MA2002 - Cálculo Avanzado y Aplicaciones
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Pauta Auxiliar 7
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$\boxed{\text{P1.}}$ Encuentre la representación cartesiana y polar de los siguientes números complejos:

(a) $z = i^{-i}$

(b) $z = i^{\ln(i+1)}$

(c) $z = \sqrt{3} + i^{-i+1}$

Respuesta:

(a) Se tiene que:

$$\begin{aligned} z &= i^{-i} \\ &= \exp(\log(i^{-i})) \\ &= \exp(-i\log(i)) \end{aligned}$$

Recordando que para $z \in \mathbb{C}$:

$$\log(z) = \log|z| + i\arg(z)$$

Con $\arg(z) \in (-\pi, \pi]$. Entonces:

$$\begin{aligned} z &= \exp(-i\log(i)) \\ &= \exp(-i\cancelto{0}{\log|i|} + \arg(i)) \\ &= \exp(\pi/2) \\ &= e^{\pi/2} \qquad \text{(forma Cartesiana)} \\ &= e^{\pi/2}e^{i0} \qquad \text{(forma Polar)} \end{aligned}$$

(b) Para este caso:

$$\begin{aligned} z &= i^{\log(i+1)} \\ &= \exp(\log(i+1)\log(i)) \\ &= \exp(\log|i+1| + i\arg(i+1))(\cancelto{0}{\log|i|} + i\arg(i)) \\ &= \exp(\log(\sqrt{2}) + i\frac{\pi}{4})(i\frac{\pi}{2})) \\ &= \exp(\frac{\pi}{8} + i\frac{\pi}{4}\log(2)) \\ &= e^{\pi/8}e^{i(\pi/4)\log(2)} \qquad \text{(forma Polar)} \\ &= e^{\pi/8}\cos((\pi/4)\log(2)) + ie^{\pi/8}\sin((\pi/4)\log(2)) \qquad \text{(forma Cartesiana)} \end{aligned}$$

(c) Propuesto.

$\boxed{\text{P2.}}$ Sea $\Omega \subseteq \mathbb{C}$ un abierto conexo y $f : \Omega \to \mathbb{C}$ una función holomorfa. Demuestre que si $\mathbb{R}e(f)$ es constante en Ω , entonces f es constante en Ω.

Respuesta: Digamos, sin pérdida de generalidad, que:

$$f = u + iv$$

Como $\mathbb{R}e(f)$ es constante, u es constante. Luego:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

Como f es holomorfa, se cumplen las condiciones de Cauchy-Riemann y entonces:

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Como estamos en un conexo, v es constante.

Se concluye.

$\boxed{\text{P3.}}$ Definamos los operadores diferenciales $\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial \bar{z}}$ mediante las fórmulas:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

(a) Pruebe que $f = u + iv$ satisface las condiciones de Cauchy-Riemann si y sólo si $\dfrac{\partial}{\partial \bar{z}} = 0$.

(b) Si $f \in H(\Omega)$, muestre que $\forall z \in \Omega, f'(z) = \dfrac{\partial f}{\partial z}(z)$.

(c) Explicite en términos de u y v a qué corresponde la ecuación $\dfrac{\partial^2 f}{\partial z \partial \bar{z}} = 0$.

(d) Dada una función $f = u + iv$ con u y v de clase C^2, se define el laplaciano de f mediante

$$\triangle f = \triangle u + i\triangle v$$

y si $\triangle f = 0$ en Ω entonces se dice que f es armónica en Ω. Deduzca que si $f \in H(\Omega)$ entonces f es armónica en Ω. Pruebe que $f \in H(\Omega)$ si y sólo si $f(z)$ y $zf(z)$ son armónicas en Ω.

Respuesta:

(a) Probaremos la doble implicancia:

- ($\Longrightarrow$) Tenemos que:

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)\end{aligned}$$

  Como f satsiface C-R:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

  Luego, parte real y parte imaginaria de $\dfrac{\partial f}{\partial \bar{z}}$ son nulas. Equivalentemente: $\dfrac{\partial f}{\partial \bar{z}} = 0$.

- ($\Longleftarrow$) Recíprocamente, si:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0$$

  Entonces la parte real y la parte imaginaria de esta expresión son nulas.
  Se obtienen directamente las condiciones de C-R.

(b) Sea $f \in H(\Omega)$ y sea $z_0 = x_0 + iy_0 \in \Omega$, cualquiera.

Por C.R, sabemos que:

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \end{aligned}$$

Luego:

$$2f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$$

Y equivalentemente:

$$\begin{aligned} f'(z_0) &= \frac{1}{2}\left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x}(x_0, y_0) - i\frac{\partial f}{\partial y}(x_0, y_0)\right) \\ &= \frac{\partial f}{\partial z}(z_0) \end{aligned}$$

(c) Calculemos:

$$\begin{aligned} \frac{\partial^2 f}{\partial z \partial \bar{z}} &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right) \\ &= \frac{\partial}{\partial z}\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\right) \\ &= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) \\ &= \frac{1}{4}\left[\left(\frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial y \partial x}\right) + i\left(\frac{\partial^2 v}{\partial x^2} - i\frac{\partial^2 v}{\partial y \partial x}\right) + i\left(\frac{\partial^2 u}{\partial x \partial y} - i\frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 v}{\partial y^2}\right)\right] \\ &= \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial y \partial x} + i\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + i\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} + i\frac{\partial^2 v}{\partial y^2}\right] \end{aligned}$$

Luego, imponer:

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$$

Es equivalente a imponer simultáneamente:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

Y:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial y \partial x} = 0$$

(d) Sea $f \in H(\Omega)$. Luego, f cumple C.R en todo punto, es decir:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Derivando:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}, \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

Como $u, v \in C^2$:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\ , \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

Y entonces:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}\ , \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$

$$\iff \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\iff \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = i\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0$$

$$\iff \quad \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + i\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0$$

$$\iff \quad \triangle u + i \triangle v = 0$$

$$\iff \quad f \text{ es armónica}$$

Y sólo falta probar la otra equivalencia que pide el enunciado.

- ($\Longrightarrow$) Sea $f \in H(\Omega)$. Luego, $zf(z) \in H(\Omega)$. Por lo recién probado, $f(z), zf(z)$ son armónicas.

- ($\Longleftarrow$) Propuesta al lector.