Pauta Súper Guía Complejos

P.1 PDQ $\forall z \in \mathbb{C}$ $z \neq -1 \wedge |z| = 1$

$$\frac{1+z}{1+\overline{z}} = z$$

En efecto

$$\frac{1+z}{1+\overline{z}} = \frac{1+z}{1+\overline{z}} \cdot \frac{z}{z}$$

$$= \frac{z(1+z)}{z+z\overline{z}}$$

$$= \frac{z(1+z)}{z+|z|^2}$$

$$= \frac{z(1+z)}{z+1}$$

$$= z$$

qed

$\boxed{P_2}$ $\quad (-1+i\sqrt{3})^{3n} + (-1-i\sqrt{3})^{3n} \quad \forall n \in \mathbb{N}$

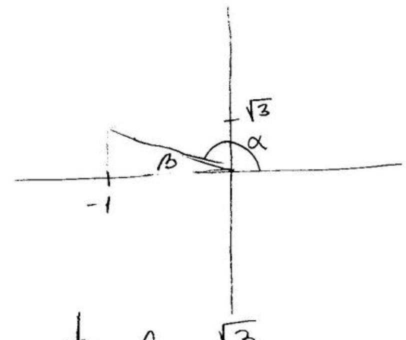
30 60 60 90
1 2 $\frac{\sqrt{3}}{2}$ 4
$\frac{1}{2}$

Sea $z = -1 + i\sqrt{3}$

$$= (z)^{3n} + (\bar{z})^{3n}$$

$$= \left(2e^{i\frac{2\pi}{3}}\right)^{3n} + \left(2e^{i\frac{-2\pi}{3}}\right)^{3n}$$

$$\sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2 \,//$$



$$\tan\beta = \frac{\sqrt{3}}{-1}$$

$$\Leftrightarrow \frac{\operatorname{sen}\beta}{\cos\beta} = \frac{\sqrt{3}}{-1}$$

$$\Leftrightarrow \frac{\operatorname{sen}\beta}{\cos\beta} = \frac{\sqrt{3}}{2} \cdot \frac{2}{-1}$$

$$\Rightarrow \beta = 60°$$

$$\Rightarrow \alpha = 120°$$

$$\alpha = \frac{2\pi}{3} \,//$$

$$= 2^{3n}\left(\left(e^{i\frac{2\pi}{3}}\right)^{3n} + \left(e^{-i\frac{2\pi}{3}}\right)^{3n}\right)$$

$$= 2^{3n}\left(e^{i\cancel{3}n\frac{2\pi}{\cancel{3}}} + e^{-i\frac{\cancel{3}n2\pi}{\cancel{3}}}\right)$$

$$= 2^{3n}\left(e^{i2\pi n} + e^{-i2\pi n}\right)$$

$$= 2^{3n}\left(\cos 2\pi n + i\operatorname{sen} 2\pi n + \cos -2\pi n + i\operatorname{sen} -2\pi n\right)$$

$$= 2^{3n}\left(\cos 2\pi n + \cancel{i\operatorname{sen} 2\pi n} + \cos 2\pi n \cancel{- i\operatorname{sen} 2\pi n}\right)$$

$$= 2^{3n}\left(2\cos 2\pi n\right) = \operatorname{Re}\left(z^{3n} + \bar{z}^{3n}\right) //$$

$$\Rightarrow \operatorname{Im}\left(z^{3n} + \bar{z}^{3n}\right) = 0 \,//$$

Notar que $\cos 2\pi n \;\; \forall n \in \mathbb{N}$ siempre es 1

$$\Rightarrow \operatorname{Re}\left(z^{3n} + \bar{z}^{3n}\right) = 2^{3n+1} \,//$$

$\boxed{P_3}$ $\omega^3 = 1 \quad \omega \in \mathbb{C} \quad \omega \neq 1$

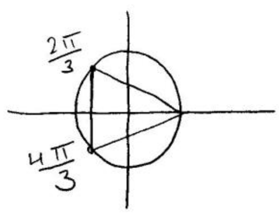
$\Rightarrow \omega = e^{\frac{2\pi r}{3}} \quad r \in \{0, 1, 2\}$

$\Leftrightarrow \omega = \begin{cases} e^0 & r=0 \longrightarrow \text{No puede ser pues } \omega \neq 1 \\ e^{\frac{2}{3}\pi} & r=1 \\ e^{\frac{4}{3}\pi} & r=2 \end{cases}$

$e^{\frac{2}{3}\pi} = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$ //

$\Rightarrow \omega = e^{\frac{2}{3}\pi} \vee e^{\frac{4\pi}{3}}$

Ambos resultados son conjugados



Como ambos resultados son raices cúbicas, entonces ω^3 y $\overline{\omega}^3$ serán 1, sin importar cual se escoja //

