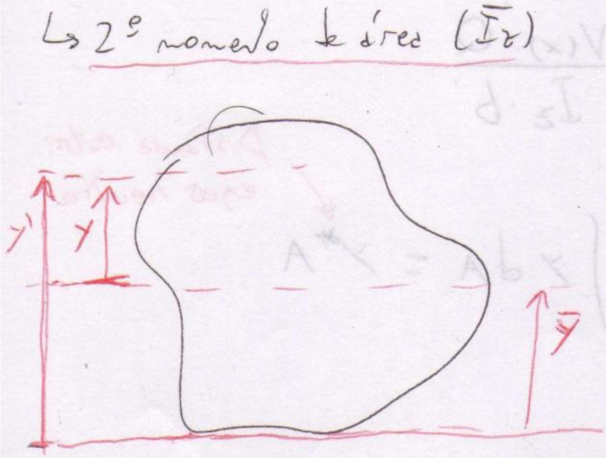
* Cálculo de propiedades de Área (I_z, Q)

↳ 2º momento de área (I_z)



- $\bar{y}$ (Eje neutro) $= \dfrac{\int y' \, dA}{A}$

- $I_z = \int y^2 \, dA = \int (y' - \bar{y}) \, dA$

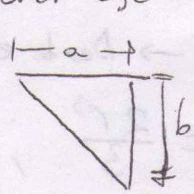
- Si hay varias fig. geométricas

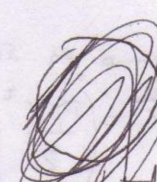
$$\bar{Y}_T = \frac{\sum_i \bar{Y}_i \cdot A_i}{\sum A_i}$$

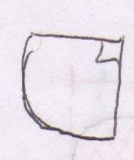
- Traslado de eje de rotación (Teorema de los ejes paralelos)

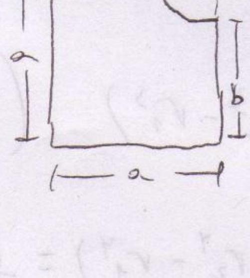
$$\bar{I}_z = I_z + \delta^2 A$$

* Encontrar eje neutro y I_z de:

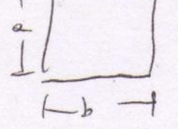




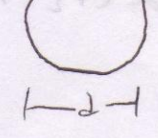




* Secciones conocidas:



$I_z = \dfrac{b a^3}{12}$



$I_z = \dfrac{\pi d^4}{64}$

- Cálculo de deflexión por integración:

$$\frac{d^4 \hat{y}}{dx^4} = -\frac{w(x)}{EI}$$ ← Fza distribuida

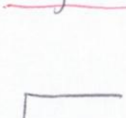
* Modelación de fzas mediante función escalón ($r(x-x_0)$) para fzas distribuidas, y delta de Dirac ($\delta(x-x_0)$) para fzas. puntuales

* Lo que ocurre en los extremos de la viga se reemplaza por condiciones de borde (son necesarias 4 cond. de borde)

* Las cond. de borde son condiciones en la deflexión en los extremos, el ángulo de deflexión ($\frac{d\hat{y}}{dx}$), el momento ($\frac{d^2\hat{y}}{dx^2}$) ó la fza. de corte ($\frac{d^3\hat{y}}{dx^3}$)

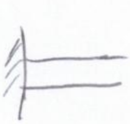
$$\hookrightarrow \boxed{M = EI\frac{d^2\hat{y}}{dx^2}} \wedge \boxed{V = -EI\frac{d^3\hat{y}}{dx^3}}$$

* Algunos bordes:



$\Rightarrow$ No hay deflexión $\rightarrow \hat{y}(0) = 0$

No hay momento $\rightarrow \frac{d^2\hat{y}}{dx^2} = 0$



$\Rightarrow$ No hay deflexión $\rightarrow \hat{y}(0) = 0$

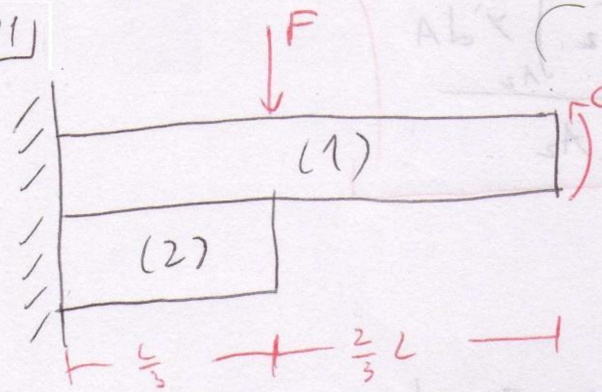
No hay ángulo $\rightarrow \frac{d\hat{y}}{dx}(0) = 0$

