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Bosons Condense and Fermions 'Exclude', But Anyons…?

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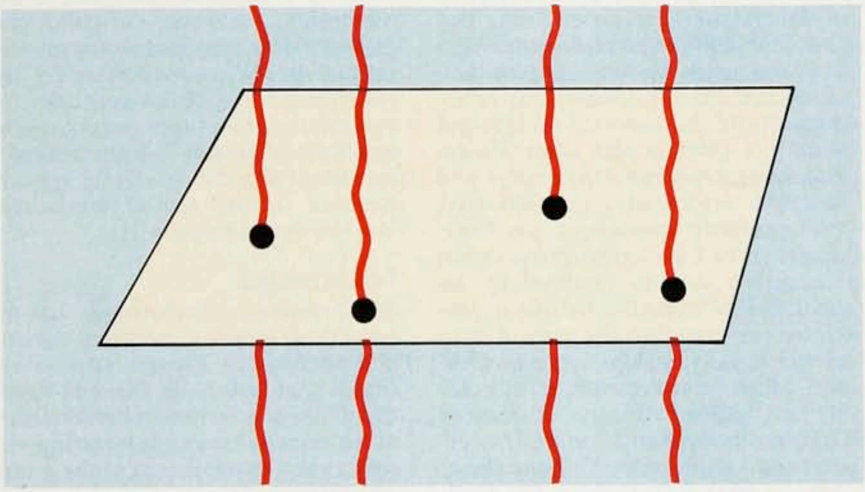
BOSONS CONDENSE AND FERMIONS 'EXCLUDE,' BUT ANYONS . . . ?

The familiar classification of particles into bosons and fermions may not hold in certain two-dimensional systems. Those systems allow the existence of particles or particle-like excitations with properties that are intermediate between those of fermions and bosons. Frank Wilczek (Institute for Advanced Study, Princeton) named such entities *anyons* in 1982, in the third of a series of papers that year in which he laid the foundation for their systematic study. The logical affirmation of the existence of anyons rests on some of the most intriguing ideas in physics, such as the magnetic monopole, the Aharonov–Bohm effect and the quantization of charge and angular momentum.



Anyons may be conceived of as particles carrying both electric charge and magnetic flux (red strings). The magnetic field is zero everywhere in the plane except at the location of the particles, and it causes no direct, classical interaction between the particles: Its role is to change the many-anyon wavefunction by a phase factor every time an anyon moves around another. The phase change is given by the Aharonov–Bohm effect.

In 1984, two years after Wilczek discussed this seemingly arcane possibility, Bertrand Halperin (Harvard University) suggested that the excitations in the theory of fractional quantum Hall effect discussed by Robert Laughlin (Stanford University) behave like anyons. Later Wilczek, Daniel Arovas (University of California, San Diego) and Robert Schrieffer (University of California, Santa Barbara) confirmed the idea. And the suggestion that the high-temperature superconductors do not behave like conventional metals in their normal, nonsuperconducting state, made by Philip Anderson (Princeton University) soon after the discovery of those materials, led Laughlin to propose that many of those unusual properties may be best described and understood in terms of the peculiar properties of a collection, or gas, of anyons.

Symmetry and statistics

A many-particle wavefunction is symmetric under permutations of identical bosons, but it is antisymmetric under permutations of identical fermions. The distinction between symmetric and antisymmetric wavefunctions under particle permutations is almost as old as quantum mechanics: It first appeared in Heisenberg's second paper on matrix mechanics, in which he extended to the study of two, coupled harmonic oscillators the ideas of matrix mechanics that he had developed for a single harmonic oscillator in his first paper.

When two anyons are interchanged, the phase of the wavefunction may change by any real value, or the wavefunction may acquire a *complex* phase factor; unlike in the case of fermions and bosons, the change in phase need not be an integral multiple of π. Furthermore, the change in phase when anyons of a given species are exchanged may be modified at the expense of introducing long-range gauge forces between the particles. In other words, the multivalued anyon wavefunction may be replaced by a Fermi- or Bose-type single-valued one, but such a reformulation of anyon physics inevitably introduces long-range forces between the particles over and above any electromagnetic or other interaction due to their electrical charge or another degree of freedom. Anyon species are therefore usually characterized by a real number that measures in multiples of π the change in phase of a many-anyon wavefunction when any two anyons regarded as *free* particles are exchanged.

