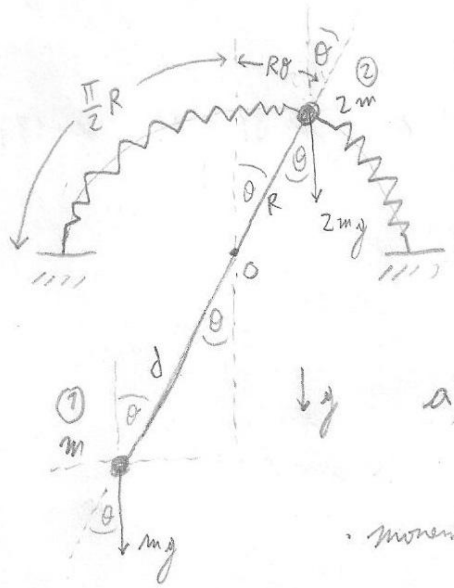
cilíndricas: $(\hat{r},\hat{\theta},\hat{z})$

Punto Aux 13



Resortes: $\left(l_0=R,\ k=\dfrac{\sqrt{2}\,mg}{\pi R^2}(2R-d)\right)$

con $(2R-d>0)$

• Teorema dinámico:

a) Queremos hacer: $\dfrac{d\vec{L}}{dt}=\sum\vec{\tau}$

· momentum: $\vec{L}=\vec{L}_1+\vec{L}_2=dm\,\hat{r}\times d\dot{\theta}\,\hat{\theta}+R2m\,\hat{r}\cdot R\dot{\theta}\,\hat{\theta}$

$$\Rightarrow \vec{L}=m\dot{\theta}(2R^2+d^2)\,\hat{z} \quad (1)$$

· Torque: $\sum\vec{\tau}=\vec{\tau}_1+\vec{\tau}_2$

$$= d\,\hat{r}\times(mg\cos\theta\,\hat{r}-mg\sin\theta\,\hat{\theta})+R\,\hat{r}\times(-mg2\cos\theta\,\hat{r}+mg\sin\theta\,\hat{\theta})$$

$$+R\,\hat{r}\times\left[-k\left(\cancel{\tfrac{\pi}{2}R}+R\theta-\cancel{l_0}\right)+k\left(\cancel{\tfrac{\pi}{2}R}-R\theta-\cancel{l_0}\right)\right]\hat{\theta}$$

$$= -mg\sin\theta\,d\,\hat{z}+2mg\sin\theta\,R\,\hat{z}-2R^2\theta k\,\hat{z}=\sum\vec{\tau} \quad (2)$$

(1) y (2) y el valor de k en: $\dfrac{d\vec{L}}{dt}=\sum\vec{\tau} \Rightarrow \cancel{m}\ddot{\theta}(2R^2+d^2)=\cancel{m}g\sin\theta(2R-d)-\theta\,\dfrac{2\sqrt{2}\cancel{R^2}\cancel{m}g}{\pi\cancel{R^2}}(2R-d)$

$$\Rightarrow \ddot{\theta}=\frac{2R-d}{2R^2+d^2}g\left(\sin\theta-\frac{2\sqrt{2}}{\pi}\theta\right) \Rightarrow \ddot{\theta}=\Lambda\left(\sin\theta-\frac{2\sqrt{2}}{\pi}\theta\right) \quad (3)$$

con $\Lambda=\dfrac{2R-d}{2R^2+d^2}$

· equilibrios: $\dot{\theta}=\ddot{\theta}=0$

(3) $\Rightarrow 0=\Lambda\left(\sin\theta_{eq}-\frac{2\sqrt{2}}{\pi}\theta_{eq}\right) \Rightarrow \sin\theta_{eq}=\frac{2\sqrt{2}}{\pi}\theta_{eq}$ $\left(\sin\frac{\pi}{4}=\sqrt{2}\,\frac{2\cancel{\pi}}{4\cancel{\pi}}\right)$

$$\Rightarrow \boxed{\theta_1=0 \quad y \quad \theta_2=\frac{\pi}{4} \quad \text{son equilibrios}} \quad (4)$$

¿Estables o inestables? → hagamos pequeñas oscilaciones en torno a los equilibrios:

