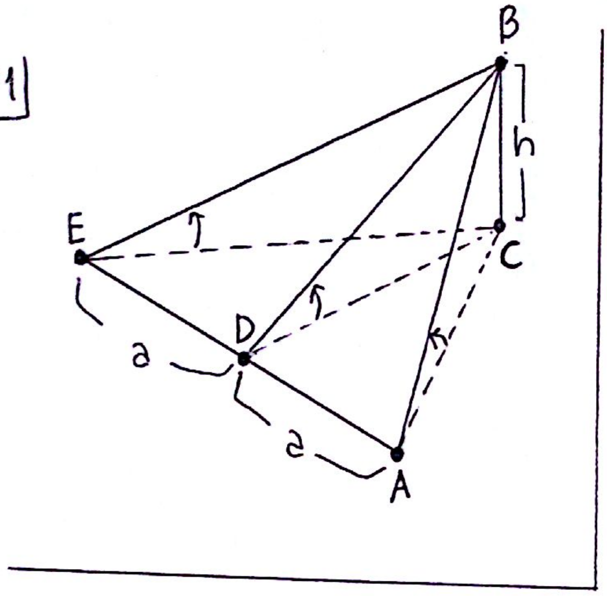
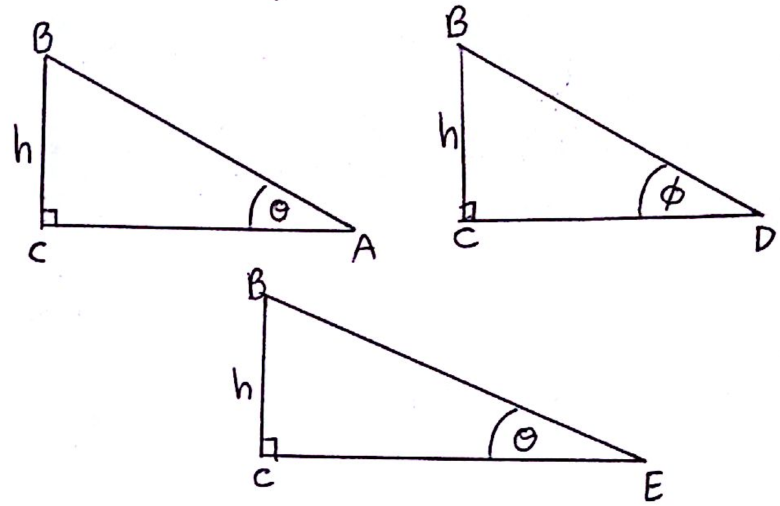
Auxiliar 7

Solución

P1



Dadas las condiciones del problema, tenemos los siguientes triángulos:

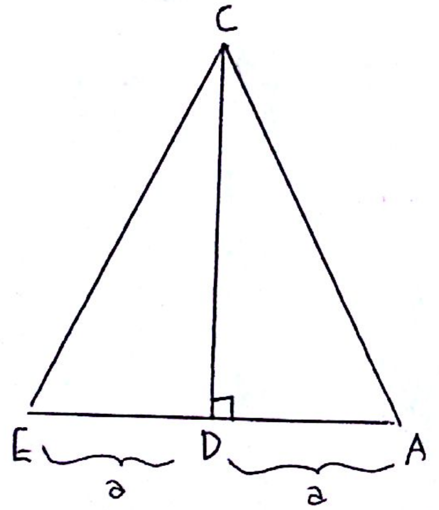


Utilizando la función tangente en cada triángulo:

$$\tan\theta = \frac{h}{AC} \Rightarrow AC = \frac{h}{\tan\theta} \quad ; \quad \tan\theta = \frac{h}{CE} \Rightarrow CE = \frac{h}{\tan\theta}.$$

$$\tan\phi = \frac{h}{CD} \Rightarrow CD = \frac{h}{\tan\phi}$$

Note que $CE = AC$, por lo que el ΔEAC es isósceles de base $\overline{AE}$. Puesto que D es punto medio de AE, se cumple $EA \perp DC$.