Ademas notar que $\omega^2 = \overline{\omega}$

$\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right)^2$

$= \frac{1}{4} - \frac{\sqrt{3}}{2} i - \frac{3}{4}$

$= \frac{-1}{2} - \frac{\sqrt{3}}{2} i$ //

$\therefore (1+\omega)^3$

$= (1 + 3\omega + 3\omega^2 + \omega^3)$

$= 2 + 3\omega + 3\overline{\omega}$

$= 2 + 6\,\mathrm{Re}(\omega)$

$= 2 + 6 \frac{-1}{2}$

$= 2 - 3 \Leftrightarrow -1$ //

$$(1+\omega^2)^9$$

$$= \left((1+\bar{\omega})^3\right)^3$$

$$= \left[\left(1+3\bar{\omega}+3\bar{\omega}^2+\bar{\omega}^3\right)\right]^3$$

$$= \left[\left(2+3\bar{\omega}+3\overline{(\omega^2)}\right)\right]^3$$

$$= \left[2+3\bar{\omega}+3\bar{\bar{\omega}}\right]^3$$

$$= \left[2+3\bar{\omega}+3\omega\right]^3$$

$$= [-1]^3 /\!/ = -1 /\!/$$

$$\therefore (1+\omega)^3 + (1+\omega^2)^9 + (1+\omega^3)^6$$

$$= -1 + -1 + (1+1)^6$$

$$= -2 + 64$$

$$= 62 /\!/$$ qed

P_3 Alternativo

$\omega \in \mathbb{C} \quad \omega^3 = 1 \quad \omega \neq 1$

$$\omega = \begin{cases} 1 \longrightarrow \text{no puede por } \omega \neq 1 \\ e^{i\frac{2\pi}{3}} \\ e^{i\frac{4\pi}{3}} \end{cases}$$

$$\Rightarrow \omega = \begin{cases} e^{i\frac{2\pi}{3}} \\ e^{i\frac{4\pi}{3}} \end{cases} \qquad \text{Sea } \omega_1 = e^{i\frac{2\pi}{3}}, \quad \omega_2 = e^{i\frac{4\pi}{3}}$$

Sabemos que $1 + \omega_1 + \omega_2 = 0$

$\Leftrightarrow -\omega_1 = 1 + \omega_2$

Además
$$\omega_1^2 = \left(e^{i\frac{2\pi}{3}}\right)^2 = e^{i\frac{4\pi}{3}} = \omega_2 \Rightarrow \omega_1^2 = \omega_2$$

Para ω_1

$$(1+\omega_1)^3 + (1+\omega_1^2)^9 + (1+\omega^3)^6$$
$$= (-\omega_1)^3 + (1+\omega_2)^9 + 2^6$$
$$= (-1) + (-\omega_1)^9 + 64$$
$$= (-1) + (-1)^3 + 64$$
$$= (-1) + (-1) + 64$$
$$= 62 \quad \text{qed}$$

Si se hace con ω_2 el resultado es el mismo.

$\underline{P_4}$ $f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}_+ \cup \{0\}$

$$f(z_1, z_2) = |z_1 + z_2|$$

PDQ $\forall z_1, z_2 \in \mathbb{C}$

$$f(z_1, z_2) \cdot f(\overline{z_1}, \overline{z_2}) \leqslant (|z_1| + |z_2|)^2$$

Em efecto $f(z_1, z_2)\, f(\overline{z}_1, \overline{z}_2)$

$$= |(z_1 + z_2)|\,|(\overline{z}_1 + \overline{z}_2)|$$

$$= |(z_1 + z_2)(\overline{z}_1 + \overline{z}_2)|$$

$$= |z_1\overline{z}_1 + z_2\overline{z}_2 + z_1\overline{z}_2 + \overline{z}_1 z_2|$$

$$\Rightarrow \left| |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 z_2)\right| \leqslant \left| |z_1|^2 + |z_2|^2 + 2|z_1 z_2| \right|$$

$$\Leftrightarrow \left| |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \cdot z_2)\right| \leqslant \left| (|z_1| + |z_2|)^2 \right|$$

$$\Leftrightarrow \left| |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \cdot z_2)\right| \leqslant (|z_1| + |z_2|)^2$$

$$\Rightarrow f(z_1, z_2)\, f(\overline{z}_1, \overline{z}_2) \leqslant (|z_1| + |z_2|)^2$$

qed

$\underline{P_5|}$ Encontrar $n \in \mathbb{N}$ que resuelva

$$\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{2n} - \left(\frac{\sqrt{3}+i}{2}\right)^{2n} = i\sqrt{3}$$

$$R = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} \qquad \tan\alpha = \frac{\frac{-1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \sqrt{\frac{3}{4} + \frac{1}{4}} \qquad \Leftrightarrow \tan\alpha = \frac{-1}{\sqrt{3}} \Rightarrow \alpha = \frac{-\pi}{6}$$

$$= 1 \qquad \tan\beta = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \Rightarrow \beta = \frac{\pi}{6}$$

$$\Rightarrow \left(e^{-i\frac{\pi}{6}}\right)^{2n} - \left(e^{i\frac{\pi}{6}}\right)^{2n} = i\sqrt{3}$$

$$\Leftrightarrow e^{-i\frac{\pi}{3}n} - e^{i\frac{\pi}{3}n} = i\sqrt{3}$$

$$\Leftrightarrow \cancel{\cos\frac{-\pi}{3}n} + i\,\mathrm{sen}\,\frac{-\pi}{3}n - \left(\cancel{\cos\frac{\pi}{3}n} + i\,\mathrm{sen}\,\frac{\pi}{3}n\right) = i\sqrt{3}$$

$$\Leftrightarrow -2i\,\mathrm{sen}\,\frac{\pi}{3}n = i\sqrt{3}$$

$$\Leftrightarrow i\,\mathrm{sen}\,\frac{\pi}{3}n = \frac{-\sqrt{3}}{2}i$$

$$\Rightarrow \frac{\pi}{3}n = k\pi + (-1)^k\left(\frac{-\pi}{3}\right)$$

$$\Leftrightarrow n = k3 + (-1)^{k+1} \qquad k \geq 1$$

//

$\boxed{P_6}$ $z \in \mathbb{C}$ $|z| > 1$ $\mathrm{Im}(z) > 0$

PDQ $\mathrm{Im}\left(z + \frac{1}{z}\right) > 0$

En efecto

$$z + \frac{1}{z}$$

$$= z + \frac{\overline{z}}{|z|^2}$$

$$= \frac{z|z|^2 + \overline{z}}{|z|^2} \quad //$$

Pero sabemos que

$$\mathrm{Im}(z + \overline{z}) = 0$$

$$\Rightarrow \mathrm{Im}(Rz + \overline{z}) \geqslant 0 \text{ si } R \geqslant 1 \quad R \in \mathbb{R}$$

