8. Semana 7

P1 **(a)**

$$\begin{aligned}
\sum_{k=n}^{m} \log\left(1+\frac{1}{k}\right) &= \sum_{k=n}^{m} \log\left(\frac{k+1}{k}\right) \\
&= \sum_{k=n}^{m} [\log(k+1) - \log(k)] \qquad \backslash\text{telescópica} \\
&= \log(m+1) - \log(n)
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} &= \sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} + \frac{1}{n!} - \frac{1}{n!} \qquad \backslash\text{sumar } 0 \\
&= \sum_{k=0}^{n-1} \frac{1}{k!(n-k)!} - \frac{1}{n!} \qquad \backslash\text{incorporar } \frac{1}{n!} \text{ a la sumatoria} \\
&= \sum_{k=0}^{n-1} \frac{1}{k!(n-k)!} + \frac{1}{n!} - \frac{1}{n!} - \frac{1}{n!} \qquad \backslash\text{sumar } 0 \\
&= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} - \frac{2}{n!} \qquad \backslash\text{incorporar } \frac{1}{n!} \text{ a la sumatoria} \\
&= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \cdot \frac{n!}{n!} - \frac{2}{n!} \\
&= \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} - \frac{2}{n!} \\
&= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} - \frac{2}{n!} \\
&= \frac{2^n}{n!} - \frac{2}{n!} \qquad \backslash \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = (1+1)^n = 2^n \\
&= \frac{2^n - 2}{n!}
\end{aligned}$$

P2

$$\begin{aligned}
\sum_{i=\frac{m(m-1)}{2}+1}^{\frac{m(m+1)}{2}} (2i-1) &= \sum_{i=1}^{\frac{m(m+1)}{2}-\frac{m(m-1)}{2}} \left(2\left(i+\frac{m(m-1)}{2}\right)-1\right) && \text{\textbackslash cambio de índice}\\
&= \sum_{i=1}^{m} (2i+m(m-1)-1)\\
&= \sum_{i=1}^{m} 2i + \sum_{i=1}^{m} m(m-1) - \sum_{i=1}^{m} 1\\
&= 2\sum_{i=1}^{m} i + m(m-1)\sum_{i=1}^{m} 1 - \sum_{i=1}^{m} 1 && \text{\textbackslash las constantes salen}\\
&= 2\cdot\frac{m(m+1)}{2} + m(m-1)m - m && \text{\textbackslash sumas conocidas}\\
&= m^2+m+m^3-m^2-m\\
&= m^3
\end{aligned}$$

P3

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{\sqrt{k(k+1)}(\sqrt{k+1}+\sqrt{k})} &= \sum_{k=1}^{n} \frac{1}{\sqrt{k(k+1)}(\sqrt{k+1}+\sqrt{k})}\cdot\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}\\
&= \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}(k+1-k)} && \text{\textbackslash racionalizar}\\
&= \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}}\\
&= \sum_{k=1}^{n} \left[\frac{\sqrt{k+1}}{\sqrt{k(k+1)}} - \frac{\sqrt{k}}{\sqrt{k(k+1)}}\right]\\
&= \sum_{k=1}^{n} \left[\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right] && \text{\textbackslash telescópica}\\
&= 1-\frac{1}{\sqrt{n+1}}
\end{aligned}$$

P4

$$\begin{aligned}
\sum_{i=0}^{n-1} x^{n-1-i}y^i &= \sum_{i=0}^{n-1} \frac{x^{n-1}y^i}{x^i} \\
&= x^{n-1}\sum_{i=0}^{n-1} \frac{y^i}{x^i} \\
&= x^{n-1}\sum_{i=0}^{n-1} \left(\frac{y}{x}\right)^i \qquad \text{\geométrica} \\
&= x^{n-1}\cdot\left(\frac{1-\left(\frac{y}{x}\right)^n}{1-\left(\frac{y}{x}\right)}\right) \\
&= x^{n}\cdot\left(\frac{1-\left(\frac{y}{x}\right)^n}{x-y}\right) \\
&= \frac{x^n-y^n}{x-y}
\end{aligned}$$

P5 **(a)** $(f*f)(n) = \sum_{k=0}^n f(k)f(n-k) = \sum_{k=0}^n 1\cdot 1 = \sum_{k=0}^n 1 = n+1.$

$(f*g)(n) = \sum_{k=0}^n f(k)g(n-k) = \sum_{k=0}^n 1\cdot(n-k) = \sum_{k=0}^n n - \sum_{k=0}^n k = n\sum_{k=0}^n 1 - \frac{n(n+1)}{2} = n(n+1) - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}.$