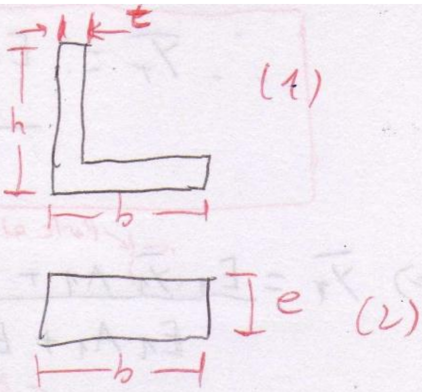
$\Rightarrow$ No hay momento $\rightarrow \frac{d^2\hat{y}}{dx^2}(0) = 0$

Balance de fzas $\rightarrow V(0) = -F = -k\hat{y}(0)$

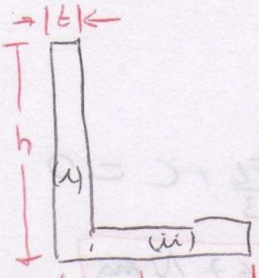
$$\therefore -EI\frac{d^3\hat{y}}{dx^3}(0) = -k\hat{y}(0)$$

P1)





a) Viga (1)



$$\bar{y}_1 = \frac{\sum_i y_i \cdot A_i}{\sum_i A_i} = \frac{\bar{y}_{(i)} A_{(i)} + \bar{y}_{(ii)} \cdot A_{(ii)}}{A_{(i)} + A_{(ii)}}$$

$$= \frac{\frac{h}{2} \cdot h \cdot t + \frac{t}{2} \cdot (t(b-t))}{ht + t(b-t)}$$

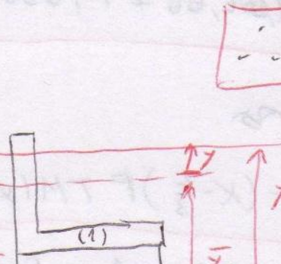
$$\Rightarrow \boxed{\bar{y}_1 = 3{,}038\ cm}$$

$$\rightarrow \bar{I}_{(i)} = I_{(i)} + \delta^2 A_{(i)} = \frac{th^3}{12} + \left(\bar{y} - \frac{h}{2}\right) \cdot t \cdot h = 249{,}249\ cm^4$$

$$I_{(ii)} = I_{(ii)} + \delta^2 A_{(ii)} = \frac{(b-t) \cdot t^3}{12} + \left(\bar{y} - \frac{t}{2}\right)^2 (b-t) \cdot t = 91{,}38\ cm^4$$

$$\boxed{\therefore I_1 = 3.406 \cdot 10^{-6}\ m^4}$$

b)



- Para calcular el eje neutro de una viga compuesta se usa:

$$E_1 \int_{A_1} y\, dA + E_2 \int_{A_2} y\, dA = 0$$

- Haciendo $y = y' - \bar{y}_T$

$$\Rightarrow E_1 \int_{A_1} (y' - \bar{y}_T)\, dA + E_2 \int_{A_2} (y' - \bar{y}_T)\, dA = 0$$

$$\Rightarrow \left[E_1 \int_{A_1} y'\, dA + E_2 \int_{A_1} y'\, dA\right] + \left[E_2 \bar{y}_T A_1 + E_2 \bar{y}_T A_2\right] = 0$$

$$\boxed{\therefore \bar{Y}_T = \frac{E_1\int_{A_1} y'\,dA + E_2 \int_{A_2} y'\,dA}{E_1 \cdot A_1 + E_2 A_2}}$$

Parte el

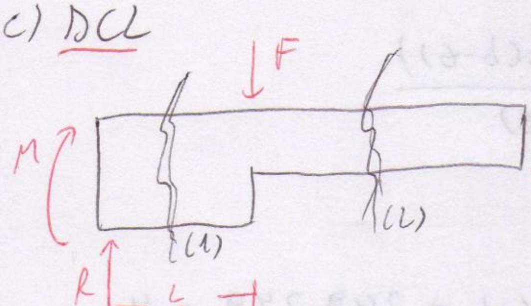
$$\Rightarrow \bar{Y}_T = \frac{E_1 \bar{y}_1 A_1 + E_2 \bar{y}_2 A_2}{E_1 A_1 + E_2 A_2} \quad \bar{y}_2 \to \frac{e}{2}$$