The antisymmetry of the many-fermion wavefunction when a pair of particles is interchanged gives rise to the Pauli exclusion principle, according to which two identical fermions must not occupy the same quantum state. So the hallmark of an ideal gas of fermions is the Fermi surface—the surface in momentum space that separates the occupied and unoccupied states in the ground state. (A large collection of particles is called an ideal gas when the particles do not interact.) By contrast, the probability for bosons to be in some state is enhanced when the state is already occupied. Thus in the ground state of an ideal gas of identical bosons, all particles *condense* in the lowest-energy state.

The simplest type of anyon is the semion; the phase of a multi-semion wavefunction changes by $\pi/2$ when two semions are exchanged. Theoretical studies show that the ground state of an ideal gas of semions is very likely superfluid. If the anyons are charged, the gas should exhibit superconductivity. The first systematic study of the properties of an anyon gas was done by Arovas, Schrieffer, Wilczek and Anthony Zee (University of Washington, Seattle) in 1985. But that first study was valid only at very high temperatures or low anyon densities and did not discover superconductivity in the many-anyon ground state. In 1988, a year after Vadim Kalmeyer (Stanford University) and Laughlin presented arguments that the magnetic excitations in CuO_2 planes of the high-temperature superconductors may be modeled by an ideal gas of semions, Laughlin presented evidence that the ground state of such a gas might be superconducting. More recent studies, most notably by Charles B. Hanna, Alexander L. Fetter (both from Stanford University) and Laughlin; by Yi-Hong Chen, Edward Witten (both from the Institute for Advanced Study), Wilczek and Halperin; and by Dung-Hi Lee and Matthew P. A. Fisher (IBM Thomas J. Watson Research Center) have explored the superconducting state of an anyon gas in great detail.

Spin and statistics

The relationship between the symmetry of a many-particle wavefunction under permutation of identical particles and the particles' intrinsic spin, a cornerstone of quantum theory, emerged in the mid-1930s in the course of attempts to develop relativistic quantum field theories. (A quantum field theory may be regarded as a relativistic quantum mechanical theory for a system of many identical particles.) According to this relationship, particles, such as the photon or the He^4 nucleus, whose intrinsic spin is zero or an integral multiple of $\hbar$ behave as bosons; those such as the electron, the neutrino and the nucleus of He^3, having spin values that are odd integral multiples of $\frac{1}{2}\hbar$, behave as fermions.

It turns out that the existence of anyons does not violate the relationship between the intrinsic spin and the symmetry of the wavefunction under permutations. But that relationship is maintained at the cost of another familiar feature of quantum mechanics, namely, the allowed eigenvalues of angular momentum.

It is almost a cliché to say that in quantum mechanics angular momentum is quantized. But that statement is incomplete: It is true only if the space has three or more dimensions. The quantization of angular momentum in half-integral units of $\hbar$ is intimately linked with the fact that in three or more dimensions rotations about different axes do not commute, so that the quantum mechanical operators for different components of angular momentum also do not commute. For a system confined to two dimensions, however, angular momentum has only one component—normal to the plane defined by the two dimensions—and it may take any real value. This is one reason that the existence of anyons is a theoretically consistent possibility only for systems confined to two spatial dimensions (see the figure on page 21).

Parastatistics

Some readers will remember that the possible existence of particles obeying neither Bose nor Fermi statistics was raised once before, in the mid-1960s. That case is now remembered mostly as an important and interesting episode in the development of the quark model for the constitution of hadrons.

