Problems

The following set of problems is designed to illustrate the conventions of the main notations. Each group of problems uses only the material of the preceding groups.

A. FREE INDEX

1. Translate the following sets of equations into index notation by using the rule of the free index.

a.
$$u_1 + v_1 + w_1 = r_1$$
$$u_2 + v_2 + w_2 = r_2$$
$$u_3 + v_3 + w_3 = r_3$$
$$u_4 + v_4 + w_4 = r_4$$

b.
$$u_1v_1 + w_1r_1 = A_{11}$$
$$u_1v_2 + w_1r_2 = A_{12}$$
$$u_2v_1 + w_2r_1 = A_{21}$$
$$u_2v_2 + w_2r_2 = A_{22}$$

c.
$$u_1v_1w_1 = A_{111} \qquad u_2v_1w_1 = A_{211} \qquad u_3v_1w_1 = A_{311}$$
$$u_1v_1w_2 = A_{112} \qquad u_2v_1w_2 = A_{212} \qquad u_3v_1w_2 = A_{312}$$
$$u_1v_1w_3 = A_{113} \qquad u_2v_1w_3 = A_{213} \qquad u_3v_1w_3 = A_{313}$$
$$u_1v_2w_1 = A_{121} \qquad u_2v_2w_1 = A_{221} \qquad u_3v_2w_1 = A_{321}$$
$$u_1v_2w_2 = A_{122} \qquad u_2v_2w_2 = A_{222} \qquad u_3v_2w_2 = A_{322}$$
$$u_1v_2w_3 = A_{123} \qquad u_2v_2w_3 = A_{223} \qquad u_3v_2w_3 = A_{323}$$
$$u_1v_3w_1 = A_{131} \qquad u_2v_3w_1 = A_{231} \qquad u_3v_3w_1 = A_{331}$$
$$u_1v_3w_2 = A_{132} \qquad u_2v_3w_2 = A_{232} \qquad u_3v_3w_2 = A_{332}$$
$$u_1v_3w_3 = A_{133} \qquad u_2v_3w_3 = A_{233} \qquad u_3v_3w_3 = A_{333}$$

2. For a range of four, write out all expressions contained in the condition of symmetry, $A_{ij} = A_{ji}$, and in the condition of skew-symmetry, $B_{ij} = -B_{ji}$.

3. Show that $A_{ij} = u_iu_j$ satisfies the condition of symmetry, whereas $B_{ij} = u_iv_j - v_iu_j$ satisfies the condition of skew-symmetry. Obtain expressions for the symmetric and skew-symmetric parts of $C_{ij} = u_iv_j$.

4. If an expression contains M free indices with range from 1 to R, what is the number N of equations described by the expression?

5. Calculate the numerical values of $C_{ij} = u_iv_j$ if $u_1 = 1, u_2 = 2, u_3 = 3, v_1 = 5, v_2 = 10, v_3 = 4$.

B. SUMMATION INDEX

6. Translate the following sets of equations into index notation by using the rule of the summation index.

a. $$ds^2 = g_{11}(dx_1)^2 + g_{12}\,dx_1\,dx_2 + g_{13}\,dx_1\,dx_3 + g_{21}\,dx_2\,dx_1 + g_{22}(dx_2)^2 + g_{23}\,dx_2\,dx_3 + g_{31}\,dx_3\,dx_1 + g_{32}\,dx_3\,dx_2 + g_{33}(dx_3)^2$$

b. $u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 = 0$

c. $\phi = u_1v_1w_1r_1 + u_1v_1w_2r_2 + u_2v_2w_1r_1 + u_2v_2w_2r_2$

d. $dS^2 = (dX_1)^2 + (dX_2)^2 + (dX_3)^2$

e. $\phi = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$

7. Verify (1.12). Also show that $W_{ij}u_iu_j = W_{(ij)}u_iu_j$, $B_{[ij]}u_iu_j = 0$.

8. For a range of three, write out in scalar notation the expression $\phi = A_{ij}A_{ji}$.

9. For a range of two, write out in scalar notation the following:

a. $\phi = u_k v_k A_{jj}$ **b.** $\phi = u_i v_i + w_j r_j$ **c.** $A_{kk} = p_j r_j$
d. $u_i = v_i + A_{ikk}$ **e.** $u_i v_j w_j = B_{ik} r_k$ **f.** $A_{ij} = B_{ik} C_{kn} D_{nj}$

10. Evaluate $\phi = u_i v_i + A_{kk}$ for $u_1 = 2$, $u_2 = 4$, $v_1 = 3$, $v_2 = 1$, $A_{11} = 5$, $A_{12} = 36$, $A_{21} = 10$, $A_{22} = 5$.