• $\theta_1 = 0 \Rightarrow \theta = \underset{0}{\theta_1} + \delta\theta \Rightarrow \dot\theta = \delta\dot\theta,\ \ddot\theta = \delta\ddot\theta,\ \sin(\theta_1+\delta\theta) = \sin\delta\theta = \delta\theta$ en (3)

$$\Rightarrow \quad \delta\ddot\theta = \Lambda\,\delta\theta - \frac{2\sqrt2}{\pi}\Lambda\,\delta\theta \Rightarrow \delta\ddot\theta - \underbrace{\Lambda\left(1-\frac{2\sqrt2}{\pi}\right)}_{\omega_1^2}\delta\theta = 0$$

• con $\Lambda > 0$ y $1 > \frac{2\sqrt2}{\pi}$

$\Rightarrow$ se tiene la forma: $\delta\ddot\theta - \omega_1^2\,\delta\theta = 0 \Rightarrow \boxed{¡\,\theta_1 = 0 \text{ es inestable!}}$

• $\theta_2 = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4} + \delta\theta \Rightarrow \dot\theta = \delta\dot\theta,\ \ddot\theta = \delta\ddot\theta,\ \sin\left(\frac{\pi}{4}+\delta\theta\right) = \underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt2}{2}}\underbrace{\cos(\delta\theta)}_{1} + \underbrace{\sin(\delta\theta)}_{\delta\theta}\underbrace{\cos\frac{\pi}{4}}_{\frac{\sqrt2}{2}}$

$$\Rightarrow \quad \delta\ddot\theta = \Lambda\frac{\sqrt2}{2}(1+\delta\theta) - \frac{2\sqrt2}{\pi}\Lambda\left(\frac{\pi}{4}+\delta\theta\right) = \Lambda\left(\cancel{\frac{\sqrt2}{2}} - \cancel{\frac{\sqrt2}{2}} + \frac{\sqrt2}{2}\delta\theta - \frac{2\sqrt2}{\pi}\delta\theta\right)$$

$$\Rightarrow \quad \delta\ddot\theta + \underbrace{\Lambda\sqrt2\left(\frac{2}{\pi}-\frac{1}{2}\right)}_{\omega_2^2}\delta\theta = 0$$

con $\Lambda > 0$ y $\frac{2}{\pi} > \frac{1}{2}$

$\Rightarrow$ se tiene la forma: $\delta\ddot\theta + \omega^2\,\delta\theta = 0$

$\Rightarrow \boxed{¡\,\theta_2 = \frac{\pi}{4} \text{ es estable!}}$

b) la frecuencia de pequeñas oscilaciones en el caso estable ($\theta_2 = \frac{\pi}{4}$) es $\boxed{\omega = \sqrt{\Lambda\sqrt2\left(\frac{2}{\pi}-\frac{1}{2}\right)}}$

• Camino potencial: $E = K + U_g + U_e \qquad U_g = 2mR\cos\theta\, g - m\, d\cos\theta\, g$

$\Rightarrow \quad E = K + U \quad$ con $l_0 = R \qquad U_e = \frac{k}{2}\left(l_0 - R\theta - \frac{\pi}{2}R\right)^2 + \frac{k}{2}\left(\frac{\pi}{2}R - R\theta - l_0\right)^2$

$$\Rightarrow \quad U(\theta) = mg\cos\theta\,(2R-d) + \frac{kR^2}{2}\left[\left(1-\theta-\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}-\theta-1\right)^2\right]$$

Equilibrio: $\frac{\partial U(\theta_{eq})}{\partial\theta} = 0 \Rightarrow \frac{\partial U}{\partial\theta} = -mg\sin\theta\,(2R-d) - kR^2\left[1-\theta-\cancel{\frac{\pi}{2}}+\cancel{\frac{\pi}{2}}-\theta-1\right]$

↓

$$\Rightarrow \quad \frac{\partial U}{\partial \theta} = -mg\sin\theta(2R-d) + 2R^2\theta\,\frac{\sqrt{2}\,mg}{\pi R^2}(2R-d) = mg(2R-d)\left(\frac{2\sqrt{2}}{\pi}\theta - \sin\theta\right)$$

$$\frac{\partial U(\theta_{eq})}{\partial \theta} = 0 \Rightarrow mg(2R-d)\left(\frac{2\sqrt{2}}{\pi}\theta_{eq} - \sin\theta_{eq}\right) = 0 \Rightarrow \theta_1 = 0\ ,\ \theta_2 = \frac{\pi}{4}$$

