

P2) $\phi_2 = \frac{a_2}{2} x^2 + b_2 xy + \frac{c_2}{2} y^2$

$\phi_4 = \frac{a_4}{12} x^4 + \frac{b_4}{6} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{6} x y^3 + \frac{e_4}{12} y^4$

a) Ec. Biarmónica $\nabla^2 \nabla^2 \phi = \Delta^2 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$

Para $\phi_2 \rightarrow \frac{\partial^4 \phi}{\partial x^4} = \frac{\partial^4 \phi}{\partial y^4} = 0$
 $\rightarrow \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \frac{\partial^2 \phi}{\partial y^2} (a_2) = 0$ } $\Delta^2 \phi_2 = 0 //$

Para $\phi_4 \rightarrow \frac{\partial^4 \phi}{\partial x^4} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{12} a_4 = 2a_4$

$\rightarrow \frac{\partial^4 \phi}{\partial y^4} = 2e_4$

$\rightarrow \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial y^2} (a_4 x^2 + b_4 y + c_4 y^2) = 2c_4$

$\rightarrow \Delta^2 \phi_4 = 2a_4 + 4c_4 + 2e_4 = 0$

* La ec. biarmónica se cumple sólo si:

$e_4 = -(a_4 + 2c_4)$

$\therefore \phi_4 = \frac{a_4}{12} x^4 + \frac{b_4}{6} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{6} x y^3 - \frac{(a_4 + 2c_4)}{12} y^4$

b) Para cualquier función de esfuerzos:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} ; \sigma_y = \frac{\partial^2 \phi}{\partial x^2} ; \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

• Para $\phi_2 \rightarrow \sigma_x = c_2$

$$\sigma_y = 2c_2$$

$$\tau_{xy} = -b_2$$

• Para $\phi_4 \rightarrow \sigma_x = c_4 x^2 + d_4 xy - (2c_4 + 2d_4)x^2$

$$\sigma_y = 2c_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = -\frac{b_4}{2} x^2 + 2c_4 xy + \frac{d_4}{2} y^2$$

$$c) \sigma_x = \sigma_{x(4^o \text{ orden})} = c_4 x^2 + d_4 xy - (2c_4 + 2d_4)x^2$$

$$\sigma_y = \sigma_{y(4^o \text{ orden})} = 2c_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = (\tau_{xy})_{4^o \text{ orden}} + (\tau_{xy})_{2^o \text{ orden}} = -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2 - b_2$$

$$\star \quad 2c_4 = b_4 = c_4 = 0$$

$$\therefore \sigma_x = d_4 xy$$

$$\sigma_y = 0$$

$$\tau_{xy} = -\frac{d_4}{2} y^2 - b_2$$

• ¿Cómo encontrar las constantes? $\rightarrow$ Valor de los esfuerzos en bordes!

$$i) \tau_{xy}(y = \pm c) = 0 \rightarrow -\frac{d_4}{2} c^2 - b_2 = 0 \rightarrow d_4 = -\frac{2b_2}{c^2}$$

$$ii) - \int_{-c}^c \tau_{xy} dx \cdot 1 = - \int_{-c}^c \left(\frac{b_2}{c^2} y^2 - b_2 \right) dy = - \frac{1}{3} \left[\frac{b_2}{c^2} (c^3 + c^3) \right] + b_2 \cdot 2c$$

Negativo en el cubo de esfuerzos!