Des de la base!

$$= \frac{E_1(\bar{Y}_1 + e)(ht + t(b-t)) + E_2 \cdot \frac{e}{2} \cdot eb}{E_1(ht + t(b-t)) + E_2\, eb}$$

$$\boxed{\therefore \bar{Y}_T = 4,6967 \text{ cm}}$$

c) DCL



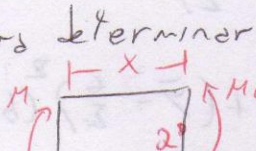
· $\sum M_z = 0 \rightsquigarrow M - \frac{FL}{3} + C = 0$

$\boxed{\therefore M = 166,667 \text{ N·m}}$

· $\sum F_y = 0 \rightsquigarrow \boxed{R = F = 1000 \text{ N}}$

· Hacemos cortes para determinar la distribución de momento interno

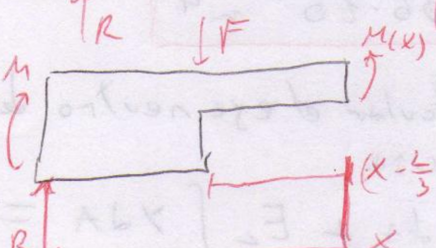
(1) $0 < x < \frac{L}{3} \Rightarrow$



· $\sum M_z = 0 \rightsquigarrow -M - Rx + M(x) = 0$

$\boxed{\therefore M(x) = 166,667 + 1000x}$

(2) $\frac{L}{3} < x < L \Rightarrow$



· $\sum M_z = 0 \rightsquigarrow$

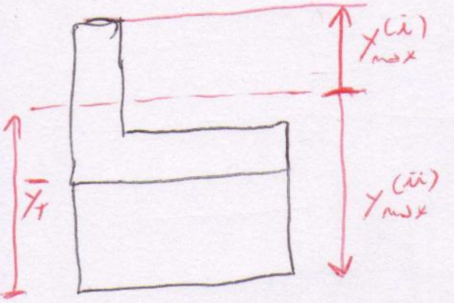
$\Rightarrow M - Rx + (x - \frac{L}{3})F + M(x) = 0$

$\boxed{\therefore M(x) = \frac{FL}{3} + M = 500 \text{ N·m}}$

· El momento máximo se da para cada intervalo (1): $M_{max} = M(\frac{L}{2}) = 166,667 + 1000 \cdot \frac{L}{3}$

$\boxed{M_{max} = 500 \text{ N·m}}$

• Para determinar Y_{max}, se hace para cada material



$$\Rightarrow Y_{max}^{(i)} = h + e - \overline{Y}_T = 0{,}143 \text{ cm (superior)}$$

$$Y_{max}^{(ii)} = -\overline{Y}_T = 4{,}6967 \text{ cm (interior)}$$

• Usando las fórmulas para Esf. máximo en vigas compuestas:

$$\boxed{\sigma_{(i),max} = \frac{-E_1}{E_1 I_1 + E_2 I_2} \cdot M_{max} \cdot Y_{max}^{(i)} = -4{,}152 [MPa] \text{ (Compresión)}}$$

$$\boxed{\sigma_{(ii),max} = \frac{-E_2}{E_1 I_1 + E_2 I_2} M_{max} \cdot Y_{max}^{(ii)} = 1{,}507 [MPa] \text{ (Tracción)}}$$

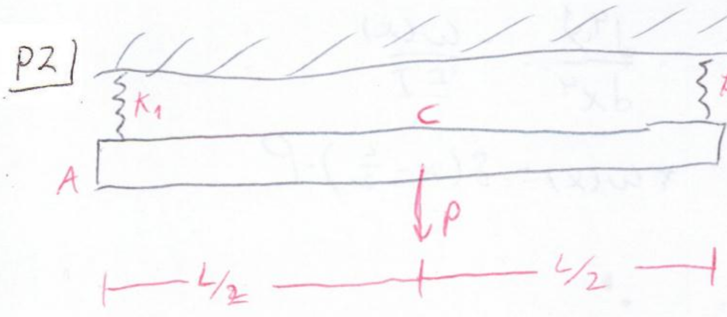
• (2): $M_{max} = 500 \text{ Nm (cte)}$

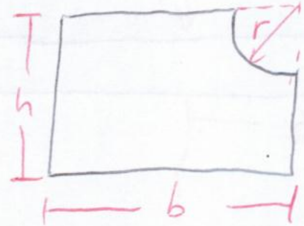
$$Y_{max} = 12 - \overline{Y}_1 = 8{,}962 [cm] \text{ (Parte superior)}$$

$$\Rightarrow \boxed{\sigma_{max} = -\frac{M_{max} \cdot Y_{max}}{I_1} = 13{,}454 [MPa]}$$

