P1 i) Como f es diferenciable, podemos escribir

$$f(x+h) = f(x) + L(h) + o(\|h\|) \quad , \text{ con } \lim_{h\to 0} \frac{o(\|h\|)}{\|h\|} = 0$$

Aplicando ω, $(\omega \circ f)(x+h) = \omega\big(f(x) + L(h) + o(\|h\|)\big)$
$= (\omega\circ f)(x) + (\omega\circ L)(h) + \omega(o(\|h\|))$

Definimos $\tilde{L}(h) = (\omega\circ L)(h)$. Veamos que es lineal:

$$\tilde{L}(\lambda h_1 + h_2) = (\omega\circ L)(\lambda h_1 + h_2) = \omega(\lambda L(h_1) + L(h_2))$$
$$= \lambda(\omega\circ L)(h_1) + (\omega\circ L)(h_2) = \lambda \tilde{L}(h_1) + \tilde{L}(h_2)$$

Basta ver que $\lim_{h\to 0} \frac{\omega(o(\|h\|))}{\|h\|} = 0$.

$$\lim_{h\to 0} \frac{\omega(o(\|h\|))}{\|h\|} = \lim_{h\to 0} \omega\left(\frac{o(\|h\|)}{\|h\|}\right) = \omega\left(\lim_{h\to 0} \frac{o(\|h\|)}{\|h\|}\right) = \omega(0) = 0$$

pues ω es continua

Así, $\omega\circ f$ es diferenciable y su diferencial es $(\omega\circ L)(\cdot)$.

ii) $f(x) = x^t A x + b^t x + c$

$\rightarrow$ $f(x+h) = (x+h)^t A(x+h) + b^t(x+h) + c$
$= x^t A x + x^t A h + h^t A x + h^t A h + b^t x + b^t h + c$
$= x^t A x + b^t x + c + x^t A h + h^t A x + b^t h + h^t A h$
$= f(x) + x^t A h + [(Ax)^t h]^t + b^t h + h^t A h$
$= f(x) + x^t A h + [x^t A^t h]^t + b^t h + h^t A h$
$= f(x) + x^t A h + x^t A h + b^t h + h^t A h$

Esto pues $A^t = A$ (A es simétrica) y $(x^t A h)^t = x^t A h$ (es un escalar).

$$\rightarrow f(x+h) = f(x) + 2x^t A h + b^t h + h^t A h$$
$$= f(x) + (2x^t A + b^t) h + h^t A h$$
$$= f(x) + \langle (2x^t A + b^t)^t, h \rangle + h^t A h$$
$$= f(x) + \langle 2Ax + b, h \rangle + h^t A h$$

Veamos que $\lim_{h\to 0} \frac{h^t A h}{\|h\|} = 0$: $|h^t A h| = |\langle h, Ah \rangle|$
$$\leq \langle h, h \rangle \langle Ah, Ah \rangle \text{ (Cauchy-Schwarz)}$$
$$= \|h\|_2^2 \, \|Ah\|_2^2$$

$$\Rightarrow 0 \leq \frac{|h^t A h|}{\|h\|_2} \leq \frac{\|h\|_2^2 \, \|Ah\|_2^2}{\|h\|_2} = \|h\|_2 \, \|Ah\|_2^2 \xrightarrow{h\to 0} 0$$

$$\therefore \lim_{h\to 0} \frac{h^t A h}{\|h\|} = 0.$$

Así, f es diferenciable y $\nabla f = 2Ax + b$.

P2 i) Sea $S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$$(x,y,z) \mapsto (f(x,y,z), g_1(x,z), g_2(x,z))$$

Luego, $H = f \circ S : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Se trata de un gradiente:

$$J_H = \nabla H = J_f(S(x,y,z)) \cdot J_S(x,y,z)$$

$$J_f(x,y,z) = \left[\frac{\partial f}{\partial x}(x,y,z) , \frac{\partial f}{\partial y}(x,y,z) , \frac{\partial f}{\partial z}(x,y,z) \right]$$

$$\Rightarrow J_f(S(x,y,z)) = \left[\frac{\partial f}{\partial x}(S(x,y,z)) , \frac{\partial f}{\partial y}(S(x,y,z)) , \frac{\partial f}{\partial z}(S(x,y,z)) \right]$$

$$J_S(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y,z) & \frac{\partial f}{\partial y}(x,y,z) & \frac{\partial f}{\partial z}(x,y,z) \\ \frac{\partial g_1}{\partial x}(x,z) & 0 & \frac{\partial g_1}{\partial x_2}(x,z) \\ \frac{\partial g_2}{\partial x}(x,z) & 0 & \frac{\partial g_2}{\partial x_2}(x,z) \end{bmatrix}$$