The proposal that hadrons may be thought of as made up of three quarks, by Murray Gell-Mann (Caltech) and, independently, George Zweig (now at Los Alamos) in 1964, offered an immediate explanation for why hadrons occur in multiplets of 8 or 10, each member of the multiplet having the same spin and parity and, very approximately, the same mass. It soon became apparent, however, that physically relevant multiplets corresponded to wavefunctions that were *symmetric* under the interchange of the constituent quarks, an obvious contradiction because quarks were postulated to have spin $\frac{1}{2}\hbar$ and therefore should behave as fermions. In the lowest-energy multiplets, the quarks were assumed to be in the ground state of orbital angular momentum, so the wavefunction was symmetric under the interchange of the position coordinates, and the only other degrees of freedom were the three quark species—only the "up," the "down" and the "strange" quarks were postulated at the time—and the two states of intrinsic spin. For example, the Δ^{++} has charge $2e$ and spin $\frac{3}{2}\hbar$, so it consists of three up quarks with spins aligned—obviously a symmetric state.

One of the several conjectures proposed to resolve this puzzle, by Oscar W. Greenberg (University of Maryland), was that quarks obey neither Fermi nor Bose statistics, but parastatistics. Unlike in Fermi statistics, in para-Fermi statistics more than one particle is allowed to occupy the same quantum state, and unlike in Bose statistics, the number of identical parafermions that may occupy a quantum state is *finite*. (Parabosons behave similarly to parafermions, but the role of symmetric and antisymmetric states is reversed.) Greenberg argued, in fact, that the physically relevant hadron wavefunctions would not violate the required antisymmetry under the interchange of quarks if three quarks could occupy the same state. The resolution of the puzzle, however, by M.-Y. Han (Duke University) and Yoichiro Nambu (University of Chicago), postulated a new, three-valued internal degree of freedom, now called color, for each quark. The dynamics of the gauge forces associated with this degree of freedom, called quantum chromodynamics, has emerged since the early 1970s as the correct description of the strong interactions between quarks.

"Anyons are different from particles obeying parastatistics," Wilczek told us. Like the wavefunctions of bosons and fermions, a many-anyon wavefunction changes by at most a complex factor when two anyons are exchanged. By contrast, the wavefunction of identical particles obeying parastatistics may consist of several independent components. The components mix under interchange of particles, so that each component is transformed under exchange of particles into a linear combination of the components. Mathematically, bosons and fermions are one-dimensional representations of the permutation group, whereas particles obeying parastatistics correspond to higher-dimensional representations of that group. Anyons also constitute a one-dimensional representation, but the group in that case is called the braid group because it describes the distinct ways in which particle world lines may "braid" around one another when the particles are exchanged.

Witten told us that particle states in two dimensions do not have to be one-dimensional representations of the braid group, but that more general representations can arise and have been used recently in different areas of superstring theory and mathematical physics.