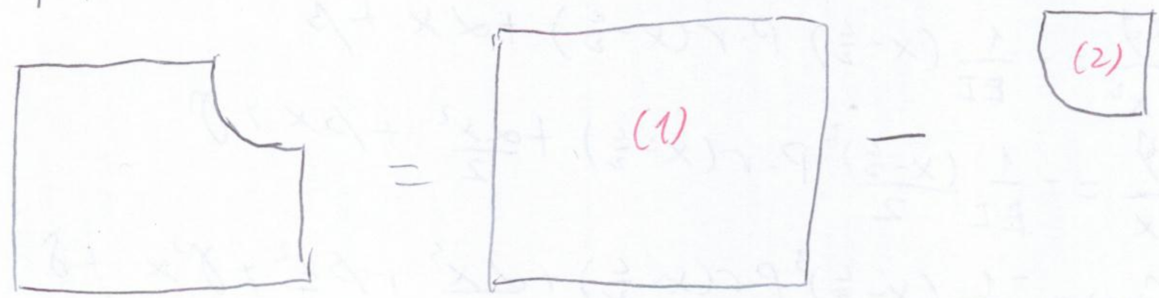
2014

P2)





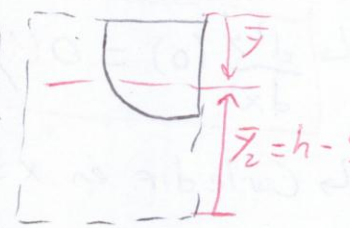
a) Propiedades de área:



(1): $\bar{y}_1 = \dfrac{h}{2} \;\wedge\; I_1 = \dfrac{b \cdot h^3}{12}$

(2):

$\cancel{\bar{y}_2 = \frac{4r}{3\pi}} \;\wedge\; I_2 = \dfrac{\pi r^4}{16}$



$$\Rightarrow \bar{y}_T = \frac{\bar{y}_1 A_1 - \bar{y}_2 A_2}{A_1 - A_2} = \frac{\frac{h}{2} \cdot hb - \overbrace{\left(h - \frac{4r}{3\pi}\right)}^{\text{Desde la base}} \cdot \frac{1}{4}\pi r^2}{hb - \frac{1}{4}\pi r^2} \cancel{= 4{,}567 \text{ cm}}$$

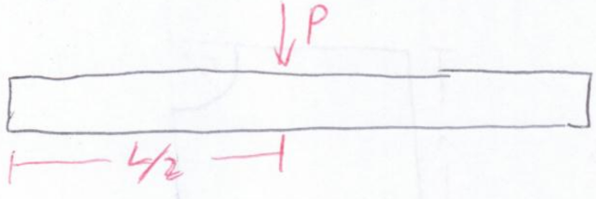
$$\boxed{\bar{y}_T = 4{,}567 \text{ cm}}$$

$$\Rightarrow \bar{I}_1 = I_1 + \delta^2 A = \frac{bh^3}{12} + \left(\frac{h}{2} - \bar{y}_T\right)^2 bh \Rightarrow \boxed{\bar{I}_1 = 1{,}278 \cdot 10^{-5} \text{ m}^4}$$

$$\bar{I}_2 = I_2 + \delta^2 A = \frac{\pi r^4}{16} + \left(\bar{y}_T - \left(h + \frac{4r}{3\pi}\right)\right)^2 \cdot \frac{1}{4}\pi r^2 \Rightarrow \boxed{\bar{I}_2 = 3{,}38 \cdot 10^{-6} \text{ m}^4}$$

$$\therefore \bar{I}_T = \bar{I}_1 - \bar{I}_2 \Longrightarrow \boxed{\bar{I}_T = 9{,}401 \cdot 10^{-6} \text{ m}^4}$$

b) Aplicando la ec. de la curva elástica



$$\frac{d^4 y}{dx^4} = -\frac{\omega(x)}{EI}$$

$$* \ \omega(x) = \delta\left(x - \frac{L}{2}\right) \cdot P$$

· Integrando:

$$\rightsquigarrow \frac{d^3 y}{dx^3} = -\frac{1}{EI} \cdot P \cdot r\left(x - \frac{L}{2}\right) + \alpha$$

$$\rightsquigarrow \frac{d^2 y}{dx^2} = -\frac{1}{EI}\left(x - \frac{L}{2}\right) \cdot P \cdot r\left(x - \frac{L}{2}\right) + \alpha x + \beta$$

$$\rightsquigarrow \frac{dy}{dx} = -\frac{1}{EI} \frac{\left(x - \frac{L}{2}\right)^2}{2} P \cdot r\left(x - \frac{L}{2}\right) + \frac{\alpha x^2}{2} + \beta x + \gamma$$

$$\Rightarrow \hat{y}(x) = \frac{-1}{6EI}\left(x - \frac{L}{2}\right)^3 P \cdot r\left(x - \frac{L}{2}\right) + \frac{\alpha x^3}{6} + \beta \frac{x^2}{2} + \gamma x + \delta$$

· Condiciones de Borde:

$$\hookrightarrow \boxed{\frac{d^2 y}{dx^2}(0) = 0 \ (1) \quad \frac{d^2 y}{dx^2}(L) = 0 \ (2)}$$

↳ Corte dif. en $x = 0$ ⇝

$$\left.\begin{aligned} &V(0) + A = 0 \\ &\Rightarrow V(0) = -A \end{aligned}\right.$$

$$* \ V(x) = -EI \frac{d^3 y}{dx^3}(x) \quad y \quad A = -k_1 \cdot \hat{y}(0)$$

$$\boxed{\therefore -EI \frac{d^3 y}{dx^3}(0) = k_1 \hat{y}(0) \ (3)}$$

↳ Corte dif. en $x = L$

$$\left.\begin{aligned} &V(L) - B = 0 \Rightarrow V(L) = B \\ &* \ B = -k_2 \hat{y}(L) \end{aligned}\right.$$

$$\boxed{\therefore EI \frac{d^3 y}{dx^3}(L) = k_2 \hat{y}(L) \ (4)}$$

• Aplicando las cond. de Borde

(1) $\frac{d^2y}{dx^2}(0)=0 \Rightarrow \boxed{\beta=0}$

(2) $\frac{d^2y}{dx^2}(L)=0 \rightsquigarrow -\frac{1}{EI}(L-\frac{L}{2})\cdot P+\alpha L=0$

$\Rightarrow \boxed{\alpha=\frac{P}{2EI}}$

(3) $-EI\frac{d^3y}{dx^3}(0)=k_1\hat{y}(0)$

$-\alpha EI=k_1\delta \rightsquigarrow \delta=-\frac{\alpha EI}{k_1}=-\frac{P}{2\cancel{EI}}\cdot\frac{\cancel{EI}}{k_1} \Rightarrow \boxed{\delta=\frac{-P}{2k_1}}$

(4) $EI\frac{d^3y}{dx^3}(L)=k_2\hat{y}(L)$

$-P+\alpha EI=k_2\left(-\frac{1}{6EI}(L-\frac{L}{2})^3P+\frac{\alpha L^3}{6}+\gamma L+\delta\right)$

~~* Despejando~~

$-P+\frac{P}{2}=k_2\left(-\frac{L^3P}{48EI}+\frac{\overset{\frac{P}{2EI}}{\alpha} L^3}{6}-\frac{P}{2k_1}\right)+k_2L\gamma$

$\boxed{\therefore \gamma=\frac{1}{k_2L}\left(-\frac{P}{2}-k_2P\left(-\frac{L^3}{48EI}+\frac{L^3}{12EI}-\frac{1}{2k_1}\right)\right)}$

• Una vez calculadas todas las constantes, evaluemos $\hat{y}(\frac{L}{2})$ para encontrar el desplazamiento en el punto C.