11. Translate the following sets of equations into index notation:

a.
$$v_1 = A_{111} + A_{122} + A_{133}$$
$$v_2 = A_{211} + A_{222} + A_{233}$$
$$v_3 = A_{311} + A_{322} + A_{333}$$

b.
$$A_{11} = B_{11}C_{11} + B_{12}C_{21}$$
$$A_{12} = B_{11}C_{12} + B_{12}C_{22}$$
$$A_{21} = B_{21}C_{11} + B_{22}C_{21}$$
$$A_{22} = B_{21}C_{12} + B_{22}C_{22}$$

c.
$$u_1 = v_1^2 w_1 + v_2^2 w_1 + v_3^2 w_1$$
$$u_2 = v_1^2 w_2 + v_2^2 w_2 + v_3^2 w_2$$
$$u_3 = v_1^2 w_3 + v_2^2 w_3 + v_3^2 w_3$$

12. For a range of three, write out in scalar notation the following:

a. $v_i = B_{ji} u_j$ **b.** $u_i = B_{ik} A_{kjj}$

13. For a range of three, obtain the scalar representations of $v_{i;j} = v_{i,j} + w_{kij} v_k$, where w_{kij} have the skew-symmetry property $w_{kij} = -w_{ikj}$. Work as though the commas and semicolons between indices were not there.

C. DIRECT NOTATION

14. If **C** is of rank two, write out in index notation the following:

$$-\mathbf{C}^3 + \mathrm{I_C}\mathbf{C}^2 - \mathrm{II_C}\mathbf{C} + \mathrm{III_C}\mathbf{1} = \mathbf{0}.$$

Here $\mathbf{C}^2 \equiv \mathbf{CC}$, $\mathbf{C}^3 \equiv \mathbf{CCC}$, and $\mathrm{I_C}$, $\mathrm{II_C}$, $\mathrm{III_C}$ are special designations of scalar coefficients (i.e. these symbols are to be left as they are).

15. Translate the following expressions into direct notation:

a. $(\mathbf{A}^2)_{ij} = A_{ik}A_{kj}$ **b.** $(\mathbf{A}^3)_{ij} = A_{ik}A_{km}A_{mj}$
c. $B_{ij} = A_{ki}A_{kj}$ **d.** $C_{ij} = A_{ik}A_{jk}$ **e.** $D_{ij} = A_{ik}A_{mk}A_{jm}$
f. $E_{kk} = A_{im}A_{jm}A_{ij}$ **g.** For a range of two, write out (a) and (b).

16. Change the following into index notation:

a. $\mathbf{G} = \mathbf{A}^T\mathbf{A}\mathbf{A}^T$ **b.** $\mathbf{A}^4 = \mathbf{AAAA}$ **c.** $\mathbf{FB} = \mathbf{BG}$
d. $\mathbf{D} = \mathbf{AB}^T\mathbf{C}$ **e.** $\mathbf{B} = \mathbf{FF}^T$ **f.** $\mathbf{C} = \mathbf{F}^T\mathbf{F}$
g. $\phi = \mathrm{tr}(\mathbf{AB})$ **h.** $\mu = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$ **i.** $\bar{\mathbf{A}} = \mathbf{QAQ}^T$
j. $\mathbf{AB} \neq \mathbf{BA}$ **k.** $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$

17. Let $\mathbf{C} = \mathbf{AB}$, $\mathbf{D} = \mathbf{A}^T\mathbf{B}$, $\mathbf{E} = \mathbf{AB}^T$, $\mathbf{F} = \mathbf{A}^T\mathbf{B}^T$, where

$$\|\mathbf{A}\| = \begin{Vmatrix} 1 & 1 & 5 \\ 6 & 2 & 3 \\ 2 & 4 & 3 \end{Vmatrix}, \qquad \|\mathbf{B}\| = \begin{Vmatrix} 3 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 6 & 1 \end{Vmatrix}$$

Calculate the arrays of **C**, **D**, **E**, **F**.

18. Let $\|\mathbf{x}^T\| = \|1, 2, 3\|$; if **A** and **B** are as in Problem 17, calculate $\mathbf{c}^T = \mathbf{x}^T\mathbf{A}^T$, $\mathbf{e} = \mathbf{Bx}$.

19. Let **A** and **B** be as in Problem 17. Calculate tr(**AB**).

D. ARITHMETIC OF FORMS

20. Substitute $u_i = B_{ij}v_j$ and $C_{ij} = p_iq_j$ into $w_m = C_{mn}u_n$.

21. Substitute $u_i = A_{ik}n_k$ into $\phi = u_kv_k$.

22. Substitute $v_i = B_{ij}A_{jkk}$ into $\phi = v_iC_{jij}$.

23. Substitute $A_{ij} = B_{ik}C_{kj}$ into $\phi = A_{mk}C_{mk}$.

E. EXCEPTIONAL CASES

24. For a range of three, write the explicit form of $f = f(A_{ij})$.

25. Write out the explicit forms of $v_i = f_i(u_j)$ for a range of three.

26. Write out the components of $A_{\underline{iikk}}$ for a range of two.

27. Write out $A_{\underline{ii}} = 0$ for a range of three.

F. ALTERNATOR AND THE KRONECKER DELTA

28. Continue and simplify wherever possible:

a. $\delta_{3k}p_k =$ **b.** $\delta_{3i}\delta_{ji} =$ **c.** $\delta_{2i}A_{ji} =$
d. $\delta_{i2}\delta_{ik}\delta_{3k} =$ **e.** $\delta_{i2}\delta_{i2} =$ **f.** $\delta_{i2}\delta_{j3}A_{ij} =$