En este caso $R = |z|^2 > 1$

$$\Rightarrow \mathrm{Im}\left(|z|^2 z + \overline{z}\right) > 0$$

$$\Rightarrow \mathrm{Im}\left(\frac{|z|^2 z + \overline{z}}{|z|^2}\right) > 0$$

$$\Rightarrow \mathrm{Im}\left(z + \frac{1}{z}\right) > 0$$

qed

$\boxed{P_6}$

alternativo

$$z = x + yi$$

$$\Rightarrow z + \frac{1}{z} = x + yi + \frac{1}{x+yi}$$

$$= x + yi + \frac{x - yi}{(\sqrt{x^2+y^2})^2}$$

$$= \left(x + \frac{x}{(\sqrt{x^2+y^2})^2}\right) + \left(yi - \frac{yi}{(\sqrt{x^2+y^2})^2}\right)$$

$$\operatorname{Im}\left(z + \frac{1}{z}\right) = yi\left(1 - \frac{1}{(\sqrt{x^2+y^2})^2}\right)$$

$$= i\left(\frac{y|z|^2 - y}{|z|^2}\right)$$

Pero $y > 0$ y $|z|^2 > 1$

$$\Rightarrow y|z|^2 - y > 0$$

$$\Rightarrow \frac{y|z|^2 - y}{|z|^2} > 0$$

qed

$\boxed{P_7}$ z_1 z_2 soluciones de $z^2 - 2z + 2 = 0$

$$\Rightarrow z_1 = \frac{2 + \sqrt{4 - 4 \cdot 2}}{2} = \frac{2 + \sqrt{-4}}{2} = \frac{2 + 2i}{2} = 1 + 1i \quad \Rightarrow z_2 = 1 - 1i$$

Si se hubiese tomado $z_1 = 1 - 1i$ y $z_2 = 1 + 1i$ el resultado es el mismo pues una vez que se pasa a la parte polar, los cos y sen se anulan del mismo modo

$$\therefore \frac{(\cot\phi + z_1 - 1)^n - (\cot\phi + z_2 - 1)^n}{z_1 - z_2}$$

$$= \frac{(\cot\phi + \cancel{1} + i - \cancel{1})^n - (\cot\phi + \cancel{1} - i - \cancel{1})^n}{2i}$$

$$= \frac{\left(\frac{\cos\phi + i\,\mathrm{sen}\,\phi}{\mathrm{sen}\,\phi}\right)^n - \left(\frac{\cos\phi - i\,\mathrm{sen}\,\phi}{\mathrm{sen}\,\phi}\right)^n}{2i}$$

$$= \frac{(\cos\phi + i\,\mathrm{sen}\,\phi)^n - (\cos\phi - i\,\mathrm{sen}\,\phi)^n}{2i(\mathrm{sen}\,\phi)^n}$$

$$= \frac{(e^{i\phi})^n - (e^{-i\phi})^n}{2i(\mathrm{sen}\,\phi)^n}$$

$$= \frac{e^{i\phi n} - e^{-i\phi n}}{2i(\mathrm{sen}\,\phi)^n}$$

$$= \frac{\cancel{\cos(\phi n)} + i\,\mathrm{sen}(\phi n) - \cancel{\cos(-\phi n)} - i\,\mathrm{sen}(-\phi n)}{2i(\mathrm{sen}\,\phi)^n}$$

$$= \frac{\cancel{2i}\,\mathrm{sen}(\phi n)}{\cancel{2i}(\mathrm{sen}\,\phi)^n} = \mathrm{sen}(\phi n)\,\mathrm{cosec}^n(\phi)$$

qed

$\boxed{P_8}$

$$z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$\tan\alpha = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$\Leftrightarrow \tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4} //$$

$$|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{\frac{2}{4} + \frac{2}{4}}$$

$$= \sqrt{1} = 1 //$$

$$\Rightarrow z = e^{i\frac{\pi}{4}} //$$

$$z^n = 1$$

$$\Leftrightarrow e^{i\frac{\pi}{4}n} = e^{i2\pi}$$

$$\Rightarrow \frac{\pi}{4}n = 2\pi + 2k\pi \quad k \in \mathbb{N}$$

$$\Rightarrow n = 8 + 8k \quad k \in \mathbb{N}$$

$$\Rightarrow n = 8x \quad x \in \mathbb{N}\setminus\{0\}$$

(multiplos de 8)

$$\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\tan\beta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$\Leftrightarrow \tan\beta = \sqrt{3} \Rightarrow \beta = \frac{\pi}{3} //$$

$$|\omega| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$= \sqrt{1} = 1 //$$

$$\Rightarrow \omega = e^{i\frac{\pi}{3}} //$$

$$\omega^n = 1$$

$$\Leftrightarrow e^{i\frac{\pi}{3}n} = e^{i2\pi}$$

$$\Rightarrow \frac{\pi}{3}n = 2\pi + 2k\pi \quad k \in \mathbb{N}$$

$$\Rightarrow n = 6 + 6k \quad k \in \mathbb{N}$$

$$\Rightarrow n = 6y \quad y \in \mathbb{N}\setminus\{0\}$$

(multiplos de 6)

$$\mathrm{mcm}(8 \text{ y } 6) = 24 //$$

"minimo común multiplo"