$(g*g)(n) = \sum_{k=0}^n g(k)g(n-k) = \sum_{k=0}^n k\cdot(n-k) = \sum_{k=0}^n nk - \sum_{k=0}^n k^2 = n\sum_{k=0}^n k - \frac{n(n+1)(2n+1)}{6} = \frac{n^2(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(n^2-1)}{6}.$

(b) $n!(f*g)(n) = n!\sum_{k=0}^n f(k)g(n-k) = n!\sum_{k=0}^n \frac{a^k}{k!}\cdot\frac{b^{n-k}}{(n-k)!} = \sum_{k=0}^n \binom{n}{k}a^kb^{n-k} = (a+b)^n.$

P6

$$
\begin{aligned}
\sum_{k=0}^{n}(1-x)^k &= \frac{1-(1-x)^{n+1}}{1-(1-x)} && \backslash\text{geométrica}\\
&= \frac{1-\sum_{k=0}^{n+1}\binom{n+1}{k}(-x)^k\cdot 1^{n+1-k}}{x} && \backslash\text{binomio}\\
&= \frac{1-\sum_{k=0}^{n+1}\binom{n+1}{k}(-x)^k}{x} && \backslash 1^{n+1-k}=1\\
&= \frac{1-(\sum_{k=1}^{n+1}\binom{n+1}{k}(-x)^k+\binom{n+1}{0}(-x)^0)}{x} && \backslash\text{soltar primer término}\\
&= \frac{1-(\sum_{k=1}^{n+1}\binom{n+1}{k}(-x)^k+1)}{x}\\
&= \frac{-\sum_{k=1}^{n+1}\binom{n+1}{k}(-x)^k}{x}\\
&= \frac{-\sum_{k=0}^{n}\binom{n+1}{k+1}(-x)^{k+1}}{x} && \backslash\text{cambio de índice}\\
&= \frac{-\sum_{k=0}^{n}\binom{n+1}{k+1}(-x)^k(-x)}{x}\\
&= \frac{x\sum_{k=0}^{n}\binom{n+1}{k+1}(-x)^k}{x}\\
&= \sum_{k=0}^{n}\binom{n+1}{k+1}(-x)^k\\
&= \sum_{k=0}^{n}\binom{n+1}{k+1}(x)^k(-1)^k
\end{aligned}
$$

P7

$$\begin{aligned}
\sum_{k=0}^{n} k7^k \binom{n}{k} &= \sum_{k=1}^{n} k7^k \binom{n}{k} \qquad \backslash \text{soltar primer término} = 0 \\
&= \sum_{k=1}^{n} \frac{k7^k n!}{k!(n-k)!} \\
&= \sum_{k=1}^{n} \frac{7^k n!}{(k-1)!(n-k)!} \\
&= \sum_{k=0}^{n-1} \frac{7^{k+1} n!}{k!(n-(k+1))!} \qquad \backslash \text{cambio de índice} \\
&= \sum_{k=0}^{n-1} \frac{7^{k+1} n!}{k!((n-1)-k)!} \\
&= n \sum_{k=0}^{n-1} \frac{7^{k+1} (n-1)!}{k!((n-1)-k)!} \qquad \backslash \text{definición factorial } n! = n(n-1)! \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} 7^{k+1} \\
&= 7n \sum_{k=0}^{n-1} \binom{n-1}{k} 7^k \cdot 1^{n-1-k} \qquad \backslash 1^{n-1-k} = 1 \\
&= 7n(7+1)^{n-1} \qquad \backslash \text{binomio} \\
&= 7n8^{n-1}
\end{aligned}$$

P8 (a)

$$\begin{aligned}
\binom{n}{i}\binom{i}{k} &= \frac{n!}{i!(n-i)!} \frac{i!}{k!(i-k)!} \\
&= \frac{n!}{k!(n-i)!(i-k)!} \\
&= \frac{n!}{k!(n-i)!(i-k)!} \cdot \frac{(n-k)!}{(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{(n-i)!(i-k)!} \\
&= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{((n-k)-(i-k))!(i-k)!} \\
&= \binom{n}{k}\binom{n-k}{i-k}
\end{aligned}$$

(b)

$$
\begin{aligned}
\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k} &= \sum_{i=k}^{n}\binom{n}{k}\binom{n-k}{i-k} && \backslash\text{usando parte anterior}\\
&= \binom{n}{k}\sum_{i=k}^{n}\binom{n-k}{i-k} && \backslash\text{constantes salen}\\
&= \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i} && \backslash\text{cambio de índice}\\
&= \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}1^{n-k-i}1^{i} && \backslash 1^{n-k-i}=1\\
&= \binom{n}{k}(1+1)^{n-k} && \backslash\text{binomio}\\
&= \binom{n}{k}2^{n-k}
\end{aligned}
$$