Anyons are conceivable

How is it even conceivable that under

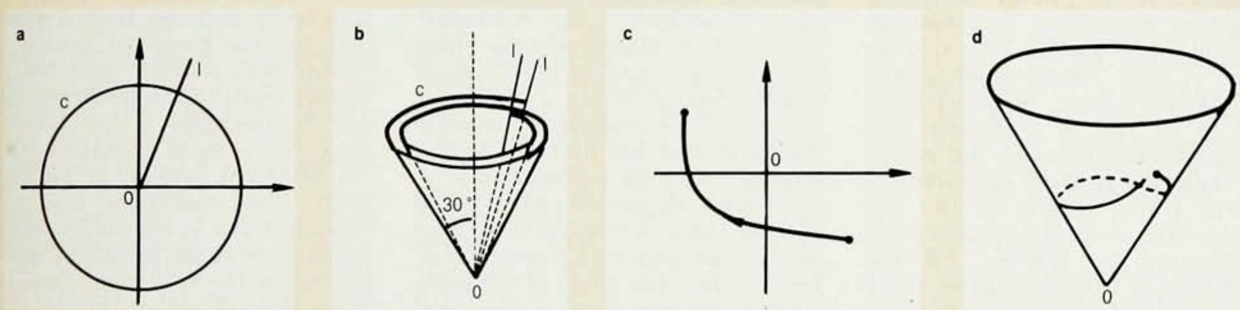
Anyons Occur Only in Two Dimensions

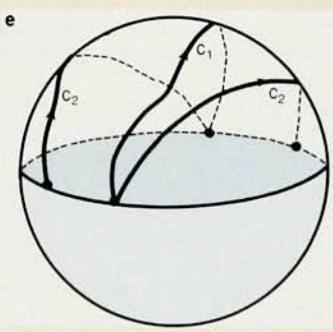
A simple exercise using a piece of paper and a pair of scissors is instructive for understanding the unique properties of the exchange of particles in two dimensions. The relative motion of a pair of particles confined to a plane has two degrees of freedom. So the configuration space of this relative motion is a plane. If the particles are identical and configurations differing by an exchange of particles are regarded as indistinguishable, a point in configuration space is equivalent to its image through the origin, and the configuration space reduces to the surface of a cone. This may be verified by cutting a piece of paper along a line l to the origin, as shown in figure **a**, and forming it into a cone as shown in figure **b**. A circle, such as c in figure **a**, will revolve twice around the cone if a point in the plane and its image through the origin are made to correspond to the same point on the surface of the cone; the half-angle of such a cone is 30°. One may also verify that trajectories that lead to exchanges of particles, such as the one in figures **c** and **d** below, correspond to closed paths, or loops, on the surface of the cone that encircle the vertex. If the particles actually pass through each other, the path goes through the origin in cofiguration space, or through the vertex of the cone.

In the geometrical approach outlined above, whether exchanging the particles twice, say, is equivalent to not exchanging them or to exchanging them a different number of times depends on whether the paths in configuration space corresponding to those exchanges are topologically equivalent—that is, on whether the paths can be deformed into one another continously without being cut or twisted—even when they may not pass through the vertex. One can verify that a loop on the cone that encircles the vertex, say, twice cannot be deformed into another that encircles the vertex a different number of times without making it pass through the vertex; nor can a loop be shrunk to zero, which would correspond to no exchange at all. This property of the configuration space is the reason that the wavefunction of particles confined to two dimensions may acquire an n-dependent phase factor when a pair of particles is interchanged.

This is, however, not so in three or higher dimensions. For aid in visualizing the configuration space in three dimensions, for example, one may without loss of generality simply consider the configuration space for the relative motion at constant separation. For identical particles, that space is the surface of a hemisphere in three dimensions with opposite points on the equator regarded as equivalent. Here, the exchange of particles corresponds to paths such as c_1 in figure **e**. When a pair is exchanged twice, c_1 is traversed twice. The curve $(c_1)^2$ can be shrunk to a point without passing it through the center (see c_2 in **e**). (The center, like the vertex of the cone, is a singular point because it corresponds to particles' passing through each other.)

[The above discussion and the figures are adapted from the paper by J. M. Leinaas and J. Myrheim (University of Oslo) in *Nuovo Cimento* **37**, 1 (1977).]





the interchange of a pair of particles the wavefunction may change by a complex factor whose square is not unity, or that its phase may change by a factor that is not a multiple of π? After all, the state of a many-particle system must be *un*changed when a pair of particles is interchanged twice, so that the square of the operator representing exchange of pairs of particles must have eigenvalue $+1$, or the operator itself must have eigenvalues $+1$ or -1, leaving us with the already discussed possibility of there being only fermions and bosons.

This apparent paradox arises only if we regard exchange of particles simply as a permutation of the particle coordinates in some appropriate wavefunction. And when we also examine carefully the paths the particles may move along when they are physically exchanged, the paradox goes away, leaving in its wake the incredible possibility of anyons. The world lines of two particles braid around one another in the time direction when the particles are exchanged, and the number of times the particles have been exchanged may be determined from topological properties of the intertwined world lines. If the particles are confined to two spatial dimensions, the world line configuration arising from, say, n exchanges may not be equivalent to the configuration arising from *no* exchange, even when n is even. If the particles may move in three or more dimensions, however, the world line configurations corresponding to an even number of exchanges are all equivalent to those when the particles are not exchanged. So the wavefunction of particles confined to two dimensions may acquire an n-dependent phase factor when a pair of particles are exchanged n times, giving rise to anyons. Clearly, the eigenvalues of the permutation operator are not sufficient to determine the statistics of identical particles in two dimensions. The topological properties of world lines in three dimensions (the third dimension here corresponds to time) are described by the braid group, which has a much richer mathematical structure than the permutation group. Ideas such as these were first discussed by Michael G. G. Laidlaw and Cécile Morette-DeWitt, in 1971, and by J. M. Leinaas and J. Myrheim, in 1977. (See the box on page 19.)