Por Pitágoras en ΔDAC, tenemos:

$$CD^2 + DA^2 = AC^2$$

$$\Leftrightarrow \frac{h^2}{\tan^2\phi} + a^2 = \frac{h^2}{\tan^2\theta}$$

$$\Leftrightarrow a^2 = \frac{h^2}{\tan^2\theta} - \frac{h^2}{\tan^2\phi} = h^2\left(\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\phi}\right)$$

Trabajemos la expresión $\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\phi}$:

$$\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\phi} = \frac{\cos^2\theta}{\operatorname{sen}^2\theta} - \frac{\cos^2\phi}{\operatorname{sen}^2\phi} = \frac{\operatorname{sen}^2\phi\cos^2\theta - \operatorname{sen}^2\theta\cos^2\phi}{\operatorname{sen}^2\theta\ \operatorname{sen}^2\phi}$$

$$= \frac{(\operatorname{sen}\phi\cos\theta + \operatorname{sen}\theta\cos\phi)(\operatorname{sen}\phi\cos\theta - \operatorname{sen}\theta\cos\phi)}{\operatorname{sen}^2\theta\ \operatorname{sen}^2\phi}$$

$$= \frac{\operatorname{sen}(\phi+\theta)\operatorname{sen}(\phi-\theta)}{\operatorname{sen}^2\theta\ \operatorname{sen}^2\phi}$$

Luego

$$a^2 = h^2\left(\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\phi}\right) = h^2\ \frac{\operatorname{sen}(\phi+\theta)\operatorname{sen}(\phi-\theta)}{\operatorname{sen}^2\theta\ \operatorname{sen}^2\phi}$$

$$\Rightarrow \quad h^2 = \frac{a^2\ \operatorname{sen}^2\theta\ \operatorname{sen}^2\phi}{\operatorname{sen}(\phi+\theta)\operatorname{sen}(\phi-\theta)} = a^2\operatorname{sen}^2\theta\ \operatorname{sen}^2\phi\ \csc(\phi+\theta)\csc(\phi-\theta)$$

Lo último recordando que $\csc(x) = \frac{1}{\operatorname{sen}(x)}$. Ahora sacamos raíz cuadrada, y puesto que "h" es una altura, nos quedamos con la solución positiva.

$$h = a\operatorname{sen}\theta\ \operatorname{sen}\phi\sqrt{\csc(\phi+\theta)\csc(\phi-\theta)}$$

P2]

- $$\frac{1-\sin(2x)}{\cos(2x)} = \frac{1-2\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)} = \frac{(\cos^2x+\sin^2x)-2\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}$$

$$= \frac{\cos^2x - 2\cos(x)\sin(x) + \sin^2(x)}{\cos^2(x)-\sin^2(x)}$$

$$= \frac{(\cos(x)-\sin(x))^2}{(\cos(x)+\sin(x))(\cos(x)-\sin(x))} = \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}$$

$$= \frac{\cos(x)(1-\tan(x))}{\cos(x)(1+\tan(x))} = \frac{1-\tan(x)}{1+\tan(x)} \ //$$

- pdq: $\sin(\alpha)+\sin(\beta)+\sin(\gamma) = 4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)$
 cuando $\alpha+\beta+\gamma=\pi$.

Usando la indicación, $\gamma = \pi-(\alpha+\beta)$ y

$$\sin(\gamma) = \sin(\pi-(\alpha+\beta)) = \sin(\pi)\cos(\alpha+\beta) - \cos(\pi)\,\mathrm{sen}(\alpha+\beta) = \mathrm{sen}(\alpha+\beta)$$

$$\cos\left(\frac{\gamma}{2}\right) = \cos\left(\frac{\pi}{2}-\left(\frac{\alpha+\beta}{2}\right)\right) = \cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) + \mathrm{sen}\left(\frac{\pi}{2}\right)\mathrm{sen}\left(\frac{\alpha+\beta}{2}\right) = \mathrm{sen}\left(\frac{\alpha+\beta}{2}\right)$$

Por lo tanto queremos demostrar que

$$\sin(\alpha)+\sin(\beta)+\sin(\alpha+\beta) = 4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$