29. For a range of three, write out in scalar notation the equations represented by

$$A_{ij} = -\varepsilon_{ijk}{}^{+}A_k, \qquad {}^{+}A_k = -\frac{1}{2}\varepsilon_{kij}A_{ij}.$$

30. Continue and simplify if possible:

a. $\delta_{ij}\varepsilon_{ijk} =$ **b.** $\delta_{ij}v_iv_j =$ **c.** $\varepsilon_{ijk}u_iv_jv_k =$
d. $\varepsilon_{ijk}a_ia_ja_k =$

31. Translate into index notation the following:

a. $\mathbf{QQ}^T = \mathbf{1}$ **b.** $\mathbf{A} = \mathbf{1A}$ **c.** $\mathbf{Q}^T\mathbf{Q} = \mathbf{1}$
d. $\phi = \text{tr}(\mathbf{1} + \mathbf{A})$ **e.** $\phi = \text{tr}(\mathbf{1} + \mathbf{1})$

32. Translate into index notation the following:

$$\begin{array}{lll}
A_{111} = 0 & A_{211} = B_{31} & A_{311} = -B_{21} \\
A_{112} = 0 & A_{212} = B_{32} & A_{312} = -B_{22} \\
A_{113} = 0 & A_{213} = B_{33} & A_{313} = -B_{23} \\
A_{121} = -B_{31} & A_{221} = 0 & A_{321} = B_{11} \\
A_{122} = -B_{32} & A_{222} = 0 & A_{322} = B_{12} \\
A_{123} = -B_{33} & A_{223} = 0 & A_{323} = B_{13} \\
A_{131} = B_{21} & A_{231} = -B_{11} & A_{331} = 0 \\
A_{132} = B_{22} & A_{232} = -B_{12} & A_{332} = 0 \\
A_{133} = B_{23} & A_{233} = -B_{13} & A_{333} = 0
\end{array}$$

33. Verify (1.24). Is there a reasonably systematic way of doing it?

34. Verify (1.30) by carrying out the summations in (1.29).

35. What are the counterparts of (1.28)–(1.30) for the alternator ε_{ij}?

36. In direct notation we denote $\varepsilon_{ijk}u_iv_jw_k$ by $[\mathbf{uvw}]$. Show that

$$[\mathbf{uvw}] = [\mathbf{vwu}] = [\mathbf{wuv}] = -[\mathbf{vuw}] = -[\mathbf{wvu}] = -[\mathbf{uwv}].$$

37. Write out in scalar notation $\phi = \varepsilon_{ijk}u_iv_jw_k$.

G. PARTIAL DERIVATIVES

38. Change $d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$ into index notation.

39. For a range of three, write out in scalar notation the following:

a. $\phi_{,ii} = 0$ **b.** $\phi_{,iikk} = 0$ **c.** $\operatorname{div}\mathbf{v} \equiv v_{i,i}$
d. $(\operatorname{curl}\mathbf{v})_i \equiv \varepsilon_{ijk}v_{k,j}$

40. Find $u_{i,i}$, $v_{j,j}$, $u_{i,j} - u_{j,i}$, $v_{i,j}$ if $u_1 = 5x_1$, $u_2 = -3x_2$. $u_3 = 7x_3$, $v_1 = 3x_1x_2$, $v_2 = 4x_2x_3$, $v_3 = 6x_3x_1$.

41. Find $\phi_{,i}$ if $\phi(\mathbf{x}) = x_1^2x_2x_3^2$.

42. Continue, and simplify wherever possible, the following:

a. $x_{k,k} =$ **b.** $(x_ix_j)_{,i} =$ **c.** $(x_ix_i)_{,k} =$ **d.** $(x_ix_j)_{,k} =$
e. $(\varepsilon_{ijk}x_jv_k)_{,m} =$ **f.** $(x_mx_mx_iA_{ij})_{,k} =$ **g.** $x_{i,j} - x_{j,i} =$

43. For a range of three, write out in scalar notation the following:

a. $x_{i.m} = \delta_{im} + u_{i.m}$ **b.** $X_{i,n} = \delta_{in} - u_{i,n}$
c. $B_{ij} = x_{i.m}x_{j.m}$ **d.** $C_{ij} = x_{m.i}x_{m.j}$
e. $2E_{ij} = u_{i.j} + u_{j.i} + u_{m.i}u_{m.j}$ **f.** Derive (e) from $2\mathbf{E} = \mathbf{C} - \mathbf{1}$ by using (d) and (a).

44. For a range of three, write out in scalar notation the following:

a. $\dot{\rho} + \rho v_{i,i} = 0$ **b.** $a_i = \partial v_i/\partial t + v_kv_{i,k}$
c. $T_{ji,j} + \rho b_i = \rho a_i$ **d.** $\rho\dot{u} + q_{i,i} = T_{ij}D_{ij}$