$\hat{y}(\frac{L}{2})=\cancel{\alpha\frac{L^3}{6}+\beta\frac{L^2}{2}+\gamma L+\delta}=\alpha\frac{(\frac{L}{2})^3}{6}+\gamma\frac{L}{2}+\delta$

• Calculando las ctes: $\alpha=1{,}4\cdot10^{-3}$

$\gamma=-7{,}4\cdot10^{-3}$

$\delta=-8{,}3\cdot10^{-3}$

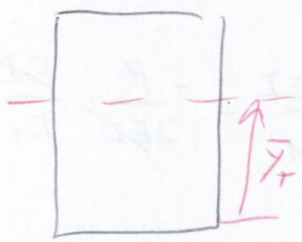
$\boxed{\therefore \hat{y}(\frac{L}{2})=-0{,}0241\ m=-2{,}41\ cm}$

~~a) $M = EI \dfrac{d^2y}{dx^2} \Rightarrow M_{max}$ ~ derivamos $M(x)$, obtenemos x para luego evaluar en $M(x)$~~

~~$\Rightarrow \dfrac{d^3y}{dx^3} = 0 \sim -\dfrac{1}{EI} P \cdot r\left(x - \frac{L}{2}\right) + \dfrac{P}{2EI}$~~

c) $M_{max} \sim \left(\dfrac{d^2y}{dx^2}\right)_{max} \sim x = \dfrac{L}{2}$

$Y_{max} \sim$



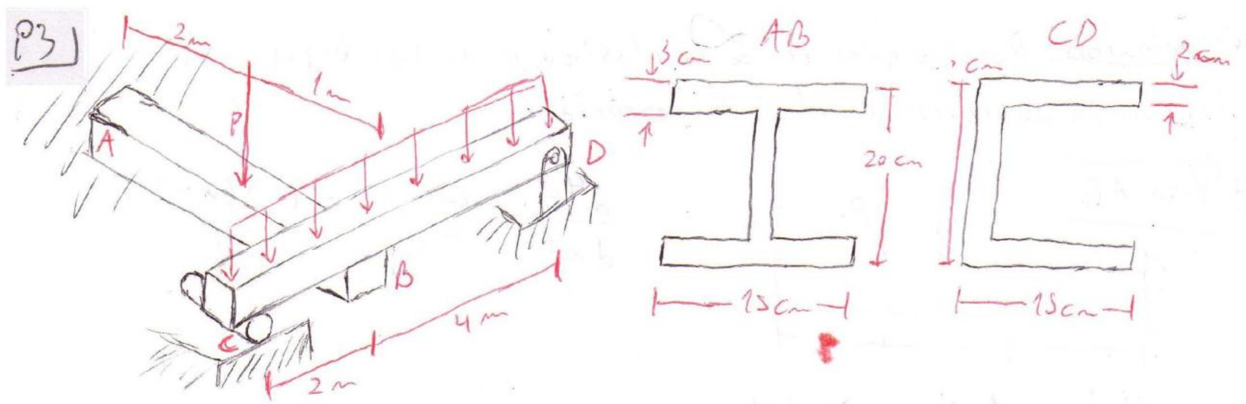
$* \; y = -\bar{Y}_T = -4{,}867 \text{ cm}$

$* \; y = h - \bar{Y}_T = 5{,}433 \text{ cm}$ → Máximo (Extr. sup)

$\therefore \sigma_{max} = -\dfrac{M}{I} y \quad \wedge \quad M = EI \dfrac{d^2y}{dx^2}$

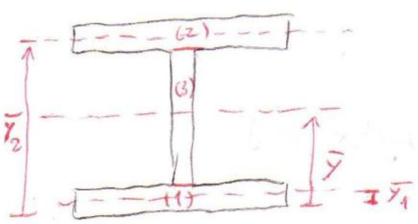
$\Rightarrow \sigma_{max} = -E y_{max} \left(\dfrac{d^2y}{dx^2}\right)_{max} \quad \Bigg\} \quad \dfrac{d^2y}{dx^2}(L/2) = \dfrac{PL}{4EI}$

$$\boxed{\therefore \sigma_{max} = -43{,}342 \text{ MPa}}$$



• Cálculo de propiedades de áreas: $(\bar{I} = I + \delta^2 \cdot A)$

* Viga AB

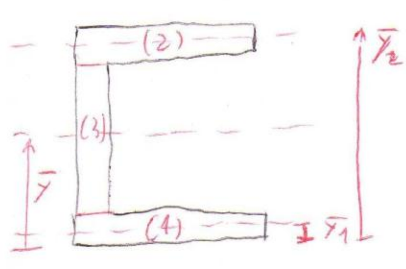


$\bar{y} = 0{,}1\ m$; $\bar{y}_1 = 0{,}015\ m$; $\bar{y}_2 = 0{,}185$

$$\bar{I}_z = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 = \frac{0{,}15 \cdot 0{,}03^3}{12} + (\bar{y} - \bar{y}_1)^2 \cdot 0{,}03 \cdot 0{,}15 + \frac{0{,}15 \cdot 0{,}03^3}{12} + (\bar{y}_2 - \bar{y})^2 \cdot 0{,}03 \cdot 0{,}15 + \frac{0{,}03 \cdot 0{,}14^3}{12}$$