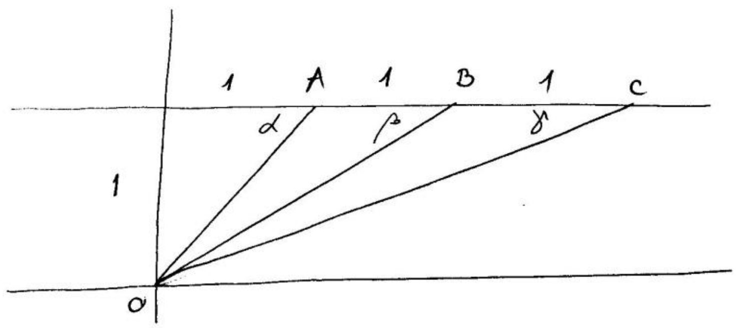
$$\Rightarrow n = 24 //$$

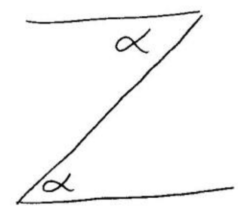
P9

a) $z_A = 1 + i = \sqrt{2}\, e^{i\alpha}$ //

$z_B = 2 + i = \sqrt{5}\, e^{i\beta}$,

$z_C = 3 + i = \sqrt{10}\, e^{i\gamma}$ //





Para todos los angulos

b) $z_A \; z_B \; z_C$

$= (1+i)(2+i)(3+i)$

$= (1+3i)(3+i)$

$= 10i$

$= 10 e^{i\frac{\pi}{2}}$ // ← Pues solo tiene parte imaginaria

$z_A \cdot z_B \cdot z_C$

$= \sqrt{2}\, e^{i\alpha} \cdot \sqrt{5}\, e^{i\beta} \cdot \sqrt{10}\, e^{i\gamma}$

$= 10\, e^{i(\alpha+\beta+\gamma)}$ //

$\Rightarrow 10\, e^{i(\alpha+\beta+\gamma)} = 10 e^{i(\frac{\pi}{2})}$

$\Rightarrow \alpha + \beta + \gamma = \frac{\pi}{2}$ qed

P10 $\quad z \in \mathbb{C} \quad |z| = 1 \quad z^{2n} \neq -1$

PDQ $\quad \dfrac{z^n}{1+z^{2n}} \in \mathbb{R}$

En efecto
$$\frac{z^n}{1+z^{2n}}$$
$$= \frac{z^n}{1+z^{2n}} \cdot \frac{\overline{z}^{\,2n}}{\overline{z}^{\,2n}}$$
$$= \frac{\left(z \cdot \overline{z}^{\,2}\right)^n}{\overline{z}^{\,2n} + \left(z\overline{z}\right)^{2n}}$$
$$= \frac{\left(z\overline{z} \cdot \overline{z}\right)^n}{\overline{z}^{\,2n} + \cancel{\left(|z|\right)^{2n}}^{\,1}}$$
$$= \underbrace{\cancel{\left(|z|^2\right)^n}}_{1} \frac{\overline{z}^{\,n}}{\overline{z}^{\,2n} + 1}$$
$$= \frac{\overline{z}^{\,n}}{\overline{z}^{\,2n} + 1}$$
$$= \frac{\overline{z^n}}{\overline{z^{2n} + 1}}$$
$$= \overline{\left(\frac{z^n}{z^{2n}+1}\right)}$$
$$\Rightarrow \frac{z^n}{z^{2n}+1} = \overline{\left(\frac{z^n}{z^{2n}+1}\right)} \Longrightarrow \frac{z^n}{z^{2n}+1} \in \mathbb{R}$$

qed

P11 Sea $z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i \;//$$

Sea $m = a_1 a_2 - b_1 b_2$

$n = a_1 b_2 + a_2 b_1$

$$\Rightarrow z_1 z_2 = m + ni \;//$$

$$S_1 = a_1^2 + b_1^2 \qquad\qquad S_2 = a_2^2 + b_2^2$$
$$= |z_1|^2 \qquad\qquad = |z_2|^2$$

$$S_1 S_2 = |z_1|^2 |z_2|^2$$
$$= |z_1 \cdot z_2|^2$$
$$= \left(\sqrt{m^2 + n^2}\right)^2$$
$$= m^2 + n^2 \quad \text{qed}$$

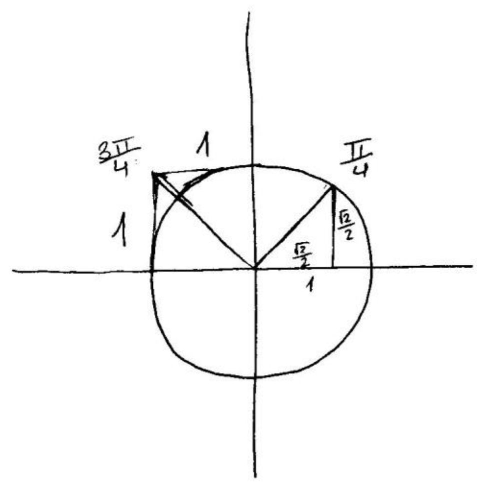
$\boxed{P_{12}}$ $|z_1| = 3 \quad arg(z_1) = \frac{\pi}{4} \Rightarrow z_1 = 3e^{i\frac{\pi}{4}}$

