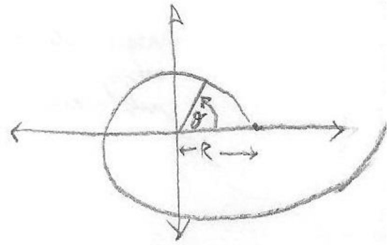
Aux 3



P1 Espiral Exponencial:

$$(1)\quad x(\theta) = R e^{k\theta}\cos(\theta)$$
$$y(\theta) = R e^{k\theta}\sin(\theta)$$

ojo: R es un parámetro constante!

$\theta \in [0, 2\pi]$ con $|\vec{v}| = v_0$

$k > 0$

a) Polares: como x e y ya están en términos de θ, sólo nos falta deducir el valor de la coordenada r.

$$\left(\begin{array}{ccc} \text{cartesianas a polares} & & \\ x & \longrightarrow & r = \sqrt{x^2+y^2} \\ y & & \tan\theta = \frac{y}{x} \end{array}\right) \Rightarrow r = \sqrt{x^2(\theta) + y^2(\theta)}$$

$$\overset{(1)}{=} \sqrt{R^2 e^{2k\theta}\cos^2\theta + R^2 e^{2k\theta}\sin^2\theta}$$

$$= \underbrace{|R e^{k\theta}|}_{\text{positivo}} \sqrt{\cancel{\cos^2\theta + \sin^2\theta}_{1}} = R e^{k\theta}$$

- Luego sabemos que en polares:

$\vec{r} = r\hat{r}$ entonces: $\Rightarrow \boxed{\vec{r}(\theta) = R e^{k\theta}\,\hat{r}}\quad (2)$

b) velocidad en polares: $\vec{v} = r\dot{\theta}\hat{\theta} + \dot{r}\hat{r}$

derivando en t

$$(2) \Rightarrow r = R e^{k\theta} \Rightarrow \dot{r} = R e^{k\theta} k\dot{\theta} = kr\dot{\theta} \Rightarrow \dot{r} = kr\dot{\theta}\quad (3)$$

>0

- La rapidez es constante! $|\vec{v}|^2 = r^2\dot{\theta}^2 + \dot{r}^2 = v_0^2$

- usando (3) $\Rightarrow \frac{\dot{r}^2}{k^2} + \dot{r}^2 = v_0^2 \Rightarrow \underbrace{\dot{r} = \frac{k v_0}{\sqrt{1+k^2}} \Rightarrow r\dot{\theta} = \frac{v_0}{\sqrt{1+k^2}}}_{(4)}$

$$\dot{r}^2\left(\frac{1+k^2}{k^2}\right) = v_0^2$$

- Luego usando (4) obtenemos la velocidad en polares: $\Rightarrow \boxed{\vec{v} = \frac{v_0}{\sqrt{1+k^2}}\left(\hat{\theta} + k\hat{r}\right)}$ (5)

ojo: (5) cumple con el enunciado! $|\vec{v}| = \frac{v_0}{\sqrt{1+k^2}}\sqrt{1+k^2} = v_0$ ✓

c) aceleración en polares: $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$$(4) \Rightarrow \quad \dot{r} = \frac{k v_0}{\sqrt{1+k^2}} = cte \quad \Rightarrow \quad \ddot{r} = 0 \quad (6)$$

$$(4) \Rightarrow \quad \dot{\theta} = \frac{1}{r}\frac{v_0}{\sqrt{1+k^2}} \quad \Rightarrow \quad \ddot{\theta} = \frac{v_0}{\sqrt{1+k^2}}\left(\frac{-\dot{r}}{r^2}\right) \overset{(4)}{=} \frac{-k v_0^2}{(1+k^2) r^2} \quad (7)$$

- usando (4), (6) y (7) obtenemos la aceleración:

$$\Rightarrow \vec{a} = \left(0 - r\frac{1}{r^2}\frac{v_0^2}{1+k^2}\right)\hat{r} + \left(\frac{2k v_0^2}{r(1+k^2)} - \frac{k v_0^2}{(1+k^2)}\frac{r}{r^2}\right)\hat{\theta}$$

$$\Rightarrow \boxed{\vec{a} = \frac{v_0}{r(1+k^2)}\left(-\hat{r} + k\hat{\theta}\right)} \quad (8)$$

d) calculamos producto punto: $\vec{v}\cdot\vec{a} \overset{(5),(8)}{=} \frac{v_0^2}{r(1+k^2)^{3/2}}\left(\underbrace{k - k}_{0}\right) = 0$