- Los esfuerzos quedan expresados como:

d) Ley de Hooke (est. plano)

$$\varepsilon_x = \frac{\sigma_x}{E} ; \quad \varepsilon_y = -\frac{\nu \sigma_x}{E} ; \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

e) Las deformaciones se definen como:

$$(a) \quad \varepsilon_x = \frac{\partial u}{\partial x} ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad (b) \quad (ii)$$

$$p_{xy} = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \text{ (in)}$$

$$\rightarrow U = \int \epsilon_x dx = -\frac{P}{EI} \int x dx = -\frac{P}{2EI} x^2 + f(y)$$

$$\rightarrow V = \int \epsilon_y dy = \frac{\nu P x}{EI} \int y dy = \frac{\nu P x y^2}{2EI} + g(x)$$

• Derivando ambas expresiones y sumándolas en (iii):

$$\frac{\partial U}{\partial y} = -\frac{P x^2}{2EI} + \frac{df}{dy} \quad \wedge \quad \frac{\partial V}{\partial x} = \frac{\nu P y^2}{2EI} + \frac{dg}{dx}$$

$$\rightarrow \gamma_{xy} = \frac{\nu P y^2}{2EI} + \frac{dg}{dx} + \frac{P x^2}{2EI} + \frac{df}{dy} = -\frac{P}{2IG} (C^2 - y^2)$$

$$\Rightarrow \underbrace{\left(\frac{dg}{dx} - \frac{P x^2}{2EI} \right)}_{\text{Constante } (\alpha)} + \underbrace{\left(\frac{\nu P y^2}{2EI} + \frac{df}{dy} - \frac{P}{2IG} y^2 \right)}_{\text{Constante } (\beta)} = -\frac{P}{2IG} C^2$$

• Se obtienen estas ^{relaciones} ~~ecuaciones~~:

$$\alpha + \beta = -\frac{P}{2IG} C^2 \quad (\text{I})$$

$$\alpha = \frac{dg}{dx} - \frac{P x^2}{2EI} \quad (\text{II})$$

$$\beta = \left(\frac{\nu P}{2EI} - \frac{P}{2IG} \right) y^2 + \frac{df}{dy} \quad (\text{III})$$

• Se utilizan las ecs. (II) y (III) para obtener las fnes. $f(y)$ y $g(x)$:

$$\rightarrow \text{Desde II): } \frac{dg}{dx} = \alpha + \frac{P x^2}{2EI} \rightarrow \boxed{g(x) = \alpha x + \frac{P x^3}{6EI} + m}$$

$$\rightarrow \text{Desde III): } \frac{df}{dy} = \beta + \left(\frac{P}{2IG} - \frac{\nu P}{2EI} \right) y^2$$

$$\hookrightarrow \boxed{f(y) = \beta y + \left(\frac{P}{IG} - \frac{\nu P}{EI} \right) \frac{y^3}{6} + n} \quad (\text{V})$$

• Luego, los desplazamientos quedan como:

$$U(x, y) = \frac{-P}{2EI} x^2 y + \beta y + \left(\frac{P}{16} - \frac{\partial P}{EI} \right) \frac{y^3}{6} + n$$

$$V(x, y) = \frac{\partial P}{2EI} x y^2 + \alpha x + \frac{P x^3}{6EI} + m$$

• Se necesita calcular las constantes α, β, m y n . Estas se obtienen de las condiciones de borde para el desplazamiento. En particular para $V(x, y)$ (Deflexión)

$$a) V(l, 0) = 0 \leadsto V(l, 0) = \alpha l + \frac{Pl^3}{6EI} + m = 0$$

$$\boxed{m = -\alpha l - \frac{Pl^3}{6EI}} \quad (*)$$

$$b) \frac{\partial V}{\partial x}(l, 0) = 0 \leadsto \frac{\partial V}{\partial x}(l, 0) = \alpha + \frac{Pl^2}{2EI} \neq 0$$

$$\boxed{\alpha = -\frac{Pl^2}{2EI}}$$

$$\text{En } (*) \leadsto \boxed{m = \frac{Pl^3}{3EI}}$$

• Luego, el campo de desplazamientos ~~para~~ verticales (deflexión) queda:

$$\boxed{V(x, y) = \frac{\partial P}{2EI} x y^2 + \frac{P}{6EI} x^3 + \frac{Pl^2}{2EI} x + \frac{Pl^3}{3EI}}$$