$|z_2| = 2 \quad arg(z_2) = \frac{\pi}{2} \Rightarrow z_2 = 2e^{i\frac{\pi}{2}}$

$$\frac{z_1\left(\frac{1}{2}+\frac{1}{2}i\right)}{z_2\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}$$

$$= \frac{3e^{i\frac{\pi}{4}}\left(\frac{1}{2}+\frac{1}{2}i\right)}{2e^{i\frac{\pi}{2}}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3}{2}e^{i\frac{-\pi}{4}} \frac{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}{(-1+1i)}$$



$$= \frac{3}{2}\cancel{e^{i\frac{-\pi}{4}}} \frac{\cancel{e^{i\frac{\pi}{4}}}}{\sqrt{2}\, e^{i\frac{3\pi}{4}}}$$

$$= \frac{3}{2\sqrt{2}} e^{i\frac{-3\pi}{4}}$$

$$= \frac{3\sqrt{2}}{4} e^{i\frac{-3\pi}{4}} \quad //$$

$$\frac{3\sqrt{2}}{4}\left(\cos\frac{-3\pi}{4} + i\,sen\frac{-3\pi}{4}\right)$$

$$= \frac{3\sqrt{2}}{4}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -\frac{3}{4} - \frac{3}{4}i \quad //$$

$\boxed{P_{13}}$

$$\sum_{k=0}^{n-1} Z_k Z_{k+1} = \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}} \cdot e^{i\frac{2\pi(k+1)}{n}}$$

$$= \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}} \cdot e^{i\frac{2k\pi+2\pi}{n}}$$

$$= \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}} e^{i\frac{2k\pi}{n}} \cdot e^{\frac{2\pi}{n}}$$

$$= e^{i\frac{2\pi}{n}} \sum_{k=0}^{n-1} e^{i\frac{4k\pi}{n}}$$

$$= e^{i\frac{2\pi}{n}} \sum_{k=0}^{n-1} \left(e^{i4\pi k}\right)^{\frac{1}{n}}$$

Pero $\underset{k\in\mathbb{Z}}{e^{i2k\pi} = e^{i4k\pi}} \Longrightarrow = e^{i\frac{2\pi}{n}} \sum_{k=0}^{n-1} \left(e^{i2k\pi}\right)^{\frac{1}{n}}$

$$= e^{i\frac{2\pi}{n}} \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}}$$

$$= e^{i\frac{2\pi}{n}} \cdot 0 = 0$$

qed

P14 $z_1, z_2 \in \mathbb{C}$ $|z_1| = 1$ $|z_2| = 1$

Al ser unitarios

$$z_1 + z_2 = -u \quad , \; u \in \mathbb{C}$$

$$z_1 \cdot z_2 = v \quad , \; v \in \mathbb{C}$$

a) PDQ $|u| \leqslant 2$

En efecto

$$|z_1| + |z_2| = 2$$

$$\Rightarrow |z_1 + z_2| \leqslant 2$$ ← Desigualdad triangular

$$\Rightarrow |-u| \leqslant 2$$

$$\Leftrightarrow |u| \leqslant 2$$ qed

PDQ $|v| = 1$

En efecto $|z_1| \cdot |z_2| = 1$

$$\Leftrightarrow |z_1 \cdot z_2| = 1$$

$$\Leftrightarrow |v| = 1$$ qed

b) PDQ $\overline{z_1} + \overline{z_2} = \dfrac{-u}{v}$

En efecto $-\dfrac{u}{v}$

$$= \frac{z_1 + z_2}{z_1 \cdot z_2}$$

$$= \frac{\cancel{z_1}}{\cancel{z_1} \cdot z_2} + \frac{\cancel{z_2}}{z_1 \cdot \cancel{z_2}}$$

$$= \frac{1}{z_2} + \frac{1}{z_1}$$

$$= \frac{\overline{z_2}}{\cancel{|z_2|^2}_{\to 1}} + \frac{\overline{z_1}}{\cancel{|z_1|^2}_{\to 1}}$$

$$= \overline{z_2} + \overline{z_1}$$

qed

c) PDQ $u = \bar{u} \cdot v$

En efecto $\overline{\bar{u} \cdot v}$

$$= \overline{(-1)(z_1+z_2) \cdot v}$$

$$= \overline{(-1)}\,\overline{(z_1+z_2)} \cdot \bar{v}$$

$$= (-1)(\bar{z}_1+\bar{z}_2) \cdot \bar{v}$$

$$= (-1)\left(\frac{-u}{\cancel{\bar{v}}}\right) \cdot \cancel{\bar{v}}$$

$$= u \quad \text{qed}$$

d) Por c) sabemos que

$$u = \bar{u} \cdot v$$

$$\Leftrightarrow \cancel{|u|}e^{i\varphi} = \cancel{|u|}e^{-i\varphi}\,\cancel{|v|}^{1}e^{i\theta}$$

$$\Leftrightarrow e^{i\varphi} = e^{-i\varphi}e^{i\theta}$$

$$\Leftrightarrow e^{i\varphi} = e^{i(\theta-\varphi)}$$

$$\Rightarrow \varphi + 2k\pi = \theta - \varphi,\; k \in \mathbb{Z}$$

$$\Leftrightarrow 2\varphi + 2k\pi = \theta,$$

$$k \in \mathbb{Z} \quad \text{qed}$$

$\boxed{P_{15}}$ $z^2 + z + 1 = 0$

$$z = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i \,//$$

$$\Rightarrow z_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2} i \qquad z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\tan\alpha = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \qquad \tan\beta = \frac{\frac{-\sqrt{3}}{2}}{\frac{-1}{2}}$$

$$\Leftrightarrow \tan\alpha = -\sqrt{3} \qquad \Leftrightarrow \tan\beta = -\sqrt{3}$$

$$\Rightarrow \alpha = \frac{2\pi}{3} \qquad \Rightarrow \beta = \frac{4\pi}{3}$$

$$\sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \qquad \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} \qquad = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$= 1 \,// \qquad = 1 \,//$$

$$\Rightarrow e^{i\frac{2\pi}{3}} = z_1 \qquad \Rightarrow e^{i\frac{4\pi}{3}} = z_2$$