$\rightarrow$ Usamos $\frac{\partial g}{\partial x_2}$ pues queremos denotar la derivada c/r a su segunda variable sin dejar lugar a confusión por notación

Luego se calcula el producto expresado anteriormente

ii) Sea $s: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto (x+y,\ g(x,y),\ h(y,x))$$

Luego, $\pi(x,y) = (f \circ s)(x,y)$, $\pi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Se trata de un jacobiano de 2×2: $J_\pi(x,y) = J_f(s(x,y)) \cdot J_s(x,y)$.

- $s(0,1) = (0+1,\ g(0,1),\ h(1,0)) = (1,1,0)$

- $J_f(x,y,z) = \begin{pmatrix} \sin(z) & 0 & x\cos(z) \\ 0 & \cos(z) & -y\sin(z) \end{pmatrix} \Rightarrow J_f(s(0,1)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

- $J_s(x,y) = \begin{pmatrix} 1 & 1 \\ -y^2\sin(x) & 2y\cos(x) \\ -\ln(\cos(y)) & x\cdot\frac{\sin(y)}{\cos(y)} \end{pmatrix} \Rightarrow J_s(0,1) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -\ln(\cos(1)) & 0 \end{pmatrix}$

$$\therefore\ J_\pi(0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -\ln(\cos(1)) & 0 \end{pmatrix} = \begin{pmatrix} -\ln(\cos(1)) & 0 \\ 0 & 2 \end{pmatrix}$$

P3 i) $b\dfrac{\partial u}{\partial y} + cu = f(x,y) \Rightarrow \dfrac{\partial u}{\partial y} + \dfrac{c}{b}u = \dfrac{1}{b}f(x,y) \quad / \cdot e^{\int \frac{c}{b}dy}$

$$\Rightarrow \frac{\partial}{\partial y}\left(u\, e^{\int \frac{c}{b}dy}\right) = \frac{1}{b} e^{\int\frac{c}{b}dy} f(x,y)$$

$$\Rightarrow \frac{\partial}{\partial y}\left(u\, e^{\frac{c}{b}y}\right) = \frac{1}{b} e^{\frac{c}{b}y} f(x,y) \quad / \int^y dz$$

$$\Rightarrow u\, e^{\frac{c}{b}y} = \frac{1}{b}\int^y e^{\frac{c}{b}z} f(x,z)\,dz + K(x)$$

$$\Rightarrow u(x,y) = e^{-\frac{c}{b}y}\left(\frac{1}{b}\int^y e^{\frac{c}{b}z} f(x,z)\,dz + K(x)\right)$$

donde K es una función de clase C^1.

ii) Como $u(x,y) = v(S(x,y))$, con $S: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (bx-ay, y) = (\xi, y)$

$$\nabla u(x,y) = \nabla v(S(x,y))\, J_S(x,y)$$

$$\left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right] = \left[\frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial y}\right]\begin{pmatrix} \frac{\partial S_1}{\partial x} & \frac{\partial S_1}{\partial y} \\ \frac{\partial S_2}{\partial x} & \frac{\partial S_2}{\partial y}\end{pmatrix} = \left[\frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial y}\right]\begin{pmatrix} b & -a \\ 0 & 1\end{pmatrix}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} = b\frac{\partial v}{\partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial y} = -a\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial y}$$

$\Rightarrow$ Reemplazando : $ab\frac{\partial v}{\partial \xi} + b\left(-a\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial y}\right) + cv = f(x,y) = f\left(\frac{\xi + ay}{b}, y\right)$

$\Rightarrow b\frac{\partial v}{\partial y} + cv = f\left(\frac{\xi+ay}{b}, y\right)$ → Misma situasao de la parte i).

iii)

$$\Rightarrow v(\xi, y) = e^{-\frac{c}{b}y}\left(\frac{1}{b}\int^y e^{\frac{c}{b}z} f\left(\frac{\xi+az}{b}, z\right)dz + K(\xi)\right)$$

$$\Rightarrow u(x,y) = e^{-\frac{c}{b}y}\left(\frac{1}{b}\int^y e^{\frac{c}{b}z} f\left(\frac{bx-ay+az}{b}, z\right)dz + K(bx-ay)\right)$$