Introducción al Algebra (11-1)
Pauta Control 6

P1 a) $z \in \mathbb{C}$, $|z| = 1 \wedge z^{2n} \neq -1$. Demostrar que $\frac{z^n}{1+z^{2n}} \in \mathbb{R}$

Sea $z = \rho e^{i\varphi}$ en que $|z| = \rho = 1 \Rightarrow z = e^{i\varphi}$, $\varphi = \arg(z)$

1.5 $\rightarrow$ Así, $\frac{z^n}{1+z^{2n}} = \frac{1}{z^{-n}+z^n} = \frac{1}{(e^{i\varphi})^{-n}+(e^{i\varphi})^n} = \frac{1}{e^{i(-n\varphi)}+e^{i(n\varphi)}}$

$e^{i(-n\varphi)} = \cos(-n\varphi) + i\,\mathrm{sen}(-n\varphi) = \cos(n\varphi) - i\,\mathrm{sen}(n\varphi)$

$e^{i(n\varphi)} = \cos(n\varphi) + i\,\mathrm{sen}(n\varphi)$

Entonces $e^{i(-n\varphi)} + e^{i(n\varphi)} = 2\cos(n\varphi)$

1.5 $\rightarrow$ Sigue que $\frac{z^n}{1+z^{2n}} = \frac{1}{2\cos(n\varphi)} \in \mathbb{R}$

b) $a_1, a_2, b_1, b_2 \in \mathbb{N}$. Se define $\Delta_1 = a_1^2 + b_1^2$; $\Delta_2 = a_2^2 + b_2^2$

Demostrar que $\exists\, m, n \in \mathbb{Z}$ tales que $\Delta_1 \Delta_2 = m^2 + n^2$.

Sean $z_1 = a_1 + b_1 i$ y $z_2 = a_2 + b_2 i$

Entonces $z_1 \cdot z_2 = a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i$

$a_1, a_2, b_1, b_2 \in \mathbb{N}$; donde $a_1 a_2 - b_1 b_2 = m \in \mathbb{Z} \wedge a_1 b_2 + a_2 b_1 = n \in \mathbb{Z}$

1.5 $\rightarrow$ Así $z_1 \cdot z_2 = m + ni \Rightarrow |z_1 z_2| = |m + ni|$

$\Rightarrow |z_1||z_2| = \sqrt{m^2+n^2}$ pero $\left.\begin{array}{l}|z_1| = \sqrt{a_1^2+b_1^2} = \sqrt{\Delta_1} \\ |z_2| = \sqrt{a_2^2+b_2^2} = \sqrt{\Delta_2}\end{array}\right\}$ hipotesis

1.5 $\rightarrow$ Sigue que $\sqrt{\Delta_1} \cdot \sqrt{\Delta_2} = \sqrt{m^2+n^2}$, es decir $\exists\, m, n \in \mathbb{Z}$ t.q. $\Delta_1 \Delta_2 = m^2 + n^2$

P2 a) Se pide forma polar y cartesiana para

$$w = \frac{z_1\left(\frac{1}{2}+\frac{1}{2}i\right)}{z_2\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}$$ donde $|z_1| = 3$, $|z_2| = 2$

$\arg(z_1) = \frac{\pi}{4}$, $\arg(z_2) = \frac{\pi}{2}$

Así $z_1 = 3e^{i\frac{\pi}{4}}$ y $z_2 = 2e^{i\pi/2}$

1.0 Entonces $$w = \frac{3e^{i\frac{\pi}{4}}\,\frac{1}{2}(1+i)}{2e^{i\pi/2}\,\frac{1}{\sqrt{2}}(-1+i)} = \frac{3e^{i\frac{\pi}{4}}\,\frac{1}{2}\sqrt{2}\,e^{i\frac{\pi}{4}}}{2e^{i\pi/2}\,\cancel{\frac{1}{\sqrt{2}}\sqrt{2}}\,e^{i\left(\frac{3\pi}{4}\right)}} = \frac{\frac{3}{2}\sqrt{2}\,e^{i\frac{\pi}{2}}}{2\,e^{i\left(\frac{\pi}{2}+\frac{3\pi}{4}\right)}}$$

Polar $\Rightarrow w = \frac{3\sqrt{2}}{4}e^{i\left(-\frac{3\pi}{4}\right)}$ donde $|w| = \frac{3\sqrt{2}}{4}$ $\wedge$ $\arg(w) = -\frac{3\pi}{4}$

1.0

Cartesiana : $w = \frac{3\sqrt{2}}{4}\left(\cos\left(-\frac{3\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{3\pi}{4}\right)\right) = \frac{3\sqrt{2}}{4}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$

1.0 $\rightarrow w = -\frac{3\sqrt{2}}{4}\frac{\sqrt{2}}{\cancel{2}}(1+i) = -\frac{3}{4}(1+i) = -\frac{3}{4} - \frac{3}{4}i$

b) $z_0, z_1, \dots z_{n-1}$ las n raíces ene-ésimas de la unidad. Dem. que $z_0z_1 + z_1z_2 + \dots + z_{n-1}\cdot z_0 = 0$

Recordar que $z_k = z_1^k$ para $k = 0, 1, \dots, n-1$ y $z_0 = z_n$

Así: $$z_0z_1 + z_1z_2 + \dots + z_{n-1}z_n = \sum_{k=0}^{n-1} z_k z_{k+1} = \sum_{k=0}^{n-1} z_1^k \cdot z_1^{k+1}$$

1.0 $$= \sum_{k=0}^{n-1} z_1^k \cdot z_1^k \cdot z_1 = z_1 \underbrace{\sum_{k=0}^{n-1}\left(z_1^2\right)^k}_{\Sigma \text{ geométrica}} = z_1\,\frac{1-\left(z_1^2\right)^n}{1-z_1^2}$$

pero $z_1^2 = z_2 \neq 1$ $\wedge$ $\left(z_1^2\right)^n = \left(z_1^n\right)^2 = 1^2 = 1$ pues $z_1^n = 1$ (z_1 raíz n-ésima d…

2.0 Así $$z_0z_1 + z_1\cdot z_2 + \dots + z_{n-1}z_0 = z_1\,\frac{1-1}{1-z_1^2} = 0$$

OBSERVACIÓN: También puede resolverse considerando $z_k = e^{i\frac{2k\pi}{n}} = (e^{i\frac{2\pi}{n}}\dots$