• Ahora para $U(x, y)$:

$$2) U(l, 0) = 0 \leadsto n = 0$$

b) Desde cc. (I)

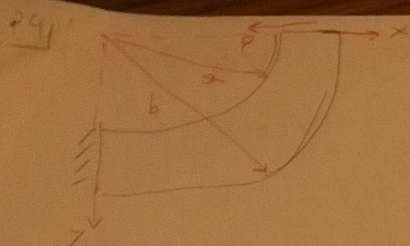
$$\alpha + \beta = -\frac{P}{2IG} C^2 \quad / \quad \alpha = -\frac{Pl^2}{2EI}$$

$$\therefore \beta = \frac{P}{2I} \left(\frac{l^2}{E} - \frac{C^2}{6} \right)$$

Luego, el campo de desplazamientos horizontal queda como:

$$U(x, y) = -\frac{P}{2EI} x^2 y + \left(\frac{P}{IG} - \frac{2P}{EI} \right) \frac{y^3}{6} + \left(\frac{l^2}{E} - \frac{C^2}{6} \right) \frac{P}{2I} y$$

* Propuesto: Encuentre la deflexión del plano medio de la viga y compárela con el valor obtenido usando la ec. de la curva elástica ($y^{(4)}(x) = -\frac{w(x)}{EI}$)



$$\phi(r, \theta) = f(r) \sin(\theta)$$

• Para que $\phi(r, \theta)$ satisfaga las ecs. de equilibrio, ésta debe cumplir con la ec. biarmónica (coord. polares)

$$\nabla^2 \nabla^2 \phi = 0$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left[\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right) \sin(\theta) \right] = 0$$

$$\Rightarrow \nabla^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right) \sin(\theta) \quad (1)$$

$$+ \underbrace{\nabla^2(\sin(\theta))}_{(2)} \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right) = 0$$

$$(1): \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left[\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right) \right] \sin(\theta)$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left[\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right) \right] \sin(\theta)$$

$$(2): \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) [\sin(\theta)] \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right)$$

$$= -\frac{1}{r^2} \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right) \sin(\theta)$$

$$\nabla^2 \varphi = \left\{ \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \left[\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right] - \frac{1}{r^2} \left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right] \right\} \sin(\theta) = 0 \quad \forall \theta$$

$$\therefore \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \left[\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} f \right] = 0$$

b) $f(r) = Ar^3 + B \cdot \frac{1}{r} + Cr + Dr \ln(r)$

$$\varphi(r, \theta) = (Ar^3 + B \cdot \frac{1}{r} + Cr + Dr \ln(r)) \sin(\theta)$$

• Reemplazando en la def. de función de esfuerzos (coord polares)

$$\begin{aligned} * \sigma_r &= \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} = \left(3Ar - \frac{B}{r^3} + \frac{C}{r} + D \left(\ln(r) + \frac{1}{r} \right) \right) \sin(\theta) \\ &+ \left(Ar + \frac{B}{r^3} + \frac{C}{r} + \frac{D \ln(r)}{r} \right) \sin(\theta) \end{aligned}$$

$$\therefore \sigma_r = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \sin(\theta)$$

$$* \sigma_\theta = \frac{\partial^2 \varphi}{\partial r^2} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) \sin(\theta)$$

$$* \tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right) = -\frac{\partial}{\partial r} \left(Ar^2 + \frac{B}{r^2} + C + D \ln(r) \right) \cos(\theta)$$

$$\tau_{r\theta} = \left(-2Ar + \frac{2B}{r^3} - \frac{D}{r} \right) \cos(\theta)$$

• Las constantes se calculan a partir de las condiciones de los del problema.