$$z_1^{\,3} = \left(e^{i\frac{2\pi}{3}}\right)^3 = e^{i2\pi} = 1$$

$$z_2^{\,3} = \left(e^{i\frac{4\pi}{3}}\right)^3 = e^{i4\pi} = 1$$

$\therefore$ z_1 y z_2 son raices cúbicas de la unidad

$\boxed{P_{16}}$ $z \in \mathbb{C} \setminus \{0\}$ $\{z_0, \dots, z_{n-1}\}$ raíces n-esimas de z

$$\sum_{k=0}^{n-1} \frac{1}{z_k}$$

$$= \sum_{k=0}^{n-1} \frac{\overline{z_k}}{|z_k|^2}$$

Puede salir puesto que es constante ya que todas las raíces n-ésimas tienen el mismo modulo

$$= \frac{1}{|z_k|^2} \sum_{k=0}^{n-1} \overline{z_k}$$

$$= \frac{1}{|z_k|^2} \overline{\sum_{k=0}^{n-1} z_k}$$

$$= \frac{1}{|z_k|^2} \overline{0}$$

$$= 0 \quad //$$

$\boxed{P_{17}}$ $\forall k \in \{1, 2, \quad , n-1\}$

$$\text{PDQ} \quad \sum_{j=0}^{n-1} (\omega_j)^k = 0$$

En efecto $\omega_j = e^{i \frac{2\pi}{n} j}$

$$= \left(e^{i \frac{2\pi}{n}}\right)^j$$

$$= (\omega_1)^j \,_{//}$$

Por lo tanto $\displaystyle\sum_{j=0}^{n-1} (\omega_j)^k = \sum_{j=0}^{n-1} \left((\omega_1)^j\right)^k$

Geométrica $\longrightarrow$ $\displaystyle = \sum_{j=0}^{n-1} \left((\omega_1)^k\right)^j$

$$= \frac{1 - (\omega_1{}^k)^n}{1 - \omega_1{}^k}$$

Pero $(\omega_1{}^k)^n = (\omega_1{}^n)^k = 1^k = 1$

Por lo tanto con $\omega_1{}^k \neq 1$

$$\frac{1-1}{1-(\omega_1)^k} = \frac{0}{1-(\omega_1)^k} = 0_{//} \quad \text{qed}$$

$\boxed{P_{18}}$ PDQ $\forall n \in \mathbb{N} \quad (1-i)^n + (1+i)^n \in \mathbb{R}$

En efecto $(1-i)^n + (1+i)^n$ sea $z = 1-i$

$$= (z)^n + (\overline{z})^n$$

$$= z^n + \overline{z^n}$$

$$= 2\operatorname{Re}(z^n)$$

$$\Rightarrow \in \mathbb{R}$$

qed

$\boxed{P_{19}}$ $z \in \mathbb{C} \quad |z| \neq 1 \quad n \geq 1$

PDQ $\dfrac{1}{1+z^n} + \dfrac{1}{1+\overline{z}^n} \in \mathbb{R}$

En efecto $\dfrac{1}{1+z^n} + \dfrac{1}{1+\overline{z}^n} = \dfrac{1+\overline{z}^n + 1 + z^n}{(1+z^n)(1+\overline{z}^n)}$

$$= \frac{2+2\operatorname{Re}(z^n)}{1+\overline{z}^n + z^n + z\overline{z}^n}$$

Todo en esta fracción es $\mathbb{R}$ $\nearrow$ $= \dfrac{2+2\operatorname{Re}(z^n)}{1+2\operatorname{Re}(z^n) + |z|^{2n}}$

$$\Rightarrow \in \mathbb{R}$$

qed

$\boxed{P_{20}}$ $\quad z_0 = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$ Resolver $z^4 = z_0$

En efecto $\quad z_0 = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$

$$\Longleftrightarrow z_0 = \frac{1+i\sqrt{3}}{2} \cdot \frac{2}{1-i\sqrt{3}}$$

$\tan\alpha = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$\tan\alpha = \sqrt{3}$

$\Rightarrow \alpha = \dfrac{\pi}{3}$

$$\Longleftrightarrow z_0 = e^{i\frac{\pi}{3}} \cdot \frac{1}{\frac{1-i\sqrt{3}}{2}}$$

conjugado de $e^{i\frac{\pi}{3}}$

$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= 1$ //

$$\Longleftrightarrow z_0 = e^{i\frac{\pi}{3}} \cdot \frac{1}{e^{i\cdot -\frac{\pi}{3}}}$$

$$\Longleftrightarrow z_0 = e^{i\frac{2\pi}{3}} \;//$$

$$z^4 = z_0$$

$$\Longleftrightarrow e^{i4\alpha} = e^{i\frac{2\pi}{3}}$$

$$\Rightarrow 4\alpha = \frac{2\pi}{3} + 2k\pi$$

$$\Longleftrightarrow \alpha = \frac{\pi}{6} + \frac{3k\pi}{6} \;//$$

$$\Rightarrow \alpha = \begin{cases} \frac{\pi}{6};\ k=0 & \frac{10\pi}{6};\ k=3 \\ \frac{4\pi}{6};\ k=1 \\ \frac{7\pi}{6};\ k=2 \end{cases}$$

$$\cos\frac{\pi}{6} + i\,\mathrm{sen}\,\frac{\pi}{6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\cos\frac{4\pi}{6} + i\,\mathrm{sen}\,\frac{4\pi}{6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\cos\frac{7\pi}{6} + i\,\mathrm{sen}\,\frac{7\pi}{6} = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

$$\cos\frac{10\pi}{6} + i\,\mathrm{sen}\,\frac{10\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