$\boxed{\vec{v}\cdot\vec{a} = 0 \iff \vec{v} \perp \vec{a}}$ ($\vec{v}$ y $\vec{a}$ son siempre ortogonales en este problema)

e) - De (4) podemos calcular $r = r(t)$: $\quad \dot{r} = \dfrac{dr}{dt} = \dfrac{k v_0}{\sqrt{1+k^2}}$

$$\Rightarrow \int_{r(t=0)=R}^{r} dr = \int_0^t \frac{k v_0}{\sqrt{1+k^2}}\, dt \Rightarrow \boxed{r(t) = \frac{k v_0}{\sqrt{1+k^2}}\, t + R} \quad (9)$$

- De (4) ahora tenemos la velocidad angular: $\quad r\dot{\theta} = \dfrac{v_0}{\sqrt{1+k^2}}$

$$\Rightarrow \dot{\theta} = \frac{\dfrac{v_0}{\sqrt{1+k_2}}}{\dfrac{k v_0}{\sqrt{1+k^2}}\, t + R} \Rightarrow \boxed{\dot{\theta}(t) = \frac{1}{k}\, \frac{1}{t + \dfrac{R\sqrt{1+k^2}}{k v_0}}} \quad (10)$$

velocidad angular

- si integramos (10) obtenemos el ángulo θ en función del tiempo:

$$(10) \Rightarrow \dot{\theta} = \frac{d\theta}{dt} \Rightarrow \int_{\theta=0}^{\theta} d\theta = \frac{1}{k}\int_0^t \frac{dt}{t + \dfrac{R\sqrt{1+k^2}}{k v_0}} = \frac{1}{k} \ln\left(t + \frac{R\sqrt{1+k^2}}{k v_0}\right)\Bigg|_0^t$$

- ajustando los logaritmos:

$$\Rightarrow \boxed{\theta(t) = \frac{1}{k} \ln\left(\frac{v_0 k}{R\sqrt{1+k^2}}\, t + 1\right)}$$

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P2 *Partícula en cono:*

a) - semiángulo $\frac{\pi}{4}$ $\Rightarrow$ $\theta = \frac{\pi}{4}$ (constante!) $\Rightarrow$ $\dot{\theta} = \ddot{\theta} = 0$ $\Big\}$ (1)

- rapidez radial igual a c $\Rightarrow$ $\underset{\text{(constante!)}}{\dot{r} = c}$ $\Rightarrow$ $\ddot{r} = 0$

- Posición inicial igual a cero $\Rightarrow$ $\dot{r} = c \Rightarrow \int_{r_0}^{r} dr = \int_0^t c\,dt$

$$\Rightarrow r(t) - \underset{0}{\cancel{r(t=0)}} = ct \qquad \Rightarrow r(t) = ct \quad (2)$$

- Trayectoria descrita por: $\phi = 2\pi \frac{r}{R} \Rightarrow \underbrace{\dot{\phi} = 2\pi \frac{\dot{r}}{R} \overset{(2)}{=} \frac{2\pi c}{R} \Rightarrow \ddot{\phi} = 0}_{(3)}$

- velocidad en esféricas: $\vec{v} = \underset{c}{\dot{r}}\hat{r} + \underset{0}{r\overset{(1)}{\dot{\theta}}}\hat{\theta} + \underset{ct}{r}\overset{(2)}{\dot{\phi}} \sin\theta\, \hat{\phi}$

$$\Rightarrow \vec{v} \overset{(3)}{=} c\hat{r} + 0 + ct\,\frac{2\pi c}{R}\,\underset{\sqrt{2}/2}{\sin\frac{\pi}{4}}\,\hat{\phi} \Rightarrow \boxed{\vec{v} = c\hat{r} + \frac{\pi c^2\sqrt{2}}{R}\,t\,\hat{\phi}} \quad (4)$$

- aceleración en esféricas: $\vec{a} =$ (CORCHO)

*pero usando (1), (2) y (3) nos queda:

$$\vec{a} = \left[0 - 0 - (ct)\left(\frac{2\pi c}{R}\right)^2\left(\sin^2\frac{\pi}{4}\right)\right]\hat{r} + \left[0 + 0 - (ct)\left(\frac{2\pi c}{R}\right)^2\left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right)\right]\hat{\theta}$$

$$+ \left[2(c)\left(\frac{2\pi c}{R}\right)\left(\sin\frac{\pi}{4}\right) + 0 + 0\right]\hat{\phi}$$

$$\Rightarrow \boxed{\vec{a} = \frac{\pi c^2\sqrt{2}}{R}\left(-\frac{\pi c\sqrt{2}}{R}\,t\,\hat{r} - \frac{\pi c\sqrt{2}}{R}\,t\,\hat{\theta} + 2\,\hat{\phi}\right)} \quad (5)$$

b) - rapidez : $|\vec{v}| = v \overset{(4)}{=} \sqrt{c^2 + \left(\frac{\pi c^2 \sqrt{2}}{R} t\right)^2} \Rightarrow \boxed{v = c\sqrt{1 + \left(\frac{\pi c\sqrt{2}}{R} t\right)^2}}$