$$i) \sigma_r(a, \theta) = 0 \Rightarrow \left(2Aa - \frac{2B}{a^3} + \frac{D}{a} \right) \sin(\theta) = 0$$

$$\therefore \left(2Aa - \frac{2B}{a^3} + \frac{D}{a} \right) = 0 \quad (i)$$

$$ii) \sigma_r(b, \theta) = 0$$

$$\therefore 2Ab - \frac{2B}{b^3} + \frac{D}{b} = 0 \quad (ii)$$

$$iii) \int_a^b \sigma_{r\theta} dr \Big|_{\theta=0} = P \quad \rightarrow -2A \int_a^b r dr - 2B \int_a^b \frac{1}{r^3} dr + D \int_a^b \frac{1}{r} dr = \frac{P}{\cos \theta} \Big|_{\theta=0}$$

$$\Rightarrow \left[-A(b^2 - a^2) + B \left(\frac{1}{b^2} - \frac{1}{a^2} \right) + D \ln \left(\frac{b}{a} \right) \right] = P \quad (iii)$$

Despejando las constantes A, B y D , se tiene:

$$\left[A = \frac{P}{2\gamma} \quad B = -\frac{Pa^2b^2}{2\gamma} \quad D = -\frac{P}{\gamma}(a^2 + b^2) \right]$$

$$\star \gamma = (a^2 - b^2) + (a^2 + b^2) \ln \left(\frac{b}{a} \right)$$

Reemplazando en las exp. de los esfuerzos

$$\sigma_r = \left(\frac{P}{\gamma} r - \frac{Pa^2b^2}{\gamma} \cdot \frac{1}{r^3} + \frac{P(a^2 + b^2)}{\gamma} \cdot \frac{1}{r} \right) \sin(\theta)$$

$$\sigma_\theta = \left(\frac{3P}{\gamma} r - \frac{Pa^2b^2}{\gamma} \cdot \frac{1}{r^3} - \frac{P(a^2 + b^2)}{\gamma} \cdot \frac{1}{r} \right) \sin(\theta)$$

$$\sigma_{r\theta} = \left(-\frac{P}{\gamma} r - \frac{Pa^2b^2}{\gamma} \cdot \frac{1}{r^3} + \frac{P(a^2 + b^2)}{\gamma} \cdot \frac{1}{r} \right) \cos(\theta)$$

• En el extremo superior ($\theta = 0$):

$$\sigma_r = \sigma_\theta = 0$$

$$\tau_{r\theta} = -\frac{P}{f} \left(r + \frac{a^2 b^2}{r^3} - \frac{1}{r} (a^2 + b^2) \right)$$

• En el extremo inferior ($\theta = \frac{\pi}{2}$):

$$\tau_{r\theta} = 0$$

$$\sigma_r = \frac{P}{f} \left(r + \frac{a^2 b^2}{r^3} - \frac{1}{r} (a^2 + b^2) \right)$$

$$\sigma_\theta = \frac{P}{f} \left(3r - \frac{a^2 b^2}{r^3} - \frac{1}{r} (a^2 + b^2) \right)$$

c) Los desplazamientos en coordenadas polares se relacionan con las def.

como:

$$\epsilon_r = \frac{\partial u}{\partial r} \quad ; \quad \epsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \quad ; \quad \gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

(I) (II) (III)

• Las def. en coord. polares se relacionan con las esfuerzos como:

$$\epsilon_r = \frac{1}{E} (\sigma_r - \nu \sigma_\theta) \quad ; \quad \epsilon_\theta = \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \quad ; \quad \gamma_{r\theta} = \frac{1}{G} \tau_{r\theta}$$

• Luego, de (I)

~~$$\frac{\partial u}{\partial r} = \frac{P}{E}$$~~

$$\begin{aligned} \leadsto \frac{\partial u}{\partial r} &= \frac{\sin \theta}{E} \left(2Ar - \frac{2B}{r^3} + \frac{D}{r} - 6\nu Ar - \frac{2\nu B}{r^3} - \frac{\nu D}{r} \right) \\ &= \frac{\sin \theta}{E} \left(2A(1-3\nu)r - \frac{2B}{r^3}(1-\nu) + \frac{D}{r}(1-\nu) \right) \end{aligned}$$