Recordemos las identidades:

$$\sin(\alpha)+\sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin(\alpha+\beta) = \sin\left(2\cdot\left(\frac{\alpha+\beta}{2}\right)\right) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Por lo tanto:

$$\sin(\alpha) + \sin(\beta) + \sin(\alpha+\beta)$$
$$= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$
$$= 2\sin\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\right]$$
$$= 2\sin\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) + \cancel{\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)} + \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) \cancel{- \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)}\right]$$
$$= 4\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) //$$

P3)

1. $\cos(x) + 1 = \sin(x)$

$\sin(x) - \cos(x) = 1$; recordemos que $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$\sin(x) - \cos(x) = 1 \quad / \cdot \frac{1}{\sqrt{2}}$

$\sin(x) \cdot \frac{1}{\sqrt{2}} - \cos(x) \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\sin(x)\cos\left(\frac{\pi}{4}\right) - \cos(x)\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

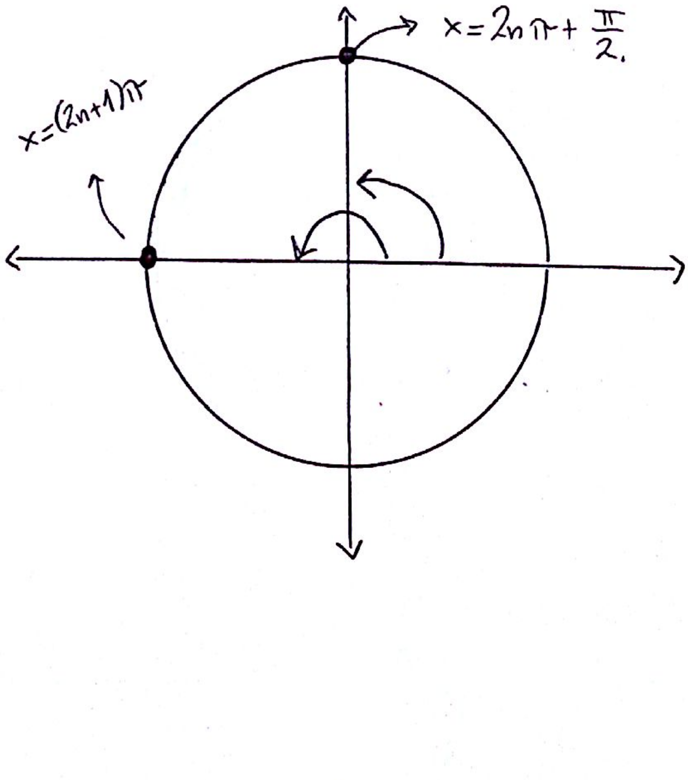
$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$; un ángulo β que cumple $\sin(\beta) = \frac{1}{\sqrt{2}}$ es $\frac{\pi}{4}$.

$\Rightarrow \quad x - \frac{\pi}{4} = K\pi + (-1)^K \beta = K\pi + (-1)^K \frac{\pi}{4}$

$\Rightarrow \quad x = K\pi + (-1)^K \frac{\pi}{4} + \frac{\pi}{4}$.

Si K es par, $(-1)^K = 1$ y $x = 2n\pi + \frac{\pi}{2}$ $(K = 2n)$.

Si K es impar, $(-1)^K = -1$ y $x = (2n+1)\pi$ $(K = 2n+1)$.



2: $\sin(2x) + \cos(2x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$ $\Rightarrow \cos(x) \neq 0$

$$2\sin(x)\cos(x) + \cos^2(x) - \sin^2(x) = \frac{\sin(x)}{\cos(x)} \quad / \cdot \cos(x)$$

$$2\sin(x)\cos^2(x) + \cos(x)\left(\cos^2(x) - \sin^2(x)\right) = \sin(x)$$

Para evitar el término $\cos^3(x)$, escribamos (recordando que [illegible]$^2(x) = 1-$ [illegible]$^2(x)$):

$$\cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x).$$

$$\Rightarrow 2\sin(x)\cos^2(x) + \cos(x) - 2\sin^2(x)\cos(x) = \sin(x)$$

$$2\sin(x)\cos^2(x) - 2\sin^2(x)\cos(x) + \cos(x) - \sin(x) = 0$$

$$2\sin(x)\cos(x)\left(\cos(x) - \sin(x)\right) + \left(\cos(x) - \sin(x)\right) = 0$$