$$\boxed{\bar{I}_z^{AB} = 7{,}256 \cdot 10^{-5}\ m^4}$$

* Viga CD



$\bar{y} = 0{,}075\ m$; $\bar{y}_1 = 0{,}01\ m$; $\bar{y}_2 = 0{,}14\ m$

$$\rightsquigarrow \bar{I}_1 = \frac{0{,}15 \cdot 0{,}02^3}{12} + (0{,}075 - 0{,}01)^2 \cdot 0{,}15 \cdot 0{,}02$$

$$\boxed{\bar{I}_1 = 1{,}278 \cdot 10^{-5}}$$

$$\rightsquigarrow \bar{I}_2 = \frac{0{,}15 \cdot 0{,}02^3}{12} + (0{,}14 - 0{,}075)^2 \cdot 0{,}15 \cdot 0{,}02$$

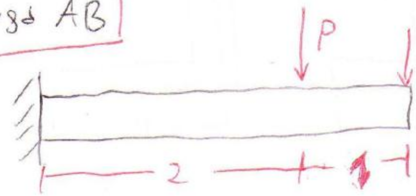
$$\boxed{\bar{I}_2 = 1{,}278 \cdot 10^{-5}}$$

$$\rightsquigarrow \bar{I}_3 = \frac{0{,}02 \cdot 0{,}11^3}{12} = 2{,}218 \cdot 10^{-6}$$

$$\boxed{\therefore \bar{I}_z^{CD} = 2{,}778 \cdot 10^{-5}\ m^4}$$

• Deflexión: Resolveremos ec. de la elástica en ambas vigas y luego igualaremos la deflexión en el punto de contacto

+ $\boxed{\text{Viga AB}}$



$$\frac{d^4y}{dx^4} = -\frac{w(x)}{EI} = -\frac{P\cdot\delta(x-2)}{EI}$$

$$\rightsquigarrow \frac{d^3y}{dx^3} = -\frac{1}{EI}P\,r(x-2) + \alpha$$

$$\rightsquigarrow \frac{d^2y}{dx^2} = -\frac{1}{EI}(x-2)P\cdot r(x-2) + \alpha x + \beta$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{2EI}(x-2)^2 P\,r(x-2) + \alpha\frac{x^2}{2} + \beta x + \gamma$$

$$\rightsquigarrow y(x) = -\frac{1}{6EI}(x-2)^3 P\,r(x-2) + \alpha\frac{x^3}{6} + \beta\frac{x^2}{2} + \gamma x + \delta$$

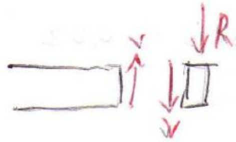
• Condiciones de Borde

$$* \; y(0) = 0 \Rightarrow \boxed{\delta = 0}$$

$$* \; \frac{dy}{dx}(0) = 0 \Rightarrow \boxed{\gamma = 0}$$

$$* \; \frac{d^2y}{dx^2}(3) = 0 \Rightarrow -\frac{1}{EI}(3-2)\cdot P + 3\alpha + \beta = 0$$

$$\boxed{\beta = \frac{1}{EI}\cdot P - 3\alpha}$$

$*$  $\Rightarrow EI\,\dfrac{d^3y}{dx^3}(3) = R$

$$-P + \alpha EI = R \Rightarrow \boxed{\alpha = \frac{1}{EI}(R+P)}$$

$$\boxed{\beta = \frac{-1}{EI}(3R+2P)}$$

$$\boxed{\therefore y(x) = -\frac{1}{EI}\left(\frac{(x-2)^3}{6}P\,r(x-2) - \frac{(R+P)}{6}x^3 + \frac{(3R+2P)}{2}x^2\right)}$$