And realizable

Although anyons are logically possible, would the possibility have interested many if it could not be realized within the laws of physics? Wilczek in 1982 was not looking for anyons. He was interested, he told us, in understanding how fractional quantum numbers arise in general. Around that time several examples of quantum mechanical operators acquiring values that are rational fractions of their fundamental quanta had become known. Wilczek pointed out to us two in particular that provided the impetus for the studies he undertook in 1982: excitations of charge $e/2$ in models for the one-dimensional conductor polyacetylene, discovered by Wu-Pei Su and Schrieffer in 1979, and dyons whose electric charge could be a rational or irrational multiple of e, discovered by Witten in 1979. (Dyons are magnetic monopoles that also carrry an electric charge.)

Wilczek's interest in fractional quantum numbers led him to consider the motion of a charged particle around a solenoid, or flux tube, because it had been recognized that the angular momentum in such motion is shifted from the canonical values by a factor proportional to the flux enclosed by the solenoid. The flux tube–charged particle composite therefore provided a concrete example of a system in which a quantum mechanical operator has values very different from its canonical spectrum. But if the angular momentum of the charge–flux system was not quantized in units of $\hbar/2$, what about the spin–statistics relation? This question led Wilczek to propose anyons.

A model for anyons in vogue nowadays conceives of them as carrying both electric charge and magnetic flux (see the figure on page 17). The magnetic flux is treated in a way similar to Dirac's treatment of the magnetic monopole, by attaching a string of magnetic flux to the particle. Unlike in the treatment of the Dirac monopole, however, the flux string does not *end* at the particle but passes through it. Furthermore, the flux in a Dirac string is chosen to be an integral multiple of the flux quantum, so that the string does not affect the physics of the surrounding particles. (Dirac strings are fictitious.) The flux in the strings attached to anyons, by contrast, is adjusted to change the statistics in accordance with the Aharonov–Bohm effect. The flux string attached to the anyon does not touch the two-dimensional plane that is the stage for anyon dynamics. Nor does the string give rise to any magnetic field in the plane (except at the location of the particle), and therefore it does not exert any direct, classical force on the anyons.

When a particle of charge $e^* = \beta e$, say, makes a circuit around a solenoid carrying a magnetic flux $\phi = \phi_0\alpha$, where $\phi_0 = hc/e$ is the quantum of flux and α and β are numbers, the particle wavefunction acquires a phase $2\pi\alpha\beta$ due to the Aharonov–Bohm effect. Conversely, the phase of a solenoid changes by the same factor when it circles a stationary charge e^*. Next, consider a particle of charge e^* that has attached to it a solenoid carrying magnetic flux ϕ. When one such particle moves around another, the wavefunction of the moving member acquires a phase of $4\pi\alpha\beta$. Finally, the interchange of two particles may be carried out in two steps: First, one particle is moved a half-circle around the other, and second, the center of mass of the two is moved back by the radius of the circle. The phase of the combined wavefunction of the two particles changes only in the first of the two steps, by $2\pi\alpha\beta$. Many people have contributed to the elucidation of this intuitive idea, but it has been especially clarified by Steven Girvin (Indiana University, Bloomington) and by Steven Kivelson (University of California, Los Angeles).

Anyons occur in nature

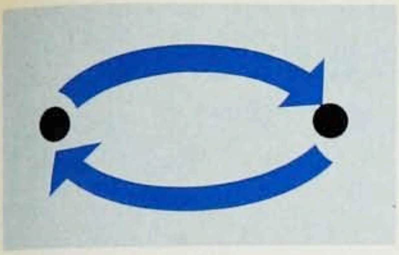
Several two-dimensional systems have been intensely studied in the laboratory over the last two decades. Of course the systems are all composed of electrons and atoms. How, then, can anyons occur, as they do in the fractional quantum Hall effect?