$$\left(2\sin(x)\cos(x) + 1\right)\left(\cos(x) - \sin(x)\right) = 0$$

$$\left(\sin(2x) + 1\right)\left(\cos(x) - \sin(x)\right) = 0$$

$$\Rightarrow \sin(2x) = -1 \quad \vee \quad \cos(x) = \sin(x) \quad / \cdot \frac{1}{\cos(x)}$$

$$\Rightarrow \sin(2x) = -1 \quad \vee \quad 1 = \tan(x)$$

Un ángulo cuyo seno es -1 es $-\frac{\pi}{2}$, y un ángulo cuya tangente es 1 es $\frac{\pi}{4}$. Luego

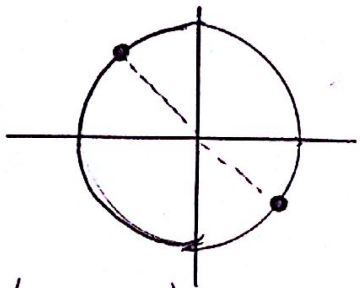
- $\sin(2x) = -1 \Rightarrow 2x = k\pi + (-1)^k\left(-\frac{\pi}{2}\right) = k\pi - (-1)^k\frac{\pi}{2}.$

$$\Rightarrow x = k\frac{\pi}{2} - (-1)^k\frac{\pi}{4}.$$

Si $k = 2n$, $x = n\pi - \frac{\pi}{4}$

Si $k = 2n+1$, $x = (2n+1)\frac{\pi}{2} + \frac{\pi}{4}$

$= n\pi + \frac{3\pi}{4}$



(note que este segundo caso está contenido en el primero)

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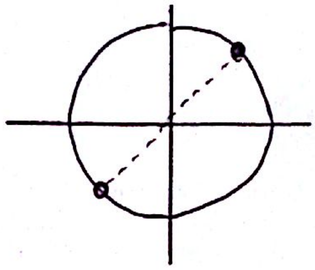
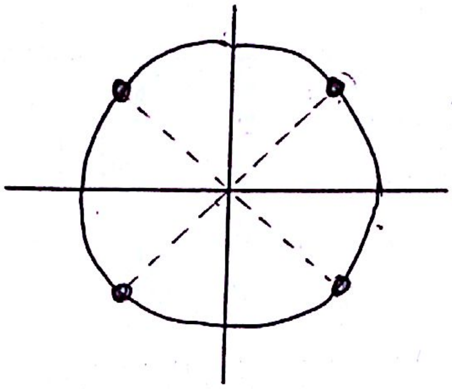
- $\tan(x) = 1 \implies x = k\pi + \frac{\pi}{4}$



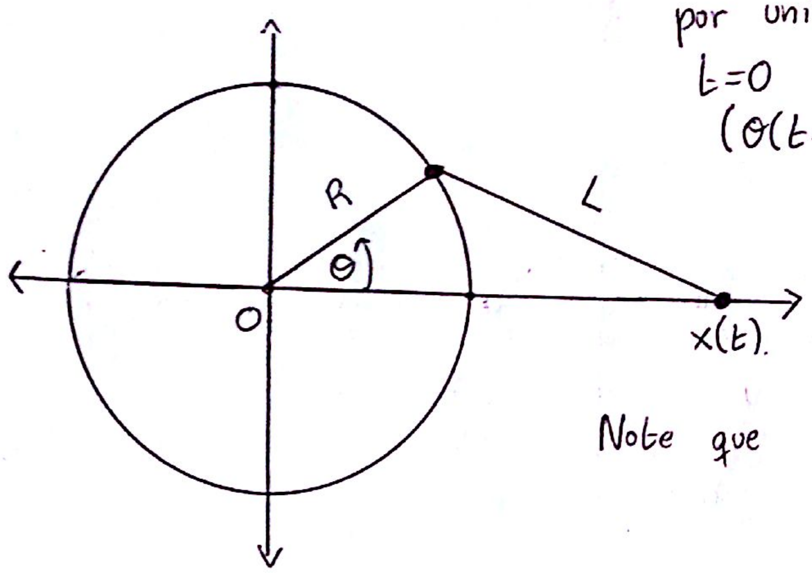
Solución general:



$$x = \frac{k\pi}{2} + \frac{\pi}{4}.$$

P4

1.-



Note que, como da 1 vuelta por unidad de tiempo y en $t=0$ la varilla está horizontal ($\theta(t=0)=0$), tenemos

$$\theta(t) = 2\pi t$$

$*$ 1 vuelta es 2π.