(6)

- con (6) podemos calcular el tiempo t^* en que la rapidez es $v = 3c$.

$$\Rightarrow v(t = t^*) = 3c = c\sqrt{1 + \left(\frac{\pi c\sqrt{2}}{R} t^*\right)^2}$$

$\Rightarrow \sqrt{8} = \frac{\pi c\sqrt{2}}{R} t^* \Rightarrow t^* = \frac{2R}{c\pi}$ (7) y dado que (2): $r = ct$

$$\Rightarrow r(t = t^*) = \cancel{c}\frac{2R}{\cancel{c}\pi} \Rightarrow \boxed{r(v = 3c) = \frac{2R}{\pi}}$$

c) - camino recorrido : $s(t) = \int_{t_0}^{t} \|\dot{\vec{r}}(t')\| \, dt'$

- nos interesa $t_0 = 0$, $t = t^*$ (7) y tenemos : $\|\dot{\vec{r}}(t)\| = v$ (6)

$$\Rightarrow \boxed{s\left(r = \frac{2R}{\pi}\right) = c \int_0^{\frac{2R}{c\pi}} \sqrt{1 + \left(\frac{\pi c\sqrt{2}}{R} t'\right)^2} \, dt'}$$

d) - El radio de curvatura está dado por : $\rho = \frac{v^3}{\|\vec{v} \times \vec{a}\|}$

- Para calcularlo usamos (4) y (5) evaluados en $t^* = \frac{2R}{c\pi}$ (tiempo de la parte b)

$$\Rightarrow \vec{v}(t^*) = c\hat{r} + \frac{\pi c^2\sqrt{2}}{\cancel{R}} \frac{2\cancel{R}}{c\cancel{\pi}} \hat{\phi} \Rightarrow \vec{v}(t^*) = c(\hat{r} + 2\sqrt{2}\hat{\phi}) \quad (8)$$

$$\Rightarrow \vec{a}(t^*) = \frac{\pi c^2\sqrt{2}}{R}\left(-\frac{\cancel{\pi}\cancel{c}\sqrt{2}}{\cancel{R}} \frac{2\cancel{R}}{\cancel{c}\cancel{\pi}}\hat{r} - \frac{\cancel{\pi}\cancel{c}\sqrt{2}}{\cancel{R}} \frac{2\cancel{R}}{\cancel{c}\cancel{\pi}}\hat{\theta} + 2\hat{\phi}\right)$$

↓

$$\Rightarrow \quad \vec{a}(t^*) = 2\frac{\pi c^2\sqrt{2}}{R}\left(-\sqrt{2}\,\hat{r} - \sqrt{2}\,\hat{\theta} + \hat{\phi}\right) \quad (9)$$

- calculamos el producto cruz:

$$\vec{v} \times \vec{a} \overset{(8),(9)}{=} 2\,\frac{\pi c^3\sqrt{2}}{R}\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 1 & 0 & 2\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 1 \end{vmatrix}$$

$$\Rightarrow \quad \vec{v} \times \vec{a} = 2\,\frac{\pi c^3\sqrt{2}}{R}\left[\underbrace{\begin{vmatrix} 0 & 2\sqrt{2} \\ -\sqrt{2} & 1 \end{vmatrix}}_{4}\hat{r} + \underbrace{\begin{vmatrix} 2\sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{vmatrix}}_{-5}\hat{\theta} + \underbrace{\begin{vmatrix} 1 & 0 \\ -\sqrt{2} & -\sqrt{2} \end{vmatrix}}_{-\sqrt{2}}\hat{\phi}\right]$$

$$\Rightarrow \quad \vec{v} \times \vec{a} = \frac{2\pi c^3\sqrt{2}}{R}\left(4\hat{r} - 5\hat{\theta} - \sqrt{2}\,\hat{\phi}\right)$$

- calculamos la norma:

$$\|\vec{v} \times \vec{a}\| = \frac{2\pi c^3\sqrt{2}}{R}\sqrt{16 + 25 + 2} = \frac{2\pi c^3}{R}\sqrt{86}$$

- De (6) tenemos $v^3 \Rightarrow \quad v(t^*) = 3c \;\Rightarrow\; v^3 = 27c^3$

$$\Rightarrow \quad \rho = \frac{v^3}{\|\vec{v} \times \vec{a}\|} = \frac{27\cancel{c^3}R}{2\pi\cancel{c^3}\sqrt{86}} \quad \Rightarrow \quad \boxed{\rho(t^*) = \frac{27R}{2\pi\sqrt{86}}}$$