The fractional quantum Hall effect is a property of an electron gas confined to two dimensions and subjected to a strong magnetic field perpendicular to the plane of the gas. The omission of "ideal" in the description of the electron gas is significant, because the effect arises only when the electron density, or the average electron–electron distance, is such as to cause strong Coulomb replusion between electrons. The effect was discovered in the electron gas trapped at the boundary of GaAs and AlGaAs in GaAs–AlGaAs heterostructures, by Daniel Tsui (Princeton), Horst Stormer (AT&T Bell Labs) and Arthur Gossard (now at the University of California, Santa Barbara). It manifests itself in the quantization of the Hall conductance at values $\nu e^2/h$ when the electron density, measured by the filling factor ν, or the number of electrons per magnetic flux quantum, has certain rational fraction values. (See PHYSICS TODAY, page 17, January 1988.)

The ν values for which the effect was first observed were of the form $1/m$, where m is an odd integer. According to the theory developed by Laughlin, the quantization occurs because the ground-state energy of the electron gas has local minima at values $1/m$, so it costs energy to add or take away particles from a system that has the preferred density. A system with density close but not exactly equal to the preferred value may be described in the Laughlin theory in terms of excitations about the preferred density, or Laughlin, state. The excitations are called quasielectrons or quasiholes depending on whether the density exceeds or falls below the preferred value. It is these excitations that behave as anyons.

Just as Laughlin was completing his theory for the $1/m$ states, experimenters threw theorists another challenge by reporting observation of the quantization of the Hall conductance at $\nu = p/q$, where p and q are relative primes and q is an odd integer. Duncan Haldane (University of California, San Diego) explained the new observations by arguing that the ground-state energy had local minima at the observed ν values because quasiholes or quasielectrons about the $1/m$ states condense into the same sort of state that Laughlin had proposed for electrons to explain the $1/m$ effect.

The next development in our understanding of the Laughlin states clarified a fundamental property of anyons mentioned above, namely, that anyons of different species (characterized, as mentioned earlier, by the change in phase under exchange of *free* particles) may be transformed into one another by singular gauge transformations. Detailed studies of the wavefunctions of quasiholes and quasielectrons demanded knowledge of their behavior under permutations. Laughlin had regarded the quasiholes as fermions, whereas Halperin and



Many-anyon wavefunctions change by the phase factor $e^{i\theta}$ when two anyons are exchanged, where θ may be any real number. But anyons are conceivable only in two dimensions. In three and higher dimensions, a rotation about the axis joining the two particles will reverse the sense in which the particles are exchanged and give a phase factor of $e^{-i\theta}$. Because of rotational invariance, therefore, the phase factor must be equal to its inverse, or θ must be 0 or π, allowing only the familiar fermions and bosons.

Haldane assumed that they were bosons in their independent theories of the $\nu = p/q$ states. Halperin, however, noted that the observed preferred density fractions could be described most simply if the quasiholes and quasielectrons were assumed to behave like anyons—if, that is, it were assumed, for example, that the phase of the wavefunction for two quasiholes or quasielectrons about the state $1/m$ changes by π/m when they are exchanged. It is only with this choice of phase that the wavefunction of quasiparticles resembles the wavefunction of free particles in a magnetic field. Initially there was some skepticism about the usefulness of Halperin's suggestion, which Arovas, Schrieffer and Wilczek verified by an explicit construction. To Wilczek, however, the news of Halperin's idea provided "one of the most exhilarating moments of my scientific career."

The explanation for why quasiholes and quasiparticles about, say, the $1/m$ state behave as anyons depends on two properties: First, as Laughlin himself pointed out, the quasiholes and quasiparticles about the $1/m$ state have charge $\pm e/m$; second, the excitations are vortices. A vortex is like a flux tube carrying a quantum of ("real" magnetic) flux, and the phase of a particle of charge e that circles it changes by 2π.