Note que $L-R \leq x(t) \leq L+R$, y $L>R$.

Por el teorema del coseno tenemos

$$L^2 = R^2 + x^2 - 2Rx\cos\theta$$

$$x^2 - 2R\cos\theta \cdot x + R^2 - L^2 = 0$$

$$\left(x^2 - 2R\cos\theta + R^2\cos^2\theta\right) + R^2 - L^2 - R^2\cos^2\theta = 0$$

$$(x - R\cos\theta)^2 = L^2 - R^2(1-\cos^2\theta) = L^2 - R^2\sin^2\theta$$

$$x - R\cos\theta = \pm\sqrt{L^2 - R^2\sin^2\theta} \qquad (*)$$

$$x = R\cos\theta \pm \sqrt{L^2 - R^2\sin^2\theta} \qquad (**)$$

$(*)$ Note que, como $L>R$, se tiene $L^2 > R^2 \geq R^2\sin^2\theta$, puesto que siempre $\sin^2\theta \leq 1$.

$(**)$, como $x(t) \geq L-R > 0$, $x(t)$ siempre debe ser "+". Como $R\cos\theta$ puede ser negativo, necesariamente nos tenemos que quedar con el "+" en el "$\pm$". Luego, recordando que $\theta = 2\pi t$:

$$x(t) = R\cos(2\pi t) + \sqrt{L^2 - R^2\sin^2(2\pi t)} \quad //$$

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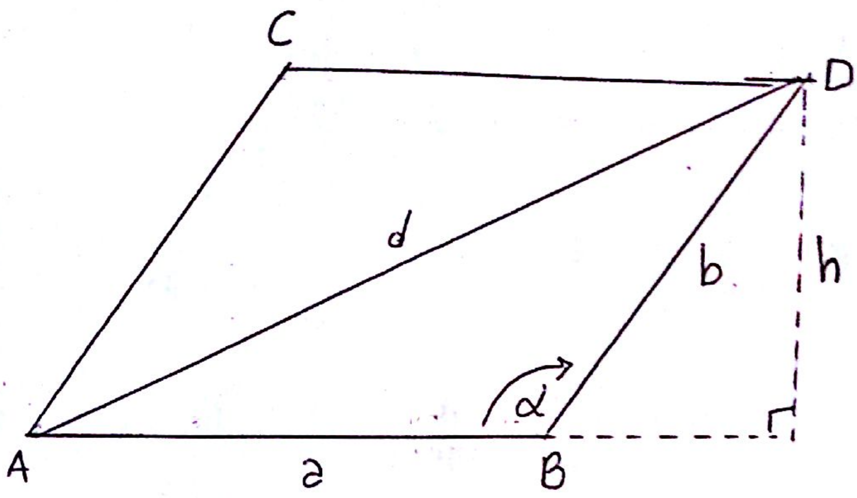
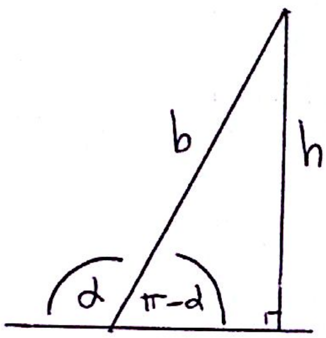
P4)

2.-



a) El área es simplemente $S = ah$. Para "h" tenemos lo sgte:



$$\operatorname{sen}(\pi-\alpha) = \frac{h}{b} \Rightarrow h = b\operatorname{sen}(\pi-\alpha)$$