//

$\boxed{P_{21}}$ $z \in \mathbb{C}$

PDQ $|z+i| = |z-i| \Leftrightarrow z \in \mathbb{R}$

$\Rightarrow|$ $|z+i| = |z-i|$

Sea $z = a+bi$

$\Rightarrow |a+bi+i| = |a+bi-i|$

$\Leftrightarrow \sqrt{a^2+(b+1)^2} = \sqrt{a^2+(b-1)^2}$

$\Leftrightarrow \cancel{a^2}+\cancel{b^2}+2b+\cancel{1} = \cancel{a^2}+\cancel{b^2}-2b+\cancel{1}$

$\Rightarrow 4b = 0$

$\Rightarrow b = 0$

$\Rightarrow z = a \Rightarrow z \in \mathbb{R}$ //

$\Leftarrow|$ $z \in \mathbb{R}$ Sea $z = a+bi$

$\Rightarrow b = 0$

$\Rightarrow z = a$

$\Rightarrow |z+i| = |a+i| = \sqrt{a^2+1}$ $\quad |z-i| = |a-i| = \sqrt{a^2+1}$

$\Rightarrow |z+i| = |z-i|$ //

$\therefore |z+i| = |z-i| \Leftrightarrow z \in \mathbb{R}$ qed

$\boxed{P_{22}}$ $z \in \mathbb{C}$ $\left|\frac{z-2}{z-1}\right| = 2$

Sea $z = a + bi$

$$\left|\frac{z-2}{z-1}\right| = 2 \Leftrightarrow \left|\frac{a+bi-2}{a+bi-1}\right| = 2$$

$$\Leftrightarrow \sqrt{\frac{(a-2)^2+b^2}{(a-1)^2+b^2}} = 2$$

$$\Rightarrow \frac{(a-2)^2+b^2}{(a-1)^2+b^2} = 4$$

$$\Leftrightarrow a^2 - 4a + 4 + b^2 = 4(a^2 - 2a + 1 + b^2)$$

$$\Leftrightarrow a^2 - 4a + \cancel{4} + b^2 = 4a^2 - 8a + \cancel{4} + 4b^2$$

$$\Leftrightarrow 3a^2 + 3b^2 = 4a$$

$$\Leftrightarrow a^2 + b^2 = \frac{4}{3}a$$

$$\Leftrightarrow a^2 + b^2 - \frac{4}{3}a = 0_{//}$$

$$\Leftrightarrow \left(a^2 - \frac{4}{3}a + \frac{4}{9}\right) + b^2 - \frac{4}{9} = 0$$

$$\Leftrightarrow \left(a - \frac{2}{3}\right)^2 + b^2 = \left(\frac{2}{3}\right)^2$$

Por lo tanto es una circunferencia de centro $\left(\frac{2}{3}, 0\right)$ y Radio $\frac{2}{3}$

$\boxed{P_{23}}$ $|z_i| = 1 \quad \forall i \in \{1, 2, \dots, p\}$

PDQ $\displaystyle\sum_{i=1}^{p} z_i = a \in \mathbb{R} \Rightarrow \sum_{i=1}^{p} \frac{1}{z_i} = a$

En efecto $\displaystyle\sum_{i=1}^{p} \frac{1}{z_1} = \sum_{i=1}^{p} \frac{\overline{z_i}}{|z_i|^2}$

$$= \sum_{i=1}^{p} \frac{\overline{z_i}}{1^2}$$

$$= \sum_{i=1}^{p} \overline{z_i}$$

$$= \overline{\sum_{i=1}^{p} z_i}$$

$$= \overline{a} \quad \text{pues } a \in \mathbb{R}$$

$$= a_{\,//}$$

qed

$\boxed{P_{24}}$ PDQ $z^n = 1 \wedge n \mid m \Rightarrow z^m = 1$

En efecto $n \mid m \Leftrightarrow \exists k \in \mathbb{N}$ tal que $m = kn$

$$\Rightarrow k \in \mathbb{N} \quad n = \frac{m}{k}$$

Por lo tan $z^n = 1$ sea $z = e^{i\alpha}$

$$\Leftrightarrow (e^{i\alpha})^n = 1$$

$$\Leftrightarrow e^{i\alpha n} = e^{i2\pi}$$

$$\Rightarrow \alpha n = 2\pi$$

$$\Leftrightarrow \alpha \frac{m}{k} = 2\pi \qquad k \in \mathbb{N}$$

$$\Leftrightarrow \alpha m = 2k\pi \quad k \in \mathbb{N}$$

$$\Rightarrow (e^{i\alpha m}) = e^{i2k\pi}$$

pero $e^{i2k\pi}$ $k \in \mathbb{N}$ $= 1$

$$\Rightarrow (e^{i\alpha})^m = 1$$

$$\Rightarrow z^m = 1$$

~~qed~~

$\boxed{P_{25}}$

$$Z_n = X_n + i\,y_n = \left(1 + i\sqrt{3}\right)^{3n}$$

$$\left(1 + i\sqrt{3}\right)^{3n} = \left(2\left(\frac{1 + i\sqrt{3}}{2}\right)\right)^{3n}$$

$$= \underbrace{\left(2\, e^{i\frac{\pi}{3}}\right)^{3n}}_{\text{Forma polar}}$$

$$\tan\alpha = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \iff \tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$X_n + i\,y_n = 2^{3n}\, e^{i\pi n}$$

