$$

* El factor 2 es porque consideramos la energía de las dos mitades de la dona!!

• Trabajemos un poco las expresiones. Aplicando Teorema de Castigliano:

$$\delta = \frac{\partial U_T}{\partial P} = \frac{\partial}{\partial P}\left(U_{Fl} + U_{tor}\right) = \left(\frac{r}{EI}\int_0^{\pi/2} \frac{\partial (M_R(\theta)^2)}{\partial P} + \frac{r}{GJ}\int_0^{\pi/2} \frac{\partial (M_\theta(\theta)^2)}{\partial P}\right) d\theta$$

$$\boxed{\delta = \frac{2\varnothing}{EI}\int_0^{\pi/2} \frac{\partial M_R}{\partial P} \cdot M_R \, d\theta + \frac{2\varnothing}{GJ}\int_0^{\pi/2} \frac{\partial M_\theta}{\partial P} M_\theta \, d\theta}$$

Depende de θ!

$$\sim \frac{\partial M_R}{\partial P} = \frac{\varnothing}{2}(\operatorname{sen}\theta - \cos\theta) - \frac{|\underset{\sim}{\rho}|}{2}\operatorname{sen}\left(\tfrac{1}{2}(\pi + \theta)\right)$$

Depende de θ!

$$\sim \frac{\partial M_\theta}{\partial P} = \frac{\varnothing}{2}(\cos\theta + \operatorname{sen}\theta) - \frac{|\underset{\sim}{\rho}|}{2}\cos\left(\tfrac{1}{2}(\pi + \theta)\right)$$

$$\Rightarrow \delta = \frac{2\cancel{\varnothing}}{EI}\left(\frac{\pi}{8} P \cancel{\varnothing}^2\right) + \frac{2\varnothing}{GJ}\left(\frac{8+3\pi}{8} P \varnothing^2\right)$$

$$\boxed{\therefore \delta = \frac{\pi P \varnothing^3}{4EI} + \frac{(8+3\pi) P \varnothing^3}{4GJ}}$$