Pero $\operatorname{sen}(\pi-\alpha) = \operatorname{sen}(\pi)\cos(\alpha) - \cos(\pi)\operatorname{sen}(\alpha) = \operatorname{sen}(\alpha)$

$$\Rightarrow \quad h = b\operatorname{sen}(\alpha) \Rightarrow S = ab\operatorname{sen}(\alpha).$$

b) Por el teorema del coseno tenemos

$$d^2 = a^2 + b^2 - 2ab\cos(\alpha)$$

Y ademas $(2a+2b) = \text{Perímetro} = 2p \Rightarrow a+b = p \Rightarrow (a+b)^2 = p^2$

$$\Rightarrow a^2+b^2 = p^2 - 2ab.$$

Luego $d^2 = p^2 - 2ab - 2ab\cos(\alpha)$

$$\Rightarrow p^2 - d^2 = 2ab + 2ab\cos(\alpha) = 2ab(1+\cos(\alpha))$$

Pero $S = ab\operatorname{sen}(\alpha) \Rightarrow ab = \dfrac{S}{\operatorname{sen}(\alpha)}$. Luego

$$\frac{p^2-d^2}{2} = \frac{S}{\operatorname{sen}(\alpha)}(1+\cos(\alpha))$$

$$\Rightarrow \quad S = \frac{p^2-d^2}{2}\left(\frac{\operatorname{sen}\alpha}{1+\cos(\alpha)}\right)$$

Necesitamos que el último factor sea $\tan\left(\frac{\alpha}{2}\right)$. Tenemos que:

$$\operatorname{sen}(\alpha) = 2\operatorname{sen}\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)$$

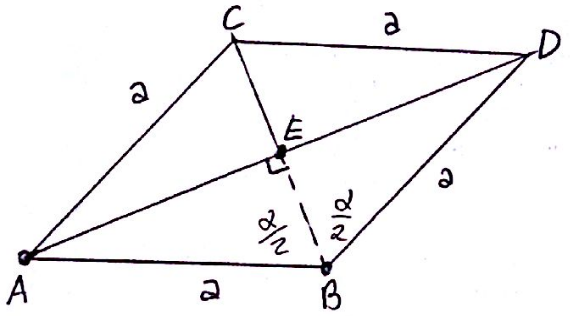
$$\cos(\alpha) = \cos^2\left(\frac{\alpha}{2}\right) - \operatorname{sen}^2\left(\frac{\alpha}{2}\right) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1.$$

Luego

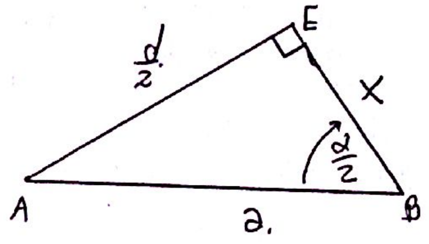
$$\frac{\operatorname{sen}(\alpha)}{1+\cos(\alpha)} = \frac{2\operatorname{sen}\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\alpha}{2}\right)} = \frac{\operatorname{sen}\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} = \tan\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow \quad S = \frac{p^2-d^2}{2}\tan\left(\frac{\alpha}{2}\right) //$$

c) Si $a=b$, entonces



BE es bisectriz debido a que es un rombo. En ΔABE tenemos



Donde $\tan\left(\frac{\alpha}{2}\right) = \dfrac{\frac{d}{2}}{x} = \dfrac{d}{2x}$ y por pitágoras $x = \sqrt{a^2 - \frac{d^2}{4}}$

Pero $a = p \Rightarrow = \frac{p}{2} \Rightarrow \quad x = \sqrt{\frac{p^2}{4} - \frac{d^2}{4}} = \frac{\sqrt{p^2-d^2}}{2.}$ y

$$\tan\left(\frac{\alpha}{2}\right) = \frac{d}{2x} = \frac{d}{\sqrt{p^2-d^2}}$$

Por lo tanto

$$S = \frac{p^2-d^2}{2}\tan\left(\frac{\alpha}{2}\right) = \frac{p^2-d^2}{2} \cdot \frac{d}{\sqrt{p^2-d^2}} = \frac{d}{2}\sqrt{p^2-d^2}$$