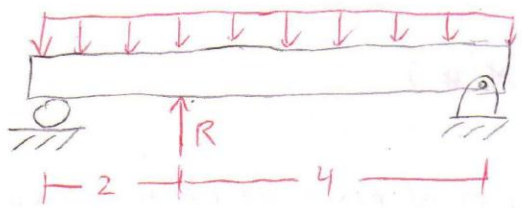
• La deflexión en el extremo de la viga:

$$y(3) = \frac{1}{EI}\left(-\frac{1}{6}P + \frac{9}{2}R + \frac{9}{2}P - \frac{27}{2}R - 9P\right)$$

$$= \frac{1}{EI}\left[\left(-\frac{1}{6}+\frac{9}{2}-9\right)P + \left(\frac{9}{2}-\frac{27}{2}\right)R\right]$$

$$\boxed{y(3) = \frac{1}{EI}\left(-4{,}667P - 9R\right)} \quad (*)$$

* $\boxed{\text{Viga CD}}$



$$\frac{d^4y}{dx^4} = -\frac{1}{EI}\left(w_0 - R\delta(x-2)\right)$$

$$\rightsquigarrow \frac{d^3y}{dx^3} = -\frac{1}{EI}\left(w_0 x - R\,r(x-2)\right) + \alpha$$

$$\rightsquigarrow \frac{d^2y}{dx^2} = -\frac{1}{EI}\left(w_0\frac{x^2}{2} - (x-2)R\,r(x-2)\right) + \alpha x + \beta$$

$$\rightsquigarrow \frac{dy}{dx^2} = -\frac{1}{EI}\left(w_0\frac{x^3}{6} - \frac{(x-2)^2}{2}R\,r(x-2)\right) + \alpha\frac{x^2}{2} + \beta x + \gamma$$

$$\rightsquigarrow y(x) = -\frac{1}{EI}\left(w_0\frac{x^4}{24} - \frac{(x-2)^3}{6}R\,r(x-2)\right) + \alpha\frac{x^3}{6} + \beta\frac{x^2}{2} + \gamma x + \delta$$

• Condiciones de Borde

* $y(0) = 0 \Rightarrow \boxed{\delta = 0}$
* $\frac{d^2y}{dx^2}(0) = 0 \Rightarrow \boxed{\beta = 0}$
* $\frac{d^2y}{dx^2}(6) = 0 \Rightarrow -\frac{1}{EI}\left(w_0\frac{6^2}{2} - (6-2)R\right) + 6\alpha = 0$

$$\boxed{\alpha = \frac{1}{6EI}\left(18w_0 - 4R\right)}$$

* $y(6) = 0 \Rightarrow -\frac{1}{EI}\left(w_0\cdot\frac{6^4}{24} - \frac{(6-2)^3}{6}R\right) + \frac{1}{\cancel{6}EI}(18w_0 - 4R)\cdot\frac{6^3}{\cancel{6}} + 6\gamma = 0$

$$\beta = \frac{1}{EI}\left(9w_0 - \frac{16}{9}R\right) - \frac{1}{EI}\left(18w_0 - 4R\right)$$

$$\boxed{y_1 = \frac{1}{EI}\left(-9w_0 + \frac{20}{9}R\right)}$$

$$\Rightarrow y(x) = -\frac{1}{EI}\left(w_0\frac{x^4}{24} - \frac{(x-2)^3}{6}R \cdot r(x-2) - \frac{x^3}{6}\left(3w_0 - \frac{2}{3}R\right) - x\left(-9w_0 + \frac{20}{9}R\right)\right)$$

• La deflexión en $x = 2$ m:

$$y(2) = -\frac{1}{EI}\left(w_0\frac{2^4}{24} - \frac{2^3}{6}\left(3w_0 - \frac{2}{3}R\right) - 2\left(-9w_0 + \frac{20}{9}R\right)\right)$$

$$= \frac{1}{EI}\left(-\frac{2}{3}w_0 + 4w_0 - \frac{8}{9}R - 18w_0 + \frac{40}{9}R\right)$$

$$\boxed{y(2) = \frac{1}{EI}\left(-\frac{44}{3}w_0 + \frac{32}{9}R\right)} \quad (**)$$

• Dado que las deflexiones son iguales en el punto de contacto, igualamos (*) con (**)

$$\frac{1}{EI_{AB}}\left(-4{,}667P - 9R\right) = \frac{1}{EI_{CD}}\left(-14{,}667w_0 + 3{,}556R\right)$$

$$R\left(\frac{3{,}556}{I_{CD}} + \frac{9}{I_{AB}}\right) = \frac{14{,}667w_0}{I_{CD}} - \frac{4{,}667P}{I_{AB}}$$

$$\boxed{\therefore R = 792{,}195 \text{ N}}$$