• Integrando:

~~$$u = \frac{\sin \theta}{E} \left(Ar^2 + \frac{B}{r^2} + \right)$$~~

• Integrando

$$U = \frac{\sin \vartheta}{E} \left(A(1-\nu)r^2 + \frac{B}{r^2}(1-\nu) + D(1-\nu)\ln(r) \right) + g(\vartheta)$$

• Reemplazando en (II)

(IV)

$$\left[\frac{\sin \vartheta}{E} \left(A(1-\nu)r^2 + \frac{B}{r^2}(1-\nu) + D(1-\nu)\ln(r) \right) + \frac{g(\vartheta)}{r} + \frac{1}{r} \frac{\partial V}{\partial \vartheta} \right]$$

$$\frac{U}{r} + \frac{1}{r} \frac{\partial V}{\partial \vartheta} = \frac{1}{E} (\sigma_\vartheta - \nu \sigma_r)$$

$$\Rightarrow \frac{\partial V}{\partial \vartheta} = \frac{r}{E} (\sigma_\vartheta - \nu \sigma_r) - U$$

$$\Rightarrow \frac{\partial V}{\partial \vartheta} = \frac{r \sin \vartheta}{E} \left(6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) - 2\nu Ar - \frac{2\nu B}{r^3} + \frac{\nu D}{dr}$$

$$- \frac{\sin \vartheta}{E} \left(Ar^2(1-\nu) + \frac{B}{r^2}(1-\nu) + D(1-\nu)\ln(r) \right) + g(\vartheta)$$

$$= \frac{\sin \vartheta}{E} \left(2Ar^2(3-\nu) + \frac{2B}{r^2}(1-\nu) + D(1-\nu) \right)$$

$$- \frac{\sin \vartheta}{E} \left(Ar^2(1-\nu) + \frac{B}{r^2}(1-\nu) + D(1-\nu)\ln(r) \right) + g(\vartheta)$$

$$= \frac{\sin \vartheta}{E} \left(Ar^2(3+\nu) + \frac{B}{r^2}(1-\nu) + D(1-\nu) - D(1-\nu)\ln(r) \right) + g(\vartheta)$$

• Integrando:

$$V = -\frac{\cos \vartheta}{E} \left(Ar^2(3+\nu) + \frac{B}{r^2}(1-\nu) + D(1-\nu) - D(1-\nu)\ln(r) \right)$$

$$+ \int g(\vartheta) d\vartheta + h(r)$$

(V)

• Reemplazando (IV) y (V) en (III)

$$\frac{1}{r} \frac{\partial V}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r} = \frac{1}{G} \nabla^2 \phi$$

$$\hookrightarrow \frac{\partial V}{\partial \theta} = + \frac{\cos \theta}{E} \left(Ar^2(1-3V) + \frac{B}{r^2}(1-V) + D(1-V) \ln(r) \right) + g'(\theta)$$

$$\hookrightarrow \frac{\partial V}{\partial r} = - \frac{\cos \theta}{E} \left(2Ar(3+V) - \frac{2B}{r^3}(1-V) - \frac{D(1-V)}{r} \right) + h'(r)$$

$$\Rightarrow \frac{\cos \theta}{E} \left(Ar^2(1-3V) + \frac{B}{r^2}(1-V) + D(1-V) \ln(r) \right) + g'(\theta)$$

$$+ \frac{\cos \theta}{E} \left(2Ar^2(3+V) - \frac{2B}{r^2}(1-V) - D(1-V) \right) + rh'(r)$$

$$+ \frac{\cos \theta}{E} \left(Ar^2(3+V) + \frac{B}{r^2}(1-V) + D(1-V) \ln(r) \right)$$

$$+ \int g(\theta) d\theta + h(r) = - \left(2Ar^2 + \frac{2B}{r^2} - D \right) \frac{\cos(\theta)}{G}$$