$$= 2^{3n}\left(\cos(\pi n) + i\,\mathrm{sen}(\pi n)\right)$$

$$\Rightarrow X_n = 2^{3n}\cos(\pi n)$$

$$X_n + 8X_{n-1} = 2^{3n}\cos(\pi n) + 8\left(2^{3(n-1)}\cos\pi(n-1)\right)$$

$$= 2^{3n}\cos(\pi n) + \cancel{8}\left(2^{3n\cancel{-3}}\cos\pi(n-1)\right)$$

$$= 2^{3n}\left(\cos(\pi n) + \cos(\pi n - \pi)\right)$$

$$= 2^{3n}\left(\cos(\pi n) + \cos(\pi n)\underbrace{\cos(\pi)}_{-1} + \sin(\pi n)\underbrace{\sin(\pi)}_{0}\right)$$

$$= 2^{3n}\left(\cos(\pi n) - \cos(\pi n)\right)$$

$$= 0 \quad \text{qed}$$

$\boxed{P_{26}}$ $\displaystyle\sum_{k=1}^{n} \frac{z_k}{|z_k|} = 0$

a) PDQ $\displaystyle\sum_{k=1}^{n} \frac{\overline{z_k}}{|z_k|}(z_k - z) = \sum_{k=1}^{n} |z_k|$

En efecto $\displaystyle\sum_{k=1}^{n} \frac{\overline{z_k}}{|z_k|}(z_k - z)$

$$= \sum_{k=1}^{n} \frac{\overline{z_k}\, z_k}{|z_k|} - \frac{z\,\overline{z_k}}{|z_k|}$$

$$= \sum_{k=1}^{n} \frac{|z_k|^{\cancel{2}}}{\cancel{|z_k|}} - \sum_{k=1}^{n} \frac{z\,\overline{z_k}}{|z_k|}$$

$$= \sum_{k=1}^{n} |z_k| - z \overline{\sum_{k=1}^{n} \frac{z_k}{|z_k|}}$$ $\rightarrow$ 0 por dato

$$= \sum_{k=1}^{n} |z_k| - z \cdot 0$$

$$= \sum_{k=1}^{n} |z_k|$$

qed

b) PDQ $\sum_{k=1}^{n} |z_k| \leqslant \sum_{k=1}^{n} |z_k - z|$

En efecto $\left| \sum_{k=1}^{n} |z_k| \right| = \left| \sum_{k=1}^{n} \frac{\overline{z}_k}{|z_k|} (z_k - z) \right|$

$$\Rightarrow \sum_{k=1}^{n} |z_k| \leqslant \sum_{k=1}^{n} \left| \frac{\overline{z}_k}{|z_k|} (z_k - z) \right|$$

[Este paso se debe a que:

$\left| \sum z_k \right| \leqslant \sum |z_k|$ en general]

$$\Leftrightarrow \sum_{k=1}^{n} |z_k| \leqslant \sum_{k=1}^{n} \left| \frac{\overline{z}_k}{|z_k|} \right| |z_k - z|$$

$$\Leftrightarrow \sum_{k=1}^{n} |z_k| \leqslant \sum_{k=1}^{n} \cancelto{1}{\frac{|z_k|}{|z_k|}} |z_k - z|$$

$$\Leftrightarrow \sum_{k=1}^{n} |z_k| \leqslant \sum_{k=1}^{n} |z_k - z|$$

qed

P_{27} $|z| = 1$ $\qquad |z+1| = 1$

Sea $z = a + bi$

$$|z| = 1 \iff \sqrt{a^2+b^2} = 1 \iff a^2 + b^2 = 1 \quad (1)$$

$$|z+1| = 1 \iff \sqrt{(a+1)^2+b^2} = 1 \iff a^2 + 2a + \cancel{1} + b^2 = \cancel{1} \iff a^2 + b^2 + 2a = 0 \quad (2)$$

$$\Rightarrow (2) - (1) \iff 2a = -1 \Rightarrow a = \frac{-1}{2}$$

$$\Rightarrow \left(\frac{-1}{2}\right)^2 + b^2 = 1 \iff b^2 = 1 - \frac{1}{4} \iff b^2 = \frac{3}{4} \Rightarrow b = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\tan\alpha = \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}} \iff \tan\alpha = -\sqrt{3} \Rightarrow \alpha = \frac{2\pi}{3}$$

$$\tan\beta = \frac{\frac{-\sqrt{3}}{2}}{\frac{-1}{2}} \iff \tan\beta = \sqrt{3} \Rightarrow \beta = \frac{4\pi}{3}$$

Sea $z_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$ $\qquad$ $z_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$

$z_1 = e^{i\frac{2\pi}{3}}$ $\qquad$ $z_2 = e^{i\frac{4\pi}{3}}$

$$\Rightarrow z_1^3 = \left(e^{i\frac{2\pi}{3}}\right)^3 \qquad \Rightarrow z_2^3 = \left(e^{i\frac{4\pi}{3}}\right)^3$$

$$= e^{i2\pi} \qquad = e^{i4\pi}$$

$$= 1 \;// \qquad = 1 \;//$$

Por lo tanto z es una raíz cúbica de la unidad.

P_{28} PDQ $\prod_{k=0}^{n-1} z_k = (-1)^{n-1}$ con z_k las raíces n-ésimas.

En efecto

$$\prod_{k=0}^{n-1} z_k$$

$$= \prod_{k=0}^{n-1} e^{i\frac{2k\pi}{n}}$$

$$= e^{\sum_{k=0}^{n-1} \frac{2k\pi}{n} i}$$

$$= e^{i\frac{2\pi}{n}\sum_{k=0}^{n-1} k}$$

$$= e^{i\frac{\cancel{2}\pi}{\cancel{n}}\frac{\cancel{n}(n-1)}{\cancel{2}}}$$

$$= e^{i\pi(n-1)}$$

$$= \left(e^{i\pi}\right)^{n-1}$$

$$= (-1)^{n-1}$$

qed