II) $\lim \frac{n^{20}}{e^n}$

Notar que este es un límite de la forma

$$\lim n^k q^n \;,\; q = \frac{1}{e} < 1,$$

luego es conocido que si $|q| < 1 \Rightarrow \underline{\underline{\lim n^k q^n = 0}}$.

Otra forma: usemos sandwich con las desigualdades de la exponencial.

Notemos que $\frac{n^{20}}{e^n} = \left(\frac{\sqrt{n}}{e^{\frac{n}{40}}}\right)^{40}$

$$\Rightarrow \quad \frac{\sqrt{n}}{e^{\frac{n}{40}}} \leq \frac{\sqrt{n}}{1+\frac{n}{40}} = \frac{\frac{\sqrt{n}}{n}}{\frac{1}{n}+\frac{1}{40}}$$

$$\Rightarrow \quad \underset{\downarrow \atop 0}{0} \leq \underset{\downarrow \text{ Teo Sandwich.} \atop 0}{\left(\frac{\sqrt{n}}{e^{\frac{n}{40}}}\right)^{40}} \leq \underset{\downarrow \atop 0}{\left(\frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}+\frac{1}{40}}\right)^{40}}$$

III $\lim \frac{\ln(n)}{n}$

$$= \lim \frac{1}{n} \ln n$$

$$= \lim \ln n^{\frac{1}{n}}$$ / de existir $\lim \ln n^{\frac{1}{n}}$ podemos intercambiar lim y ln

$$= \ln \underbrace{\lim \sqrt[n]{n}}_{1}$$

$$= \ln(1) = 0.$$

IV $\lim \frac{\ln(1+e^n)}{n}$

$$\frac{\ln(1+e^n)}{n} = \ln(1+e^n)^{1/n}$$

$$e^n \leq (1+e^n)^{\cancel{1/n}} \leq e^n + e^n \quad / (\ )^{1/n}$$

$$\sqrt[n]{e^n} \leq (1+e^n)^{1/n} \leq \sqrt[n]{2} \cdot \sqrt[n]{e^n}$$

$$\Rightarrow \quad \underset{\substack{\downarrow \\ e}}{e} \leq \underset{\substack{\downarrow \\ e}}{(1+e^n)^{1/n}} \leq \underbrace{\underbrace{\sqrt[n]{2}}_{\to 1} \cdot e}_{\substack{\downarrow \\ e}}$$

Teo Sand.

luego $\lim \ln (1+e^n)^{1/n}$

$= \ln \left( \lim (1+e^n)^{1/n} \right)$

$= \ln(e) = 1$ //

v) $\lim \log_{1+\frac{1}{n}} (\sqrt[n]{e})$

usamos cambio de base:

$$\lim \log_{1+\frac{1}{n}} \sqrt[n]{e} = \lim \frac{\ln \sqrt[n]{e}}{\ln \left(1+\frac{1}{n}\right)}$$

$$= \lim \frac{\frac{1}{n} \ln e}{\ln\left(1+\frac{1}{n}\right)}$$

$$= \ln e \; \underbrace{\lim \left( \frac{\frac{1}{n}}{\ln\left(1+\frac{1}{n}\right)} \right)}_{\text{conocido } = 1}$$

$$= \ln(e) = 1.$$

vi) $\lim n\left\{ n^2\left(\sqrt[n^2]{e}-1\right)-1\right\}$.

usemos sandwich:

primero: $\sqrt[n^2]{e} = e^{\frac{1}{n^2}}$

$$\Rightarrow \quad e^{\frac{1}{n^2}} \leq \frac{1}{1-\frac{1}{n^2}} \quad /-1$$

$$e^{\frac{1}{n^2}}-1 \leq \cancel{\frac{n^2}{n^2-1}} \quad \frac{n^2}{n^2-1}-1 = \frac{\cancel{n^2}-\cancel{n^2}+1}{n^2-1}$$

$$\Rightarrow \quad e^{\frac{1}{n^2}}-1 \leq \frac{1}{n^2-1} \quad /\cdot n^2$$

$$n^2\left(e^{\frac{1}{n^2}}-1\right) \leq \frac{n^2}{n^2-1} \quad /-1$$

$$n^2\left(e^{\frac{1}{n^2}}-1\right)-1 \leq \frac{n^2}{n^2-1}-1 = \frac{\cancel{n^2}-\cancel{n^2}+1}{n^2-1} \quad /\cdot n$$

$$n\left\{n^2\left(e^{\frac{1}{n^2}}-1\right)-1\right\} \leq \underbrace{\frac{n}{n^2-1}}_{\xrightarrow[n\to\infty]{} 0} \quad (\star)$$

Falta acotar por abajo

$$\frac{1}{n^2} + 1 \leq e^{\frac{1}{n^2}} \quad /-1$$

$$\frac{1}{n^2} \leq e^{\frac{1}{n^2}} - 1 \quad / n^2$$

$$1 \leq n^2\left(e^{\frac{1}{n^2}} - 1\right) \quad /-1$$

$$0 \leq n^2\left(e^{\frac{1}{n^2}} - 1\right) - 1 \quad / \cdot n$$

$$0 \leq n\left\{n^2\left(e^{\frac{1}{n^2}} - 1\right) - 1\right\} \quad (\star\star)$$

Juntando $(\star)$ y $(\star\star)$ se concluye que

$$\boxed{\lim n\left\{n^2\left(e^{\frac{1}{n^2}} - 1\right) - 1\right\} = 0}$$