• De esta relación se obtiene la sig. ecuación

$$\int g(\theta) d\theta + g'(\theta) + rh'(r) - h(r) = - \frac{4D \cos \theta}{E}$$

• La cual puede dividirse en dos:

$$\boxed{rh'(r) - h(r) = 0} \quad (*)$$

$$\int g(\theta) d\theta + g'(\theta) = - \frac{4D \cos \theta}{E} \quad \bigg/ \quad \frac{d}{d\theta}$$

$$\boxed{g(\theta) + g''(\theta) = \frac{4D \sin \theta}{E}} \quad (**)$$

E //

la constante D, se
finalmente:

solución a ambas EDO's es

$$(A) \quad h(r) = \alpha r$$

$$(B) \quad g(\theta) = \underbrace{-\frac{2D}{E} \cos \theta}_{\text{Soln particular}} + \underbrace{\beta \sin \theta + \delta \cos \theta}_{\text{Soln homogénea } (\Delta g = 0)}$$

• Finalmente, las dos potencialidades quedan expresadas como:

$$U(r, \theta) = \frac{\sin \theta}{E} \left(A r^4 (1-\nu) + \frac{b}{r^2} (1-\nu) + D(1-\nu) \ln(r) \right) - \frac{2D}{E} \cos \theta + \beta \sin \theta + \delta \cos \theta$$

$$V(r, \theta) = -\frac{\cos \theta}{E} \left(A r^4 (1+\nu) + \frac{b}{r^2} (1+\nu) + D(1+\nu) - D(1+\nu) \ln(r) \right) + \frac{2D}{E} \sin \theta + \frac{D(1+\nu)}{E} \cos \theta + \beta \cos \theta - \delta \sin \theta + \alpha r$$

• Aún falta determinar las constantes α , β y δ . En particular necesitamos determinar β y δ para obtener el desplazamiento radial en el extremo superior. Luego, aplicando cond. de borde en el extremo inferior para la deflexión (V), se tiene que:

$$\theta = \frac{\pi}{2} \rightarrow V$$

$$a) \quad V(r, \pi/2) = 0 \leadsto \frac{2D}{E} \cdot \frac{\pi}{2} - \delta + \alpha r = 0$$

$$b) \quad \frac{\partial V}{\partial r}(r, \pi/2) = 0 \leadsto \boxed{\alpha = 0}$$

$$\therefore \boxed{\delta = \frac{\pi D}{E}}$$

• Para finalizar, analizamos el desplazamiento U en el extremo inferior, o sea, $\theta = 0$

$$\leadsto U(r, 0) = \delta = \frac{\pi D}{E} \quad // \quad \text{Reemplazando el valor de la constante } D, \text{ se tiene finalmente:}$$

$$U(r,0) = \frac{-\pi P}{E \delta} (a^2 + b^2)$$

$$U(r,\theta) = - \frac{P \pi (a^2 + b^2)}{E (a^2 + b^2 + (a^2 + b^2) \ln(\frac{b}{a}))}$$

Propuesto 1. Sea $h = b - a$. Suponga que $h \ll a$, luego, es posible hacer la siguiente expansión del logaritmo

$$\log\left(\frac{b}{a}\right) = \log\left(1 + \frac{h}{a}\right) = \frac{h}{a} - \frac{1}{2} \frac{h^2}{a^2} + \frac{1}{3} \frac{h^3}{a^3} \dots$$

Demuestre que la solución para $U(r,0)$ es

$$U(r,0) = - \frac{3\pi a^3 P}{E h^3} \quad \left(\begin{array}{l} \text{Idéntico a una sola utilidad} \\ \text{Zudo, por eso, Esigliano} \end{array} \right)$$

Propuesto 2. Utilice la fn. de esfuerzos

$$\phi = f(r) \cos \theta$$

para el mismo ejemplo, ahora considerando una fuerza vertical en el extremo superior y un momento aplicado allí.