Anyon superconductivity

Soon after the discovery of the high-temperature superconductors, Anderson pointed out that the key to understanding them may lie in their CuO_2 planes, and especially in the exchange interactions between the magnetic moments on the copper ions. Anderson argued that the relevant magnetic excitations in the CuO_2 planes were spinons—spin-$\frac{1}{2}$ and charge-0 excitations obeying Fermi statistics. Next, Kivelson, Daniel Rokhsar (University of California, Berkeley) and James Sethna (Cornell) proposed that the existence of spinons implies another, bosonic type of excitation called holons, with spin 0 and charge e.

Excitations in conventional magnets consisting of ions carrying spin $\frac{1}{2}$ have spin 1 and behave as bosons, because the excitation is created by flipping, say, an "up" spin "down." How then do spinons arise in CuO_2 planes? Kalmeyer and Laughlin proposed an analogy between the resonating-valence-bond state for the magnetic moments in CuO_2 discussed by Anderson and the preferred density states in the fractional quantum Hall effect to argue that holons and spinons would behave like anyons.

The suggestion by Kalmeyer and Laughlin gave the impetus for extensive studies of an ideal gas of anyons. There is now considerable evidence, both from analytic calculations and from numerical simulations by Girvin and Geoffrey Canright (Indiana University), that the ground state of charged anyons is superconducting. Furthermore, the anyon superconductor is expected to behave like a conventional superconductor in many ways; most notably, the anyon superconductor also shows the Meissner effect. As Laughlin first pointed out, particles will pair up in the superconducting phase of semions. If the semions have charge e, then the magnetic flux in that superconducting state will be quantized in units of $hc/2e$. If the phase angle under exchange is p/q, then according to Lee and Fisher, the ground state will be a superconductor, in which q anyons will bind if pq is even.

"We have opened a new chapter in theoretical physics," Wilczek remarked about the research that led to the recognition of superconductivity in an anyon gas. "But," he candidly admits, "it remains to be seen whether these ideas apply to any physical system." The anyon gas breaks parity and time-reversal symmetries. This may be seen most simply from the fact that the sign of the phase change under permutations of anyons depends on the sense—clockwise or anticlockwise—in which the anyons are moved. The breakdown of parity and time-reversal symmetries allows one to predict a number of unusual properties for the anyon gas. One such prediction, made recently by John March-Russel (Harvard), Halperin and Wilczek, is the generation of local magnetic fields near charge inhomogeneities. Muon-spin rotation experiments, which provide a very sensitive probe of local magnetic fields in solids, are now under way at TRIUMF, the Canadian facility at British Columbia, to test this prediction.

Anyon superconductivity may be related to the flux phases suggested by Piers Coleman (Rutgers University) and discussed extensively by Gabriel Kotliar (Rutgers) and by Ian Affleck (now at the University of British Columbia) and John Marston (Princeton). Those phases were proposed as possible solutions for the two-dimensional Heisenberg model, which is the simplest model for copper moments in CuO_2 layers. The name and the possible similarity of these phases to anyons arise because they are characterized by the spontaneous generation of "flux"—in this case, that of a field associated with a local gauge symmetry. This flux is nonzero when that symmetry is spontaneously broken. Unlike the anyon superconductor, however, the flux phases originally proposed do not break parity and time-reversal symmetries.

In the new light

Development of the concept of anyons as particle-like excitations has opened the way for a reexamination of familiar paradigms in physics. Consider, for example, the Kondo effect, one of the foremost problems in many-body physics in the 1960s and 1970s, in which a magnetic ion interacting with an ideal Fermi gas loses its magnetic moment at low temperatures. According to Anderson, the spin of the magnetic ion in the Kondo effect may be regarded as having "disappeared into the Fermi surface." In that sense, the Kondo effect may be the simplest and most studied example of "transmutation" of quantum numbers or statistics. Furthermore, according to Rokhsar, the quasiparticle excitation about the conventional superconducting state of John Bardeen, Leon Cooper and Schrieffer may be regarded as a spinon.

Commenting on this new, revised view of well-studied phenomena in condensed matter physics, Anderson told us he is sure that some mechanism that "disengages spin, statistics and charge is operating in the high-temperature superconductors." But is the mechanism the one that gives rise to anyons, or one of the more familiar ones? "That we don't know yet," Anderson said.

—Anil Khurana ■