¿estabilidad? $\quad \dfrac{\partial^2 U}{\partial \theta^2} = mg(2R-d)\left(\dfrac{2\sqrt{2}}{\pi} - \cos\theta\right)$

$$\frac{\partial^2 U}{\partial \theta^2}(\theta_1 = 0) = \underbrace{mg(2R-d)}_{>0}\underbrace{\left(\frac{2\sqrt{2}}{\pi} - 1\right)}_{<0} < 0 \Rightarrow$$

máximo del potencial

$\Rightarrow \boxed{\theta_1 \text{ es inestable}}$

$$\frac{\partial^2 U}{\partial \theta^2}\left(\theta_2 = \frac{\pi}{4}\right) = \underbrace{mg(2R-d)}_{>0}\underbrace{\left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{2}}{2}\right)}_{>0} > 0 \Rightarrow$$

mínimo del potencial.

$\Rightarrow \boxed{\theta_2 \text{ es estable}}$

¿y la frecuencia de p. oscilaciones?

$$\delta\ddot\theta + \omega^2\delta\theta = 0 \quad /\delta\dot\theta$$

$$\Rightarrow \delta\ddot\theta\,\delta\dot\theta + \omega^2\delta\theta\,\delta\dot\theta = 0$$

$$\Rightarrow \frac{d}{dt}\underbrace{\left(\frac{\delta\dot\theta^2}{2} + \omega^2\frac{\delta\theta^2}{2}\right)}_{\mathcal{E}(\delta\theta)} = 0$$

$\Rightarrow$ hay que reescribir la energía en $\theta = \frac{\pi}{4} + \delta\theta$ a segundo orden.

$$E = \frac{\delta\dot\theta^2}{2}m(2R^2+d^2) + mg(2R-d)\left[\cos\left(\frac{\pi}{4}+\delta\theta\right) + \frac{\sqrt{2}}{2\pi}\left(\Big(\overbrace{1-\frac{\pi}{4}-\frac{\pi}{2}}^{1-\frac{3\pi}{4}}-\delta\theta\Big)^2 + \Big(\overbrace{\frac{\pi}{2}-\frac{\pi}{4}-\delta\theta-1}^{(\frac{\pi}{4}-1)-\delta\theta}\Big)^2\right)\right]$$

$$\mathcal{E} = \frac{E}{m(2R^2+d^2)} = \frac{\delta\dot\theta^2}{2} + \underbrace{g\frac{2R-d}{2R^2+d^2}}_{\Lambda}\left[\frac{\sqrt{2}}{2}\underbrace{\cos\delta\theta}_{1-\frac{\delta\theta^2}{2}} - \frac{\sqrt{2}}{2}\underbrace{\sin\delta\theta}_{\delta\theta} + \frac{\sqrt{2}}{2\pi}\left(\left(1-\frac{3\pi}{4}\right)^2 + \delta\theta^2 - 2\left(1-\frac{3\pi}{4}\right)\delta\theta + \delta\theta^2 + \left(\frac{\pi}{4}-1\right)^2 - 2\left(\frac{\pi}{4}-1\right)\delta\theta\right)\right]$$

$$= \frac{\delta\dot\theta^2}{2} + \Lambda\left[\cancel{\frac{\sqrt{2}}{2}} - \frac{\sqrt{2}}{2}\frac{\delta\theta^2}{2} - \cancel{\frac{\sqrt{2}}{2}\delta\theta} + \frac{\sqrt{2}}{\pi}\delta\theta^2 - \frac{\sqrt{2}}{\pi}\delta\theta\left(\cancel{1}-\cancel{1}+\frac{\pi}{4}-\frac{3\pi}{4}\right) + \cancel{\frac{\sqrt{2}}{2\pi}\left[\left(1-\frac{3\pi}{4}\right)^2 - \left(\frac{\pi}{4}-1\right)^2\right]}\right]$$

$$\mathcal{E} = \frac{\delta\dot\theta^2}{2} + \Lambda\sqrt{2}\left(\frac{2}{\pi} - \frac{1}{2}\right)\frac{\delta\theta^2}{2} \Rightarrow \boxed{\omega^2 = \Lambda\sqrt{2}\left(\frac{2}{\pi} - \frac{1}{2}\right)}$$